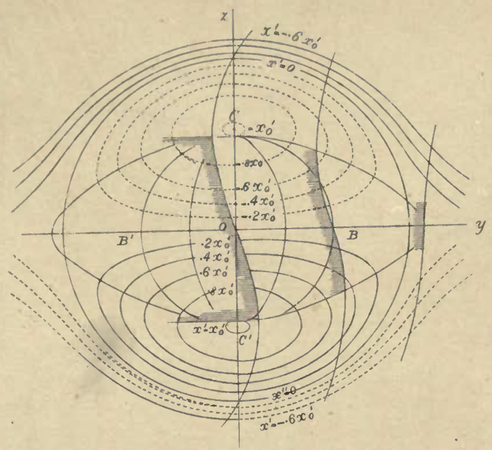

FIG. I.

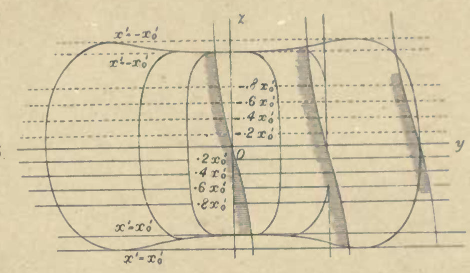

FIG. 3.

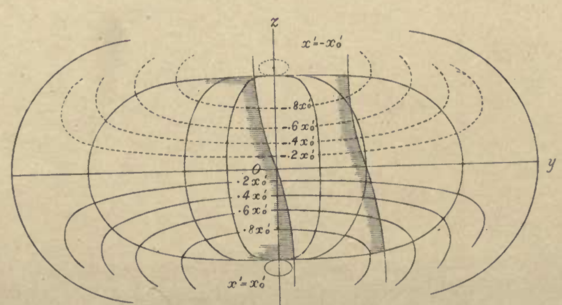

FIG. 2.

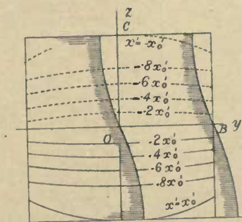

FIG. 4.

Drawn by G Baker after Saint-Venant.

Frontispiece to Vol. II.

Distorted Cross-Sections of Beams under Flexure.

(*Extract from Vol. II. of Todhunter's History of the Theory of Elasticity.*)

THE ELASTICAL RESEARCHES

OF

BARRÉ DE SAINT-VENANT

EDITED

FOR THE SYNDICS OF THE UNIVERSITY PRESS

BY

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TO THE DISCIPLES OF

BARRÉ DE SAINT-VENANT

WHO HAVE SO WORTHILY CARRIED ON HIS WORK

I SHOULD WISH TO DEDICATE THIS ACCOUNT

OF THEIR MASTER'S RESEARCHES.

Il appartenait à cette noble race d'esprits tout à la fois force, ornement et consolation de notre espèce, qui, ensemble ou isolés, laissent après eux, dans l'histoire, une traînée lumineuse, à laquelle s'allument sans cesse de nouvelles intelligences, pour former un courant toujours plus vaste de vraie et haute civilisation.

PREFACE[1].

THE portion of the second volume of the *History of Elasticity* now published as an extract is devoted solely to the later researches of Saint-Venant. It may appear to the reader that an undue amount of space has been allotted to this one scientist, but it must be remembered that these researches extend over a period of *thirty-five* years, and in a certain sense contain in themselves the history of elasticity during those years. Their importance arises not only from what they themselves contribute to our subject, but from the fact that so much of the important work of the last three decades has been suggested by or has followed the lines of Saint-Venant's papers. The contributions of such elasticians as Kirchhoff and Clebsch, as Boussinesq and Lévy to both elasticity and plasticity frequently take their starting point from an idea of Saint-Venant's, while a host of minor memoirs help to fill up the gaps in his work. Thus our Chapter X. may be taken as the frame, the stoutness and solidity of which will enable us the more readily to build up the remainder of our history. By explaining the ideas, definitions and methods of the modern science of elasticity, it allows of a more succinct and orderly account of the contemporary work.

[1] This preface as well as the index refer only to the present extract.

guide in this direction, till the years have given time for the completion of the whole *History*.

The responsibility, however, for the space and the manner in which Saint-Venant's researches have been dealt with rests entirely with the editor. Hence any disproportion, any possibly crude opinions, any erroneous criticism in this part of the work must be attributed to him and not to the author. The articles in this part due to the late Dr Todhunter are only three or four in number and these refer to the less important memoirs. They are marked as in Volume I. by the omission of square brackets round the article number, *e.g.* 101, 102, 103. This editorial preponderance requires some explanation, which will be found partly in a consideration of Dr Todhunter's original plan, partly in the dates of Saint-Venant's memoirs. As I have pointed out in the preface to Volume I. the original plan of this work was a first volume devoted to 'Theory' and a second to 'History.' Thus it arose that the important memoirs on *Torsion* and on *Flexure* were incorporated in the 'Theory' volume, and the manner in which they appear in that text book does not allow of their transfer to the 'History.' Large parts of these memoirs were at the same time omitted as having little mathematical interest, for example the entire discussions of the "fail-limit" formula and of combined strain: see Arts. 5, 52—60, etc. of this Part. Further the memoirs on *Impact* (written before 1870) were dismissed as falling outside the limits which Dr Todhunter had set to his work. Thus he wrote of the memoirs on transverse impact, that they form "an interesting investigation of a mechanical problem, but it does not belong to our subject." The large section I have devoted to the *Leçons de Navier* would also not have formed part of the original plan. The only reference to this work which occurs in Dr Todhunter's MS. is the following: "The third edition of Navier's

Leçons sur l'application de la Mécanique... was published I think in 1863 or 1864 with notes by Saint-Venant, but I have not seen it."

Of the majority of the other memoirs falling before 1870, there exist in most cases only brief extracts in French from the introductory paragraphs. This would be sufficient to shew, without the occasional remarks for changes scattered about the pages, that the MS. is not in the form in which Dr Todhunter would finally have published it. To these considerations must be added those arising from the fact that all the memoirs and papers after 1871, including some of the most important of Saint-Venant's researches on intermolecular action, plasticity, impact and above all the *Annotated Clebsch* (see our Arts. 245—408), fall beyond the date to which Dr Todhunter had carried his work.

Such then are the reasons which have led to the reconstruction of this portion of the *History*. I have felt the heavy responsibility involved in adopting this course. But it has seemed to me that the best memorial to the first Cambridge historian of mathematics would be that the last history bearing his name should have the widest possible sphere of usefulness. That usefulness will, I am firmly convinced, be best obtained by its comprehensive character, by its attempt to be a *Repertorium* of elasticity rather than an *Historique Abrégé* of its purely mathematical side. I recognise fully with my kindly French critic of the *Bulletin* the importance of the latter, but its possibility will only be actual on the completion of the former. To follow, even at a distance, Wiedemann rather than Bertrand must be the editor's duty; not an easy one it must be confessed, but one requiring time and labour rather than historical talent and insight. In the words of the German proverb: *wer giebt was er hat, der ist werth, dass er lebt.*

For assistance in the revision of the proofs of this part I have in the first place to thank Mr C. Chree of King's College, Cambridge. His suggestions and corrections have been of extreme value to me, and his criticisms enabled me to remove many imperfections. Mr W. H. Macaulay has again lent me his aid in the discussion of mathematical and other difficulties; while I am much indebted to M. Flamant, Professeur à l'École des Ponts et Chaussées, for the generosity with which he has devoted a portion of his busy time to reading the proofs of this account of his friend and master's researches. His corrections have been of much service, especially in the French portions of the work. For the temporary index attached to this part I am, so far as the titles are concerned, personally responsible. Perhaps, only the writer of a book is in a position to prepare an efficient set of titles. I have, however, to thank Miss L. Eckenstein for the more laborious task of alphabetically arranging the titles and for a verification of the entries.

KARL PEARSON.

UNIVERSITY COLLEGE, LONDON,
December 29, 1888.

CONTENTS.

CHAPTER X.

SAINT-VENANT, 1850—1886.

References throughout this extract to the articles of the *first* volume have an asterisk affixed, *e.g.* Art. 128*. Numbers without an asterisk refer to the articles of the extract.

ERRATA.

Art. 4 (δ) dele reference to Hopkins.

Art. 171 (*a*) for *neutral line* read *neutral axis.*

CHAPTER X.

SAINT-VENANT, 1850—1886.

SECTION I. *Torsion.*

[1.] WE commence our second volume with some account of the later work of the great French elastician whom we are justified in placing beside Poisson and Cauchy. From the last memoir referred to in our first volume till June 13, 1853 we have nothing to report. A slight note, however, entitled: *Divers résultats relatifs à la torsion,* which was read to the *Société philomathique* (*Bulletin,* February 26, 1853, or *L'Institut,* no. 1002, March 16, 1853), sufficiently indicates that our author had been diligently at work during these years on his new theory of torsion. On the 13th of June, 1853, his epoch-making memoir was read to the Academy (*Comptes rendus,* T. XXXVI. p. 1028). The memoir was inserted in T. XIV. of the *Mémoires des Savants étrangers,* 1855, pp. 233—560, under the title:

Mémoire sur la Torsion des Prismes, avec des considérations sur leur flexion, ainsi que sur l'équilibre intérieur des solides élastiques en général, et des formules pratiques pour le calcul de leur résistance à divers efforts s'exerçant simultanément.

We have referred to it in our first volume as the memoir on *Torsion,* and shall continue to do so.

The memoir was referred by the Academy to a committee consisting of Cauchy, Poncelet, Piobert and Lamé. Their report drawn up by Lamé (*Comptes rendus,* T. XXXVII., December 26, 1853, pp. 984—8) speaks very highly of the memoir. We cite the concluding words:

Le travail dont nous venons de rendre compte, mérite des éloges à plus d'un titre : par les nombres et les résultats nouveaux qu'il offre aux arts industriels, il constate, une fois de plus, l'importance de la théorie de l'équilibre d'élasticité ; par l'emploi de la méthode mixte, il indique comment les ingénieurs, qui veulent s'appuyer sur cette théorie, peuvent utiliser tous les procédés actuellement connus de l'analyse mathématique ; par ses tables, ses épures, et ses modèles en relief[1], il donne la marche qu'il faut nécessairement suivre, dans ce genre de recherches, pour arriver à des résultats immédiatement applicables à la pratique ; enfin, par la variété de ses points de vue, il offre un nouvel exemple de ce que peut faire la science du géomètre, unie à celle de l'ingénieur. (p. 988.)

The report gives a succinct account of the memoir. A second account by Saint-Venant himself will be found in : *Notice sur les travaux et titres scientifiques de M. de Saint-Venant*, Paris, 1858, pp. 19—31, and 71—80. This work together with one of the same title published in 1864, when Saint-Venant was again a candidate for the *Institut*, gives an excellent *résumé* of our author's researches previous to 1864. We shall refer to them briefly as *Notice* I. and *Notice* II.

[2.] The memoir itself is principally occupied with the torsion of *prisms*, a great variety of cross-sections being dealt with. This particular problem in torsion has been termed by Clebsch : *Das de Saint-Venantsche Problem* (*Theorie der Elasticität*, S. 74), and following him we shall term it *Saint-Venant's Problem.* The memoir consists of thirteen chapters.

3. The first chapter occupies pp. 233—236 ; and gives an introductory sketch of the contents of the memoir. If the values of the shifts of the several points of an elastic body are given the stresses can be easily found by simple differentiation. But the inverse problem—to find the shifts when the stresses are given—has not been generally solved, because we do not yet know how to integrate the differential equations which present themselves. Saint-Venant accordingly proposes the adoption of a *mixed method* (*méthode mixte ou semi-inverse*), which consists in assuming a part of the shifts and a part of the stresses, and then determining by an exact analysis what the remaining shifts and the remaining

[1] Copies of these numerous models are at present deposited in the mathematical model cases at University College. They represent much better than the poor woodcuts of the original memoir the distortion of the various cross-sections.

stresses must be. Before proceeding to the torsion of prisms Saint-Venant illustrates this *mixed method* in the third and fourth chapters of his memoir by applying it to simple problems.

[4.] The second chapter occupies pp. 236—288; it analyses strain and stress and investigates the general formulae for the equilibrium of elastic bodies. In 1868 Saint-Venant contributed to Moigno's *Statique* another elementary discussion of the fundamental formulae of elasticity; the later work is somewhat fuller and contains the more matured views of the author; the earlier is, however, very good. I will note the leading features of the treatment adopted:

(α) On p. 236 Saint-Venant defines the shifts as the *déplacements moyens* or as the *déplacements des centres de gravité de groupes d'un certain nombre de molécules.* He thus starts from the molecular standpoint, but this definition does not appear to be absolutely necessary to the course of his reasoning.

(β) On pp. 237—248 we have the analysis of strain. Here the slides first defined by Navier and Vicat (see our Vol. I. p. 877), and then theoretically considered by Saint-Venant in the *Cours lithographié* (see our Art. 1564*), are for the first time introduced by name and directly from their physical meaning into a general theory of elasticity. The slide of two lines primitively rectangular is defined as the *cosine of the angle between them* after strain (p. 238).

(γ) On p. 239 Saint-Venant carefully limits his researches to very small strains within the elastic limit, so that what he says later (pp. 281—288) on the conditions of *rupture*, must when applied to his torsion problems be interpreted only of the elastic limit. Indeed, as for certain materials, set is produced by any initial loading below the yield-point and is not practically dangerous (i.e. the material is not 'enervated,' to use Saint-Venant's language), we can only look upon the conditions of torsional rupture given in the memoir as of value when either (1) the material is elastic and *follows Hooke's Law* nearly up to rupture (cf. the steel bar H of the plate p. 893 of our Vol. I.), or, (2) the material has a state of ease extending almost up to the yield-point.

(δ) On pp. 242—5 we have the general expressions for s_r and $\sigma_{rr'}$. The first is due to Navier in his memoir of 1821, the second is attributed by Saint-Venant to Lamé (*Leçons...l'élasticité*, 1852, p. 46) but as we have seen it had been previously given by Hopkins in 1847 (see our Art. 1368*). From the second flows naturally a discussion of principal and maximum slide, together with a proof of Saint-Venant's theorem that a slide is equal to a stretch and a squeeze of half the magnitude of the slide in the bisectors of the slide angles (see our Art. 1570*).

Finally the strain is expressed for *small* shifts in terms of the shift-fluxions (pp. 246—8). There is reference in a footnote to the strain-values for *large* shifts (see our Art. 1618*).

(ϵ) We next pass to an analysis of stress on pp. 248—254. Stress is defined from the molecular standpoint as follows:

Nous appellerons donc en général *Pression, sur un des deux côtes d'une petite face plane imaginée à l'intérieur d'un corps ou à la limite de séparation de deux corps, la résultante de toutes les actions des molécules situées de ce côte sur les molécules du côté opposé, et dont les directions traversent cette face;* toutes ces forces étant supposées transportées parallèlement à elles-mêmes sur un même point pour les composer ensemble. (p. 248.)

The reader will find it interesting to follow the evolution of the stress-definition by comparing this with Arts. 426*, 440*, 546*, 616*, 678—9* and 1563*.

From this definition Saint-Venant deduces Cauchy's theorems (see our Arts. 606* and 610*) and an expression for $\widehat{rr'}$. On p. 253 p_{rr} is erroneously printed for $p_{rr'}$.

In a footnote to p. 254 a generalisation of the expression for $\widehat{rr'}$ is obtained. Suppose x, y, z to be any three concurrent but non-rectangular lines, and let x', y', z' be lines normal respectively to the planes yz, zx, xy. Then in our notation:

$$\begin{aligned}\widehat{rr'} = {} & \frac{\cos rx'}{\cos xx'}\left(\widehat{xx}\frac{\cos r'x'}{\cos xx'} + \widehat{xy}\frac{\cos r'y'}{\cos yy'} + \widehat{xz}\frac{\cos r'z'}{\cos zz'}\right) \\ & + \frac{\cos ry'}{\cos yy'}\left(\widehat{yx}\frac{\cos r'x'}{\cos xx'} + \widehat{yy}\frac{\cos r'y'}{\cos yy'} + \widehat{yz}\frac{\cos r'z'}{\cos zz'}\right) \\ & + \frac{\cos rz'}{\cos zz'}\left(\widehat{zx}\frac{\cos r'x'}{\cos xx'} + \widehat{zy}\frac{\cos r'y'}{\cos yy'} + \widehat{zz}\frac{\cos r'z'}{\cos zz'}\right).\end{aligned}$$

The proof is easily obtained by the orthogonal projection of areas.

(ζ) Saint-Venant next proceeds to express the relations between stress and strain (pp. 255—262). It cannot be said that this portion of his work is so satisfactory as the later treatment in Moigno's *Statique* (see p. 268 *et seq.*) or the full discussion of the generalised Hooke's Law in his edition of *Clebsch* (pp. 39—41). In fact the linearity of the stress-strain relations is obtained in the text by assumption: *Admettons donc avec tout le monde que les pressions sont fonctions linéaires des dilatations et des glissements tant qu'ils sont très-petits* (p. 257). A long footnote (pp. 257—261) treats the matter from the standpoint of central intermolecular action. Appeal is made to Cauchy (*Exercices de mathématiques* t. IV. p. 2: see our Art. 656*) for the reduction of the 36 coefficients to 15. Saint-Venant, however,—consistent rari-constant elastician as he has always been—retains the multi-constant formulae, remarking:

Mais des doutes ont été élevés sur le principe de cette réductibilité des 36 coefficients à 15 inégaux. Bien que ce doute ait pour motif principal une autre manière de l'établir, et qu'il ne paraisse atteindre, tout au plus, que les corps régulièrement cristallisés dont nous n'aurons pas à nous occuper dans la suite de ce mémoire, et, même, ceux seulement de ces corps où des groupes atomiques éprouveraient des rotations ou des déformations particulières lorsque l'on déforme l'ensemble, nous conserverons en général, à l'exemple de M. Lamé, l'indépendance des coefficients, ce qui, comme il l'a remarqué, ne rend pas plus compliquées les solutions analytiques des problèmes.

The reference to atomic rotations was suggested by Cauchy's paper of 1851: see our Art. 681*.

(η) We have next to deal with the reduction in the number of coefficients which arises in certain symmetrical distributions of homogeneity or in cases of isotropy. Saint-Venant adopts Cauchy's definitions of homogeneity and isotropy, which should have found a place in our first volume under Art. 606* (see the *Exercices* t. IV. p. 2):

On dit alors que le corps est *homogène*, ou que *l'élasticité y est la même dans les mêmes directions en tous ses points* (p. 263).

On the other hand a body is *isotrope* when it has *une élasticité constante ou égale en tous sens autour du point* (p. 272).

Saint-Venant refers to a *semi-polaire* distribution of elastic homogeneity as an example of elastic distribution. He has, as we shall see later, thoroughly treated the entire subject in a memoir of May 21, 1860.

The various cases in which one or more planes of symmetry exist are worked out, but I think brevity as well as uniformity of method are gained by adopting Green's expression for the internal work due to the strains.

(θ) As an example of Saint-Venant's method in this section we may take the following problem. He has shewn that in the case of one plane of symmetry, that of yz, the shears perpendicular to this plane reduce to:

$$\widehat{xy} = f\sigma_{xy} + h\sigma_{zx}, \quad \widehat{zx} = e\sigma_{zx} + h\sigma_{xy} \quad \text{..................(i)},$$

where

$$f = |xyxy| \quad h = |xyzx| = |zxxy|$$
$$e = |zxzx|,$$

in the umbral coefficient notation: see Vol. I. p. 885.

Now by a suitable change of axes these shears can be expressed each in terms of a single slide. This problem is not reproduced in Moigno's *Statique*.

Turn the axes of yz round x through an angle β, then we easily find:

$$\left.\begin{aligned} \widehat{z'x} &= -\widehat{xy}\sin\beta + \widehat{zx}\cos\beta \\ \widehat{xy'} &= \widehat{xy}\cos\beta + \widehat{xz}\sin\beta \end{aligned}\right\} \quad \text{......................(ii)}.$$

$$\left.\begin{aligned} \sigma_{xy} &= \sigma_{xy'}\cos\beta - \sigma_{xz'}\sin\beta \\ \sigma_{xz} &= \sigma_{xy'}\sin\beta + \sigma_{xz'}\cos\beta \end{aligned}\right\} \quad \text{......................(iii)}.$$

Substitute from (iii) in (i) and then the values so deduced in (ii). We obtain

$$\left.\begin{aligned}\widehat{xy'} &= \left(\frac{f+e}{2} + \frac{f-e}{2}\cos 2\beta + h\sin 2\beta\right)\sigma_{xy'}\\ &\quad + \left(-\frac{f-e}{2}\sin 2\beta + h\cos 2\beta\right)\sigma_{xz'}\\ \widehat{xz'} &= \left(\frac{f+e}{2} - \frac{f-e}{2}\cos 2\beta - h\sin 2\beta\right)\sigma_{xz'}\\ &\quad + \left(-\frac{f-e}{2}\sin 2\beta + h\cos 2\beta\right)\sigma_{xy'}\end{aligned}\right\}\text{.........(iv).}$$

Obviously, if we take $\tan 2\beta = \dfrac{2h}{f-e}$ we reduce this last pair of equations to

$$\left.\begin{aligned}\widehat{xy'} &= f_1\,\sigma_{xy'}\\ \widehat{xz'} &= e_1\,\sigma_{xz'}\end{aligned}\right\}\text{.................................(v),}$$

where f_1 and e_1 are roots of the quadratic $\mu^2 - (f+e)\,\mu + fe - h^2 = 0$.

Such is substantially Saint-Venant's reduction. It is obvious, however, that this result follows at once when a known problem as to the invariants of a conic is applied to the work-function.

(ι) A remark as to isotropy on p. 272 may be reproduced as bearing on the uni-constant controversy:

Mais l'isotropie paraît rare. Non-seulement les corps fibreux, tels que bois, les fers étirés ou forgés, mais même les corps grenus ou vitreux, refroidis de la surface au centre après leur fusion, peuvent présenter des élasticités différentes en divers sens.

Saint-Venant refers to the experiments and remarks of Regnault, Savart and Poncelet already noted in our first volume: see Arts. 332*, 978* and 1227*.

(κ) On pp. 272—8 we have deductions of the body-stress equations, the body-shift equations and the surface-stress equations.

On p. 276 Saint-Venant deduces the body-shift equation for a planar distribution of elasticity such as he requires for his torsion problem.

He takes for the shears the expressions found in Equation (v) above, and for the traction $\widehat{xx}$ perpendicular to the planar system the expression

$$\widehat{xx} = as_x + bs_y + cs_z + d\sigma_{yz} + e\sigma_{zx} + f\sigma_{xy},$$

with *six independent* constants. Substituting in the body-stress equation $\dfrac{d\widehat{xx}}{dx} + \dfrac{d\widehat{xy}}{dy} + \dfrac{d\widehat{xz}}{dz} = X$, and expressing the strain in terms of the shift-fluxions, he finds:

$$a\frac{d^2u}{dx^2}+f_1\frac{d^2u}{dy^2}+e_1\frac{d^2u}{dz^2}+f\frac{d^2u}{dxdy}+e\frac{d^2u}{dxdz}$$
$$+(f_1+b)\frac{d^2v}{dxdy}+(e_1+c)\frac{d^2w}{dxdz}+d\left(\frac{d^2v}{dxdz}+\frac{d^2w}{dxdy}\right)+f\frac{d^2v}{dx^2}+e\frac{d^2w}{dx^2}=X.$$

C'est la seule équation dont nous aurons besoin pour les problèmes sur la torsion, comme on verra.

It will be noted that it contains *eight* independent constants, and that X is a body-*force*, not a body-*acceleration*, and acts *towards* the origin. It is needless to say that Saint-Venant much reduces the number of his constants before he applies this equation to his problem. In Moigno's *Statique* (p. 637) he adopts in place of X the more usual notation of $-\rho X$ where ρ is the density.

[5.] The concluding pages of this chapter (pp. 278—288) contain matter which appears here for the first time, and which, as it is of considerable interest, deserves an article to itself. The section is entitled: *Conditions de résistance à la rupture éloignée ou à une altération progressive et dangereuse de la contexture des corps.*

(a) We have already noted the misleading character of this title: see Art. 4. (γ). In the first place initial loads frequently produce set which although neither progressive nor dangerous may alter the shape or elastic homogeneity of the body; and in the second place, if the body be in a state of ease, still in many cases the generalised Hooke's law will be far from holding even approximately up to the elastic limit. Saint-Venant recognises the first point by distinguishing between small sets, "qui ne font qu'*écrouir* le corps ou rendre plus stable l'arrangement de ses parties" (p. 278) and large sets, which he holds either augment progressively so that "la matière *s'énervera* bientôt" (p. 239), or else by change of form destroy the value of a structure. But he hardly seems to have taken note of the second point, for he does not hesitate on pp. 280 and 286 to use stretch- and slide- moduli which connote a proportionality of stress and strain. The same point recurs in almost each torsion problem, where a *condition de non-rupture ou de stabilité de la cohésion* is given (e.g. pp. 351, 396 etc.). It is essentially a limit to the proportionality of stress and strain which is in each case given, but this limit in many materials has no sensible existence or may in the case of a material which does not possess an extended state of ease be safely passed.

(b) One further remark before we proceed to Saint-Venant's process. He starts from the formula (p. 280)

$$s_r = s_x\cos^2\alpha + s_y\cos^2\beta + s_z\cos^2\gamma + \sigma_{yz}\cos\beta\cos\gamma + \sigma_{zx}\cos\gamma\cos\alpha + \sigma_{xy}\cos\alpha\cos\beta \quad\text{.............(i)},$$

but on p. 242 he has obtained this by supposing the stretches and slides to be so small that their squares may be neglected. It is conceivable that in some materials before rupture and, possibly, before a dangerous set is reached, this might not be allowable.

(*c*) Our author begins by noticing that the proper limit to be taken for the stability of a material is a *stretch* and not a *traction* limit. He attributes to Mariotte[1] the first recognition of this fact "que c'est le degré d'extension qui fait rompre les corps" and remarks that although it is legitimate, and occasionally convenient, to take a traction limit given by $T = E\bar{s}$ where $\bar{s}$ is the stretch-limit and E the stretch-modulus, T need not be the stress across any plane, whatever, at the point in question.

Et cette sorte de notation est sans inconvénient si l'on n'oublie pas que T *représente simplement le produit* $E\bar{s}$, ou la force capable de donner (aussi par unité superficielle) à ce même petit prisme supposé isolé, la dilatation limite $\bar{s}$ relative à sa situation dans le corps, mais qu'il ne représente que *quelquefois* et *non toujours* l'effort intérieur ou la pression supportée normalement par sa section transversale pendant qu'il fait partie du corps. (p. 280.)

This remark is all the more important as the distinction has been neglected by Lamé, Clebsch and more recent elasticians: see our Arts. 1013*, 1016* footnotes and 1567*.

(*d*) The stretch in any direction being given by the equation (i) above, we have next to ask what in an aeolotropic body is the distribution of limiting stretch? Saint-Venant having regard to equation (i) *assumes* it to be *ellipsoidal* in character; in other words he takes

$$\bar{s} = \bar{s}_x \cos^2\alpha + \bar{s}_y \cos^2\beta + \bar{s}_z \cos^2\gamma,$$

where $\bar{s}_x$, $\bar{s}_y$, $\bar{s}_z$ are three constants to be determined by experiment, and the axes of ellipsoidal distribution are chosen as those of coordinates. The condition of safety now reduces to the maximum value of $s/\bar{s}$ being = or < 1. By the ordinary max.-min. processes of the Differential Calculus we obtain for $s/\bar{s}$ the equation:

$$4\bar{s}_x\bar{s}_y\bar{s}_z\left(\frac{s}{\bar{s}} - \frac{s_x}{\bar{s}_x}\right)\left(\frac{s}{\bar{s}} - \frac{s_y}{\bar{s}_y}\right)\left(\frac{s}{\bar{s}} - \frac{s_z}{\bar{s}_z}\right) - \sigma^2_{yz}\bar{s}_x\left(\frac{s}{\bar{s}} - \frac{s_x}{\bar{s}_x}\right)$$
$$- \sigma^2_{zx}\bar{s}_y\left(\frac{s}{\bar{s}} - \frac{s_y}{\bar{s}_y}\right) - \sigma^2_{xy}\bar{s}_z\left(\frac{s}{\bar{s}} - \frac{s_z}{\bar{s}_z}\right) - \sigma_{yz}\sigma_{zx}\sigma_{xy} = 0 \ldots\ldots\text{(ii)}.$$

The roots of this equation are known to be real and we must have the greatest of them = or < 1.

Suppose the material is subject only to a sliding strain, then $s_x = s_y = s_z = \sigma_{zx} = \sigma_{xy} = 0$. Hence it follows that

$$\frac{s}{\bar{s}} = \frac{\sigma_{yz}}{2\sqrt{\bar{s}_y\bar{s}_z}}.$$

In other words if $\bar{s}$ is the limit of s, then $2\sqrt{\bar{s}_y\bar{s}_z}$ is the limit of σ_{yz} or gives the slide-limit. Let us represent it by $\bar{\sigma}_{yz}$.

[1] *Traité du mouvement des eaux*, sixième et troisième alinéa du *second discours*.

Similarly we have $\bar{\sigma}_{zx} = 2\sqrt{\bar{s}_z\,\bar{s}_x}$ and $\bar{\sigma}_{xy} = 2\sqrt{\bar{s}_x\,\bar{s}_y}$.

Saint-Venant then rewrites his equation (ii) as:

$$\left(\frac{s}{\bar{s}} - \frac{s_x}{\bar{s}_x}\right)\left(\frac{s}{\bar{s}} - \frac{s_y}{\bar{s}_y}\right)\left(\frac{s}{\bar{s}} - \frac{s_z}{\bar{s}_z}\right) - \left(\frac{\sigma_{yz}}{\bar{\sigma}_{yz}}\right)^2\left(\frac{s}{\bar{s}} - \frac{s_x}{\bar{s}_x}\right) - \left(\frac{\sigma_{zx}}{\bar{\sigma}_{zx}}\right)^2\left(\frac{s}{\bar{s}} - \frac{s_y}{\bar{s}_y}\right)$$
$$- \left(\frac{\sigma_{xy}}{\bar{\sigma}_{xy}}\right)^2\left(\frac{s}{\bar{s}} - \frac{s_z}{\bar{s}_z}\right) - 2\,\frac{\sigma_{yz}\,\sigma_{zx}\,\sigma_{xy}}{\bar{\sigma}_{yz}\,\bar{\sigma}_{zx}\,\bar{\sigma}_{xy}} = 0 \quad \text{.........(iii)}.$$

He remarks that this equation may be adopted as if the six limiting strains $\bar{s}_x$, $\bar{s}_y$, $\bar{s}_z$, $\bar{\sigma}_{yz}$, $\bar{\sigma}_{zx}$, $\bar{\sigma}_{xy}$, were all independent, and the values of the slide-limits $\bar{\sigma}$ had to be found by experiment. At any rate equations of the form $\bar{\sigma}_{yz} = 2\sqrt{\bar{s}_y\,\bar{s}_z}$ need only be used when there is an absence of experimental data. (p. 284.)

(*e*) In the following paragraph (25) Saint-Venant explains how we are to find $s/\bar{s}$ for every point in the body and then take its maximum value for all these points,

> l'on obtiendra, en l'égalant à l'unité, la condition nécessaire et justement suffisante de la résistance du corps à la rupture (p. 284).

We have noted that this language is hardly exact. The point where this maximum takes place is called after Poncelet *point dangereux*, a name which it is convenient to render by *fail-point*. This term will not necessarily connote rupture, but merely a point at which 'linear elasticity[1]' first *fails*. The consideration of this point leads Saint-Venant to a concise definition of the solid of equal resistance:

> Souvent il y a plusieurs *points dangereux*, ou plusieurs points pour lesquels la plus grande valeur de $s/\bar{s}$ est la même, d'après la manière dont les forces sont appliquées. Lorsque, dans un corps de forme allongée, il y a un pareil point à chacune de ses sections transversales, ce corps est dit d'*égale résistance:* tels sont les prismes lorsqu'ils sont simplement étendus ou tordus par des forces appliquées aux extrémités.

(*f*) We have next the application of (iii) to the case of torsion about x as axis. Here

$$s_x = s_y = s_z = \sigma_{yz} = 0,$$

whence it follows

$$s/\bar{s} = \sqrt{(\sigma_{xy}/\bar{\sigma}_{xy})^2 + (\sigma_{xz}/\bar{\sigma}_{xz})^2}.$$

We have thus the limiting condition

$$1 = \text{or} > \left(\frac{\sigma_{xy}}{\bar{\sigma}_{xy}}\right)^2 + \left(\frac{\sigma_{xz}}{\bar{\sigma}_{xz}}\right)^2.$$

It is obvious that the principal slide in any direction $\sqrt{\sigma_{xy}^2 + \sigma_{xz}^2}$ is given by the ray of an ellipse of which $\bar{\sigma}_{xy}$ and $\bar{\sigma}_{xz}$ are the

[1] I use the words 'linear elasticity' in the sense in which 'perfect elasticity' has been used by the writers of mathematical text-books, i.e. to connote the elasticity which obeys the generalised Hooke's Law or the linearity of the stress-strain relation.

semi-axes. Saint-Venant uses throughout his memoir a slightly different form. Let μ_1, μ_2 be slide-coefficients and S_1, S_2 the shears capable of producing the slides $\bar{\sigma}_{xy}$ and $\bar{\sigma}_{xz}$; then the condition of *non-rupture par glissement* (i.e. of no *failure* of linear elasticity) is expressed by

$$1 = \text{or} > \left(\frac{\mu_1 \sigma_{xy}}{S_1}\right)^2 + \left(\frac{\mu_2 \sigma_{xz}}{S_2}\right)^2.$$

The chapter concludes with a few general remarks on the physical characteristics of rupture by torsion.

[6.] The third chapter occupies pp. 288—99: it relates to the simple case of a prism on any base, whose terminal faces and sides are subjected to any uniform tractive loads. Lamé and Clapeyron in their memoir of 1828 (see our Art. 1011*) had treated the simple case of isotropy. Saint-Venant as an example of the *mixed* or *semi-inverse* method gives the solution for the case when there are three planes of elastic symmetry, the intersection x of one pair being parallel to the axis of the prism. He assumes that the tractions are constant and the shears zero throughout. This satisfies the body stress-equations; the constant values of the tractions are in this case given by the surface stress-equations. The stress-strain relations then give in terms of the elastic constants and the loads the values of the shift-fluxions. We thus arrive at a system of simple linear partial differential equations, whose solution is extremely easy. The complete solution gives for each shift a part proportional to the corresponding coordinate and a general integral which is only the resolved part of the most general displacement of the prism treated as a rigid body. On p. 292 Saint-Venant determines the value of the stretch-modulus when the tractive load on the sides of the prism is zero, and on p. 293 he considers the simple cases of (1) the axis of the prism being an axis of elastic symmetry, and (2) the material being isotropic: see our Art. 1066*. On p. 293 we have a remark that some writers have doubted the exactness of the above results, considering them only as plausible but not necessarily unique. Saint-Venant asserts that they are unique, which is undoubtedly true in this case, but I am not quite satisfied with the nature of his proof, for it would at first sight apply to any elastic body. It depends essentially on the following line of reasoning: Take any particular integrals of the equations of elasticity u_0, v_0, w_0, put the shifts equal to $u_0 + u'$, $v_0 + v'$, $w_0 + w'$; we now obtain equations of elasticity without body-force or surface-load. "On verra que u', v', w' seront les déplacements des points d'un prisme qui ne serait sollicité que par des forces nulles. *Ces déplacements seraient nuls eux-mêmes.* Nos expressions offrent donc la solution complète et unique." (p. 294.) This is true for the prism, but it does not always follow that where there are no surface- or body-forces, the body is without strain, or has only rigid displacement. For example, take a cylindrical shell, a spherical membrane of small thickness, or an anchor ring of small cross section, and turn them inside out, we have a state of strain with no applied force.

On p. 295 Saint-Venant shows that his results for the prism still hold if the shifts are large, but their fluxions remain small.

[7.] A method of solving a still more general problem is indicated on p. 296. Suppose a homogeneous aeolotropic body of any shape to be subjected to a surface-load L which is the resultant of $\widehat{xx}$, $\widehat{yy}$, $\widehat{zz}$, $\widehat{yz}$, $\widehat{zx}$, $\widehat{xy}$; these stresses being given constant values throughout the body and at the surface. Then we have six equations from which to find in terms of the 21 elastic constants the six strains. These are six simple partial differential equations which give at once the shifts. Saint-Venant suggests how the stretch-modulus for any direction may thus be obtained as a function of the 36 (21 or 15) elastic coefficients: see our Arts. 135—7, 198 (*c*), 306—8 and 796*.

[8.] The final section of this chapter (§ 33, pp. 297—9) relates to a point which Saint-Venant has frequently taken occasion to refer to. The principle involved is the following:

C'est que *le mode d'application et de répartition des forces vers les extrémités des prismes est indifférent* aux effets sensibles produits sur le reste de leur longueur, en sorte qu'on peut toujours, d'une manière suffisamment approchée, remplacer les forces qui sont appliquées, par des forces statiques *équivalentes*, ou ayant mêmes moments totaux et mêmes résultantes avec une répartition justement telle que l'exigent les formules d'extension, de flexion, de torsion, pour être parfaitement exactes. (*Notice* I. p. 22.)

Saint-Venant does not clearly state the portion of the prism over which he holds the influence of distribution to extend, the term *sur le reste de leur longueur* is somewhat vague. In the memoir itself he uses the words *en excluant seulement les points* *très-proches de ceux où agissent les forces* (p. 299). We can perhaps, however, reach some conception of the field to which he supposes the influence to extend by paying attention to a footnote on p. 22 of *Notice* I.

Suppose the terminal of a prism subjected to any system of load statically equivalent to that distribution which produces the system of strains theoretically calculated. Impose upon the terminal two equal and opposite loads having the theoretical distribution. One of these will produce the theoretical strains, the other will be in statical equilibrium with the actual load distribution. The terminal is thus acted upon by two equivalent and opposite systems of force. These systems will produce certain small shifts in the end of the prism, and these shifts measure the extent to which the prism is influenced by the difference between the theoretical and practical distributions. Saint-Venant tells us in his footnote that the influence of forces in equilibrium acting on a small portion of a body extend very little beyond the parts upon which they act.

L'auteur a fait deux expériences de ce genre sous les yeux de l'Académie en lisant un de ses mémoires. Elles ont consisté simplement à pincer avec des tenailles un prisme de caoutchouc, et à dilater transversalement une lanière mince de même matière, en tirant ses bords en deux sens opposés. Tout le monde peut les répéter et voir que l'impression ou l'élargissement *ne se fait point sentir à des distances excédant la profondeur dans le premier cas et l'amplitude dans le second.*

The reader will find this matter still further treated of in the *Navier*, pp. 40—41 and the *Clebsch*, pp. 174—7. The principle is of first-class importance, as it is scarcely possible in a practical structure to ensure any given theoretical distribution of load. The terminals will generally take a form which lies beyond theoretical investigation and only the statical equivalent of the load system will be really ascertainable, e.g. the tractive load on a bar may be applied by means of a nut carrying a weight, the nut itself being supported by the thread of a screw cut on the bar.

[9.] Saint-Venant's fourth chapter deals with the problem of flexure by the *semi-inverse* process. The important results here first published were afterwards considered at greater length in the well-known memoir on flexure: see our Art. 69 *et seq.*

Throughout the chapter the writer supposes three principal planes of elasticity, one of which coincides with the cross-section, and the two others intersect in the line of sectional centroids, i.e. in the axis of the prism. He thus makes use of formulae which in his notation apparently involve twelve independent coefficients, but these he at once reduces to three independent moduli (E, ϵ, ϵ') and two coefficients (f, e): see pp. 303, 311—313.

As Saint-Venant justly remarks:

La détermination exacte et générale des déplacements des points d'un prisme sous l'action de forces qui tendent à le *fléchir*, a échappé jusqu'à présent aux recherches les plus laborieuses des géomètres. (p. 299.)

But although his solution does not solve the problem for *all* terminal distributions of load, it is yet as close an approximation in practice as, say, Coulomb's solution of the torsion of a circular cylinder. It cannot be too often repeated that the distributions of tractive and shearing loads, such as occur in theory, are not attainable in practice, and that we must be content with their statical equivalent over small areas (see our Art. 8). But let us hear Saint-Venant himself:

Aussi les résultats ci-dessus ne sont pas applicables d'une manière tout à fait rigoureuse.

Mais l'analyse précédente nous prouve toujours que si sur deux sections quelconques, extrêmes ou non, les forces sont appliquées et distribuées de cette manière, il en sera absolument de même sur toutes les sections intermédiaires, et que les déplacements, dans toute l'étendue du prisme, seront représentés par les autres expressions trouvées ci-dessus. Les formules donnent donc l'état de choses vers lequel converge l'état intérieur réel du prisme à mesure que l'on considère des parties plus éloignées de ses extrémités ou des points d'application des forces qui font fléchir.

Il s'établit ici, dans l'espace, une sorte d'état permanent semblable à celui qui est produit, dans le temps, par l'action continue de causes constantes qui finissent par effacer l'effet des causes initiales d'un grand nombre de phénomènes. (p. 314.)

Saint-Venant's solution of the problem of flexure is thus the real solution of the problem, for were any other solution obtained it could differ from his only by terms which would be really insignificant as compared with the differences in terminal loading which must occur, not only between theory and practice, but between any two practical cases of flexure. It is just as reasonable or unreasonable to quarrel with Coulomb's torsion solution as with Saint-Venant's flexure results.

[10.] With regard to the uniqueness of the solution obtained by the semi-inverse method—supposing the theoretical shearing and tractive loads were applied to the terminals—Saint-Venant has some remarks on p. 307 which it is well to consider. After remarking that the shifts satisfy all the conditions and equations of the problem, he continues:

> Et ils sont les seuls qui y satisfassent, car le problème des déplacements est complètement déterminé si, en donnant les pressions et tractions sur tous les points de la surface, on suppose fixes l'un des points du prisme (le point O), et les directions d'un élément linéaire et d'un élément plan qui y passent (un élément sur l'axe des z et un élément sur le plan yz) en sorte qu'il ne puisse y avoir ni translation ni rotation générale à ajouter aux déplacements provenant de la flexion. (p. 307.)

He then proceeds as on p. 294 to put the shifts equal to the particular solutions found plus additional unknown parts (u', v', w'), these latter he argues must be zero as they are shifts due to a zero system of loading as appears by the vanishing of the load terms from the equations on substitution. This sketch of a proof of the uniqueness of solution of the equations of elasticity has been adopted and expanded by Clebsch: see Kap. I. § 21 of his *Theorie der Elasticität.* I have suggested above that there is need of applying the proof with some caution: see Art. 6.

[11.] In treating the problem of flexure Saint-Venant assumes the longitudinal shifts and the lateral loading, hence he deduces the transverse shifts and the terminal loading. The values of the longitudinal shifts were doubtless suggested by the Bernoulli-Eulerian solution of the problem, but in this chapter they appear to arise very naturally from the consideration of the simpler case of uniform flexure, or the bendings of each longitudinal 'fibre' into a circular arc; see pp. 292—304.

Saint-Venant makes two generalisations of his problem. The first (p. 306) to the case when besides terminal shearing load, there is also terminal tractive load. It is necessary, however, to remark that when such load is negative, and the prism of considerable length as compared with the dimensions of cross-section, the question of the *buckling* action of such load arises. This is a point to which we have referred in our first volume: see Art. 911*. Saint-Venant does not allude to it. The second generalisation is to the case of large shifts, or as it is here termed: *Extension de cette solution à une flexion aussi grande qu'on veut.* I cite the following remarks as suggestions which have been adopted by later writers (e.g. Kirchhoff):

Les formules donnant u, v, w ne s'appliquent, comme les équations différentielles dont on les a tirées, qu'à des déplacements très-petits ne produisant qu'une petite flexion. On peut cependant en tirer des déplacements d'une grandeur aussi considérable qu'on veut, tels que ceux d'une verge élastique longue et mince qu'on ploie au point de faire presque toucher les deux bouts, ce qui est très-possible sans altérer aucunement la contexture de sa matière, car les déplacements relatifs et les déformations peuvent rester petits dans chacune des portions d'une longueur bien moindre que le rayon ρ de la courbure, dans lesquelles on peut diviser par la pensée un pareil corps; et c'est leur accumulation qui produit, à l'extrémité, des déplacements considérables (p. 308).

12. The section of the chapter pp. 308—313 which deals with the general problem of flexure is reproduced in the memoir in Liouville's *Journal* and will be considered later: see our Arts. 69 *et seq.*

Two results are given on p. 312 without demonstration. The first of these relates to the case of an elliptic section; it coincides with equation (56) of the memoir in Liouville's *Journal* (see our Art. 86, Eqn. 25) when we put C the constant of that equation zero. The second of these relates to the case of a rectangular section; it is an approximation: the memoir in Liouville's *Journal* gives the exact solution, but not this approximation. It is however easy to supply the steps which lead to the approximation. In equation (91) of the memoir in Liouville's *Journal* the exact value of $F(y, z)$ is given depending on $F_1(y, z)$ which is determined by (102). If we were to expand $F_1(y, z)$ in powers of y and z, the term which involves z only would disappear by (103); then the next two terms would involve y^2z and z^3 respectively. This suggests our taking a form like that of (85) in the memoir on *Torsion* as an approximation; take this and calculate $\widehat{xz}$, that is $G'\left(\frac{dw}{dx}+\frac{du}{dz}\right)$. We find this to be

$$G'g_0+\frac{G'P}{2EI}\left\{(K-2\epsilon)y^2+(E-fK)\frac{z^2}{e}\right\};$$

then in order that this may vanish when $y=0$ and $z=c$ we must have

$$g_0=-\frac{Pc^2(E-fK)}{2EeI}.$$

Then Saint-Venant assumes that $\int \widehat{xz}\, d\omega = -P$; and this leads to the value of K which he uses in this case: see p. 312 line 3 from the foot.

13. On p. 311 Saint-Venant says that $F = 0$, and $dF/dz = 0$, when $y = 0$ and $z = 0$. Suppose that h and k denote very small quantities; then the value of u at the origin being denoted by u_0 the value at a point very near the origin would be

$$u_0 + \left(\frac{du}{dy}\right)_0 h + \left(\frac{du}{dz}\right)_0 k.$$

Now $\left(\frac{du}{dy}\right)$ is zero since u is an even function of y, so that if we have $\left(\frac{du}{dz}\right)_0$ zero as well as u_0 then the value of u vanishes all over *an element* near the origin.

[14.] Pp. 316—318 are deserving of close attention; they give results which were partially published in the memoir of 1843 (see our Art. 1581*) and which followed up the suggestion of Persy: see our Art. 811*. Saint-Venant namely finds the plane of flexure when the load-plane does not coincide with the plane of one set of principal axes of the cross-sections.

Let Oz, Oy be the principal axes at O the centroid of any cross-section of area ω; let κ_z, κ_y be the swing-radii about these axes, and ϕ, ψ the angles which the load and flexure planes make respectively with the plane through Oz and the axis of the prism. Then Saint-Venant easily shews that:

$$\tan\psi = \frac{\kappa_z^2}{\kappa_y^2}\tan\phi\,; \quad \frac{1}{\rho} = \frac{M}{E\omega}\sqrt{\frac{\cos^2\phi}{\kappa_y^4} + \frac{\sin^2\phi}{\kappa_z^4}},$$

where $1/\rho$ is the curvature, M the bending moment and E the longitudinal stretch-modulus[1].

Assuming that only longitudinal stretch produces danger, Saint-Venant deduces that if $s_0 = T_0/E$ be the limit of safe stretch then

$$M = \text{or} < \text{the minimum of } \frac{T_0\,\omega}{z\dfrac{\cos\phi}{\kappa_y^2} + y\dfrac{\sin\phi}{\kappa_z^2}}.$$

For the rectangle ($2b \times 2c$) we have

$$M = \text{or} < \frac{4T_0 b^2c^2}{3\,(b\cos\phi + c\sin\phi)},$$

[1] The first equation expresses geometrically that the plane of flexure is perpendicular to the diameter of the momental ellipse (neutral axis) conjugate to the plane of loading: see our Art. 171.

for the ellipse $(2b \times 2c)$

$$M = \text{or} < \frac{T_0 \pi b^2 c^2}{4\sqrt{b^2 \cos^2\phi + c^2 \sin^2\phi}}.$$

Such results as these he has reproduced and considerably added to in his edition of *Navier*, pp. 52—60, pp. 122—126 and 128—136. Indeed, we may affirm that Saint-Venant was the first to insist on the practical importance of investigating the relation between the planes of flexure and of loading, when the latter plane is not one of inertial symmetry.

[15.] The chapter concludes with the deduction of Saint-Venant's all-important discovery that the cross-sections of a beam under flexure do not remain plane even within the limit of elasticity. There is also an investigation of the change in the cross-sectional contour (pp. 318—323). We shall return to these points later, but meanwhile may quote the concluding words of the chapter as some evidence of the satisfaction which Saint-Venant legitimately felt at the results of his new process:

On voit, par ce chapitre IV, que la méthode mixte de solution des problèmes de l'équilibre des corps élastiques peut, non-seulement confirmer des résultats connus, en apprenant à quelles conditions ils sont exacts, mais encore les compléter, et donner sur les circonstances de la flexion des résultats nouveaux.

[16.] Saint-Venant's fifth chapter defines torsion and deduces the general equations by the semi-inverse method; it occupies pp. 323—333.

The definition of torsion which does not involve the maintenance of the primitive planeness of the cross-sections is contained in the following paragraph:

Et nous nous donnerons *une partie des déplacements* ou de leurs rapports, en ce que nous supposerons que ces déplacements ont produit une *torsion* autour d'un axe parallèle à ses arêtes, torsion qui consiste *en ce que les déplacements transversaux des divers points appartenant primitivement à une même section quelconque perpendiculaire à l'axe ne diffèrent de ceux des points homologues d'une autre section, que par une rotation d'un même angle pour tous, autour du même axe;* en sorte que les points qui se correspondaient primitivement sur les droites parallèles à l'axe puissent être ramenés à se correspondre encore, en les faisant tourner d'un angle qui est le même pour les points des deux mêmes sections (p. 324).

We will now sketch the method by which our author reaches the general equations of torsion.

[17.] The axis of torsion will be taken as axis of x; the direction of torsion will be from the axis of y towards that of z. The area of a cross-section will be denoted by ω, and we shall write $\omega\kappa_z^2 = \int y^2 d\omega$, $\omega\kappa_y^2 = \int z^2 d\omega$, these being the sectional moments of inertia. The torsion referred to unit of length will be τ; that is, if we draw the radius-vector of a displaced point in one section, and also that of the homologous point in a section at distance ξ from the first, then the second radius-vector makes with a parallel to the first an angle of which the circular measure is $\tau\xi$; this angle is measured from the axis of y to that of z. This language implies that the torsion is constant, but the meaning of τ, when it is not constant, will be assigned in the same manner as before at any point, provided we consider ξ as infinitesimally small.

The above definition of torsion leads us at once to the results:

$$dv/dx = -\tau z, \quad dw/dx = \tau y \quad \text{(i).}$$

The consideration that there is no lateral load gives for every point of a sectional contour the equation

$$\widehat{xz}\, dy - \widehat{xy}\, dz = 0 \quad \text{(ii).}$$

On p. 329 Saint-Venant fixes a point, line and elementary plane as in our Art. 10, and remarks that the total torsion between the terminal sections may be considerable provided each short element into which we may divide the prism by two cross-sections receives only a small distortion relative to itself, the length of the prism being great as compared with the linear dimensions of the section. The total shifts can then be obtained by summation from the solutions of the above equations for each short element.

Referring to the equations in our Art. 4 (θ) we easily obtain

$$\widehat{xy} = f_1\,(du/dy - \tau z), \quad \widehat{xz} = e_1\,(du/dz + \tau y) \quad \text{(iii).}$$

Whence if M be the moment of all the stresses on a cross-section about the axis of x,

$$M = \int_0^\omega d\omega\,[e_1\,(du/dz + \tau y)\,y - f_1\,(du/dy - \tau z)\,z] \quad \text{(iv).}$$

It will be seen that this agrees with the old theory—which gave $M = e_1\tau \int_0^\omega d\omega\,(y^2 + z^2)$,—only when $e_1 = f_1$ and $du/dz = du/dy$. This, since du/dx is assumed constant, amounts to $u = 0$, or the old theory that the cross-sections remain plane and perpendicular to the axis. Substituting in the equation of our Art. 4 (κ), and in (ii) above, we find for body and surface shift-equations:

$$\left.\begin{aligned} ad^2u/dx^2 + f_1 d^2u/dy^2 + e_1 d^2u/dz^2 + fd^2u/dxdy + ed^2u/dxdz \\ + (ey - fz)\,d\tau/dx = 0 \\ e_1\,(du/dz + \tau y)\,dy - f_1\,(du/dy - \tau z)\,dz = 0 \end{aligned}\right\} \quad \text{(v).}$$

Saint-Venant (p. 331) at once simplifies these equations by taking $d^2u/dx^2 = f\,d^2u/dxdy = e\,d^2u/dxdz = 0$; these follow at once from the supposition that du/dx, or the longitudinal stretch, is constant or zero, or again from the second supposition that it is constant only along lines parallel to the axis of torsion and that a principal plane of elasticity is perpendicular to this axis (i.e. $e = f = 0$).

In general we shall adopt the notation $e_1 = \mu_2$, $f_1 = \mu_1$, so that our equations become

$$\left.\begin{array}{r} \mu_1\,d^2u/dy^2 + \mu_2\,d^2u/dz^2 = 0 \\ \mu_2\,(du/dz + \tau y)\,dy - \mu_1\,(du/dy - \tau z)\,dz = 0 \end{array}\right\} \ldots\ldots\ldots \text{(vi)}.$$

Saint-Venant for the purpose of simplifying the form of his results takes $\mu_1 = \mu_2 = \mu$ in the following four chapters. Further to avoid the complexity which would be initially introduced by treating at the same time the problem of flexure Saint-Venant takes

$$s_x = s_y = s_z = \sigma_{xy} = 0.$$

We shall see in the sequel that Clebsch has combined the two problems of torsion and of flexure by preserving the general form of the equations.

The next four chapters of the memoir VI.—IX. are occupied with the torsion of prisms of various cross-sections. I shall briefly give the results here for the purpose of reference; the reader will find little difficulty in deducing the proofs for himself, if the original memoir be not accessible. At the same time I shall draw attention to one or two important points involved in Saint-Venant's discussion.

[18.] The sixth chapter occupies pp. 333—352, and is entitled: *Torsion d'un prisme ou cylindre à base elliptique.*

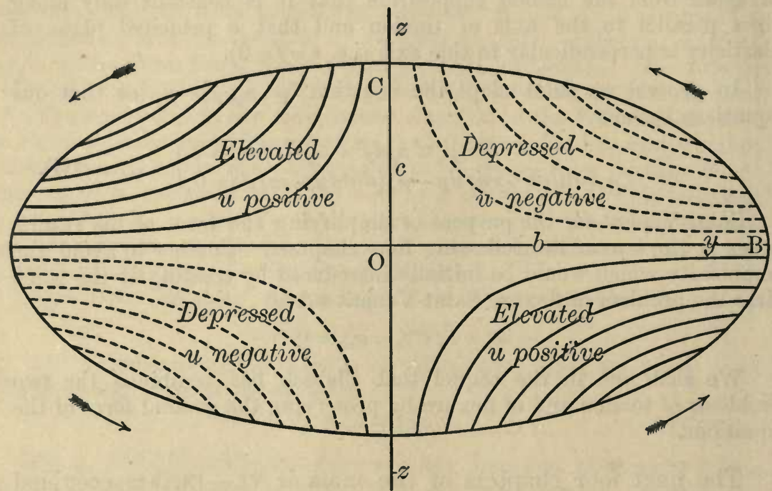
The following results are obtained, the axes of the cross-section being $2b$ and $2c$, and the notation being otherwise as before:

$$u = -\frac{b^2 - c^2}{b^2 + c^2}\tau yz, \quad \left.\begin{array}{l} v = -\tau xz \\ w = \quad \tau xy \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

$$M = \mu\tau\frac{\pi b^3c^3}{b^2 + c^2} = \frac{4\mu\tau\omega}{1/\kappa_y^2 + 1/\kappa_z^2} \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

$$\left.\begin{array}{c} \widehat{xx} = \widehat{yy} = \widehat{zz} = \widehat{yz} = 0. \\ \widehat{xy} = -\mu\tau\dfrac{2b^2z}{b^2 + c^2}, \quad \widehat{xz} = \mu\tau\dfrac{2c^2y}{b^2 + c^2} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}.$$

We see at once from (i) that the primitively plane sections suffer distortion (*gauchissement*), and become hyperbolic paraboloids. In the accompanying figure the contour lines of these surfaces of distortion are marked; broken lines denoting depressions.



The principal slide σ is given by

$$\sigma^2 = \sigma_{xy}^2 + \sigma_{xz}^2 = \left(\frac{2\tau}{b^2+c^2}\right)^2 (b^4z^2 + c^4y^2) \text{ .... (iv).}$$

The *point dangereux* or *fail-point* is obtained by making $b^4z^2 + c^4y^2$ a maximum, thus it is at the extremity of the minor axis, i.e. is the point *nearest to the axis of torsion.*

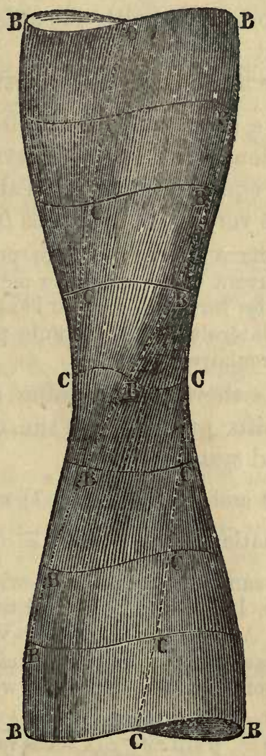
From (iv) we obtain by means of our Art. 5 (f), if $S_1 = S_2 = S_0$:

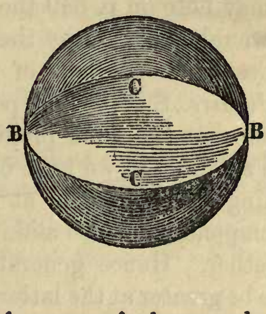
$$S_0 = \text{or} > \mu\tau\, 2b^2c/(b^2+c^2) \text{ .... (v),}$$

whence it follows that $M = \text{or} < \dfrac{\pi bc^2 S_0}{2} \left(= \text{or} < \dfrac{2\omega\kappa_y^2 S_0}{c}\right)$......... (vi).

The general appearance of the prism under torsion is given in the figures on the next page, the torsion being diagrammatically exaggerated.

[19.] There are one or two important points to be noticed in this chapter. In the first place Saint-Venant solves equation (vi) of Art. 17 by a series ascending in powers of y and z; one term (a'_2yz) suffices for the elliptic cross-section, he makes use of others later. Secondly he points out pp. 339—341 that his results agree with the theory of Coulomb only in the case of a circular section, for every other elliptic cross-section the value of the torsional moment is *smaller* than that given by the old theory and there is





distortion. He shews by numerical examples on p. 352 how much sooner the safe limit is reached in the true than in the old theory.

[20.] On pp. 341—343 we have, thirdly, a footnote on Cauchy's suggestions that the torsion τ should be made to vary transversally: see our Art. 684*. Saint-Venant shews that this would require,—at least in the case of a circular cross-section and an axis of elasticity coinciding with the axis of figure—a shearing load at each element of lateral surface. This is a supposition which could hardly be attained in any practical case.

[21.] Fourthly we have on pp. 342—345 a very concise and admirable consideration of the point referred to in our Art. 9; namely, the *practical* equivalence of statically equipollent systems of terminal loading at very short distances from the terminals.

Nos résultats relatifs à la torsion d'un prisme elliptique par des couples quelconques peuvent être adoptés au même titre et avec la même confiance qu'on adopte les formules, soit de l'extension simple, soit de la flexion par des forces latérales, et la formule plus analogue du cas de torsion des cylindres circulaires (p. 345).

In all these cases there is the same assumption as to the equivalence of the shifts produced by the theoretical and by the actual equipollent load systems.

[22.] Fifthly §§ 59 and 60 (pp. 346—7) may be noted. The first deduces from the equations $\int_0^\omega \widehat{xy}\, d\omega = \int_0^\omega \widehat{xz}\, d\omega = 0$ that the axis of torsion for the shifts assumed must coincide with the line of sectional centroids[1]: see our Art. 181 (*d*). The second treats of the case of large torsional shifts, see our Art. 17, p. 18. Saint-Venant remarks that the values $v = -\tau xz$ and $w = \tau xy$ of our Art. 18, equation (i), no longer hold, but by an easy process of summation (p. 347) we find the new values:

$$\begin{aligned} v &= -z \sin \tau x - y (1 - \cos \tau x) \\ w &= y \sin \tau x - z (1 - \cos \tau x) \end{aligned}.$$

[23.] Lastly we may note on p. 349 the general argument by which Saint-Venant would explain why the *fail-points* are those nearest and not farthest from the axis of torsion as in the old theory (*la théorie ordinaire*, St-V.). He points out that at the extremity of the major axis the slide produced by the distortion of the plane section is zero and so we have only the slide produced by the 'fibres becoming helical,' while at the extremity of the minor axis the two components of the slide both exist and compound, *operating* together. Hence generally we see how it is possible for the slide to be greater at the latter than the former point.

[1] This paragraph was cancelled in the copies of the memoir remaining in Saint-Venant's possession.

[24.] Chapter VII. of the memoir (pp. 352—360) is occupied with the analytical solution of the equation $u_{zz} + u_{yy} = 0$. The first form obtained is that in a series of exponentials and sines or cosines of multiples of y and z.

The second is in terms of cylindrical coordinates. Let $y = r\cos\phi$, $z = r\sin\phi$, then:

$$u = \Sigma A_n r^n \cos n\phi + \Sigma B_m r^m \sin m\phi,$$

$$M = \mu\tau \int r^2 d\omega - \mu\Sigma n A_n \int r^n \sin n\phi d\omega$$
$$+ \mu\Sigma m B_m \int r^m \cos m\phi d\omega.$$

These results are obvious. Special cases of uni-axial and bi-axial symmetry lead to the vanishing of certain coefficients.

[25.] Chapter VIII. (pp. 360—413) deals with the important case of the torsion of prisms of rectangular cross-section ($2b \times 2c$).

The chapter opens with some account of Cauchy's memoir of 1829—30 (see our Art. 661*) which had led Saint-Venant to recognise the general distortion of the cross-sections in the torsion problem. Cauchy had found as an approximation $u = -\frac{b^2 - c^2}{b^2 + c^2}\tau yz$, Saint-Venant's expression for the shift parallel to the axis in the case of an ellipse. This really is only an approximation when b and c are very unequal. It makes the greatest slides take place at the corners, but when we note that $\widehat{xy} = \widehat{yx}$ and $\widehat{xz} = \widehat{zx}$, then since $\widehat{yx}$ and $\widehat{zx}$ are zero on the lateral surfaces, it follows that at the angles the nullity of $\widehat{xy}$ and $\widehat{xz}$ connotes that the stress can only be tractive to the cross-section, or that:

> il n'y a, en ces points, aucun glissement, et la section a dû se ployer de manière à rester normale aux quatre arêtes saillantes devenues courbes (p. 362).

This perpendicularity of the cross-section to the sides, at projecting points or angles, holds for all prisms. The recognition of it led Saint-Venant to the investigation of a more exact expression for the torsion of rectangular prisms than that discovered by Cauchy.

[26.] The equations to be solved are

$$\begin{cases} d^2u/dy^2 + d^2u/dz^2 = 0, \\ du/dy = \tau z \text{ for all values of } z \text{ between } c \text{ and } -c \text{ when } y = \pm b, \\ du/dz = -\tau y \text{ for all values of } y \text{ between } b \text{ and } -b \text{ when } z = \pm c. \end{cases}$$

At the suggestion of Wantzel, Saint-Venant reduced these equations to a known form by the substitution of $u = -\tau yz + u'$, when they become

$$\begin{cases} d^2u'/dy^2 + d^2u'/dz^2 = 0, \\ du'/dy = 2\tau z \text{ for all values of } z \text{ between } c \text{ and } -c \text{ when } y = \pm b, \\ du'/dz = 0 \text{ for all values of } y \text{ between } b \text{ and } -b \text{ when } z = \pm c. \end{cases}$$

These equations can be solved by the assumption

$$u' = \Sigma A_m (e^{my} - e^{-my}) \sin mz$$

and the usual determination of the constants by Fourier's Theorem.

[27.] Saint-Venant obtains the following general results:

$$\text{(i)} \begin{cases} u = \tau bc \left[-\dfrac{yz}{bc} + \dfrac{1}{2}\left(\dfrac{4}{\pi}\right)^3 \dfrac{c}{b} \Sigma_{n=1}^{n=\infty} \dfrac{(-1)^{n-1}}{(2n-1)^3} \dfrac{\sinh \dfrac{(2n-1)\pi y}{2c}}{\cosh \dfrac{(2n-1)\pi b}{2c}} \sin \dfrac{(2n-1)\pi z}{2c} \right] \\[2ex] \quad = \tau bc \left[\dfrac{yz}{bc} - \dfrac{1}{2}\left(\dfrac{4}{\pi}\right)^3 \dfrac{b}{c} \Sigma_{n=1}^{n=\infty} \dfrac{(-1)^{n-1}}{(2n-1)^3} \dfrac{\sinh \dfrac{(2n-1)\pi z}{2b}}{\cosh \dfrac{(2n-1)\pi c}{2b}} \sin \dfrac{(2n-1)\pi y}{2b} \right]. \end{cases}$$

$$\text{(ii)} \begin{cases} \widehat{xx} = \widehat{yy} = \widehat{zz} = \widehat{yz} = 0, \\[1ex] \widehat{xy} = -\mu\tau c \left\{ 2\dfrac{z}{c} - \left(\dfrac{4}{\pi}\right)^2 \Sigma_{n=1}^{n=\infty} \dfrac{(-1)^{n-1}}{(2n-1)^2} \dfrac{\cosh \dfrac{(2n-1)\pi y}{2c}}{\cosh \dfrac{(2n-1)\pi b}{2c}} \sin \dfrac{(2n-1)\pi z}{2c} \right\}, \\[2ex] \quad = -\mu\tau c \left(\dfrac{4}{\pi}\right)^2 \dfrac{b}{c} \Sigma_{n=1}^{n=\infty} \dfrac{(-1)^{n-1}}{(2n-1)^2} \dfrac{\sinh \dfrac{(2n-1)\pi z}{2b}}{\cosh \dfrac{(2n-1)\pi c}{2b}} \cos \dfrac{(2n-1)\pi y}{2b}. \\[2ex] \widehat{xz}^{1} = \mu\tau b \dfrac{c}{b} \left(\dfrac{4}{\pi}\right)^2 \Sigma_{n=1}^{n=\infty} \dfrac{(-1)^{n-1}}{(2n-1)^2} \dfrac{\sinh \dfrac{(2n-1)\pi y}{2c}}{\cosh \dfrac{(2n-1)\pi b}{2c}} \cos \dfrac{(2n-1)\pi z}{2c}. \\[2ex] \quad = \mu\tau b \left\{ 2\dfrac{y}{b} - \left(\dfrac{4}{\pi}\right)^2 \Sigma_{n=1}^{n=\infty} \dfrac{(-1)^{n-1}}{(2n-1)^2} \dfrac{\cosh \dfrac{(2n-1)\pi z}{2b}}{\cosh \dfrac{(2n-1)\pi c}{2b}} \sin \dfrac{(2n-1)\pi y}{2b} \right\}. \end{cases}$$

$$\text{(iii)} \begin{cases} M = \mu\tau bc^3 \left\{ \dfrac{16}{3} - \left(\dfrac{4}{\pi}\right)^5 \dfrac{c}{b} \Sigma_{n=1}^{n=\infty} \dfrac{1}{(2n-1)^5} \tanh \dfrac{(2n-1)\pi b}{2c} \right\} \\[2ex] \quad = \mu\tau b^3 c \left\{ \dfrac{16}{3} - \left(\dfrac{4}{\pi}\right)^5 \dfrac{b}{c} \Sigma_{n=1}^{n=\infty} \dfrac{1}{(2n-1)^5} \tanh \dfrac{(2n-1)\pi c}{2b} \right\}. \end{cases}$$

[28.] It will be noted that Saint-Venant obtains in each case double values for his quantities which are unsymmetrical in b

[1] Saint-Venant puts sinh for cosh in the denominator here by a misprint (p. 368, equation 159).

and c. Symmetrical values may be at once obtained by adding and halving his solutions. Or, symmetrical values may be obtained directly by the assumption of the particular integral

$$u = A_p \sinh \frac{\pi p z}{b} \cos \frac{\pi p y}{c} + B_p \sinh \frac{\pi p y}{c} \cos \frac{\pi p z}{c},$$

where p is a positive integer.

It will be found that the surface conditions are then very easily satisfied, and the symmetrical forms of the results thus deduced possess for some cases practical advantages.

Saint-Venant next proceeds to consider special cases of rectangular cross-section which will occupy us in the following seven articles.

[29.] *Cas où l'un des côtés du rectangle est très-grand par rapport à l'autre.* (pp. 372—375.)

From the first of the expressions for M, we obtain

$$M = \frac{16}{3} \mu\tau b c^3 (1 - 0{\cdot}630249\, c/b),$$

and for a first approximation to u

$$u = -\tau z y.$$

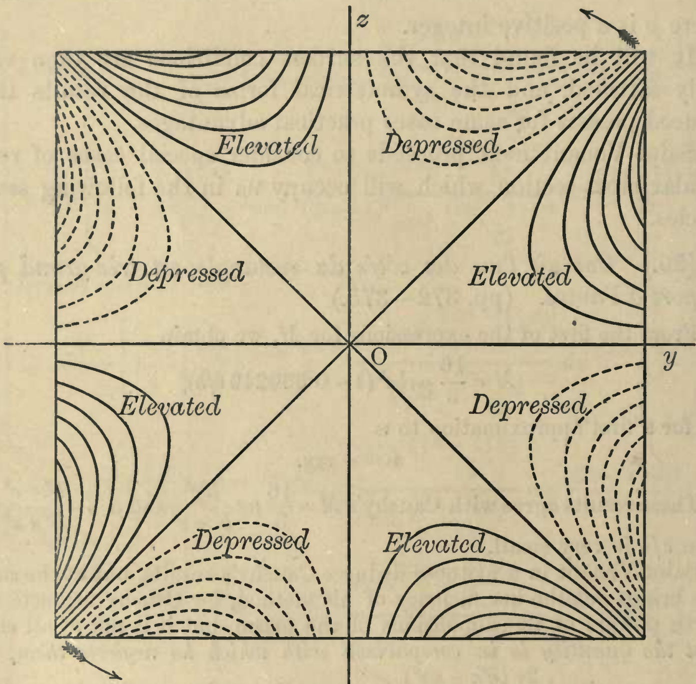
These results agree with Cauchy's $M = \frac{16}{3}\mu\tau \frac{b^3c^3}{b^2+c^2}$ and $u = -\frac{b^2-c^2}{b^2+c^2}\tau y z$ when c/b is very small.

Saint-Venant in a footnote deduces Cauchy's results, but at the same time brings out the insufficiency of his method, for Cauchy neglects the fourth powers of the dimensions of the prism, but it is not at all clear *what the quantity is in comparison with which he neglects them*, for the term omitted $\frac{2\tau (y^3z - yz^3)}{3(b^2+c^2)}$ seems really of the same order as that retained $-\frac{b^2-c^2}{b^2+c^2}\tau y z$ (p. 375).

[30.] On pp. 376—98 we have the full discussion of the prism of square cross-section. The numerical results are calculated from the tables for the hyperbolic functions given by Gudermann[1]. They are calculated from both expressions obtained in Art. 27. Saint-Venant seems to have taken from three to eight terms of his series, but he has not entered upon any investigation as to whether those series satisfy Seydel's condition of *equal convergence*.

[1] *Theorie der Potenzial- oder cyklisch-hyperbolischen Functionen*, S. 263.

The values of u are calculated and given in a table on p. 377. The accompanying figures give the contour lines of the distorted cross-section and the boundaries of the cross-section as cutting the lateral faces of the distorted prism in elevation (diagrammatically exaggerated).



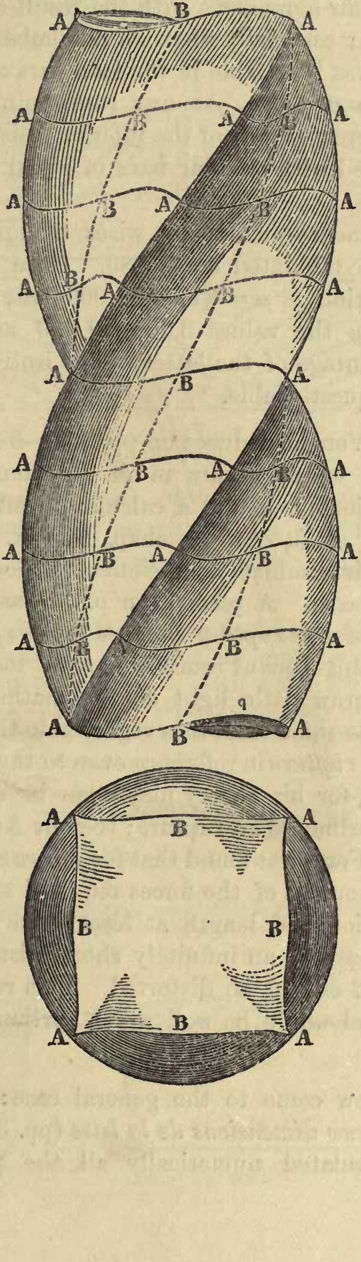
For numerical values we have,

$$M = \cdot 843462\ \mu\tau\omega^2 b^2/3$$
$$= \text{or} < 1\cdot 66532\ S_0 b^3.$$

$\sigma = 1\cdot 350630\ b\tau$ is the maximum slide and occurs at the middle points of the sides of the cross-section, which are thus the *fail-points*. These values are all less than those obtained from the old theory.

[31.] On pp. 382—387 Saint-Venant refers to the experiments of Duleau[1] and Savart[2] as confirming his results. From Duleau's experiments on circular bars the mean value of μ obtained was 6,659,230,000 kilogs. but from his experiments on

[1] See our Art. 229*. [2] See our Art. 334*.

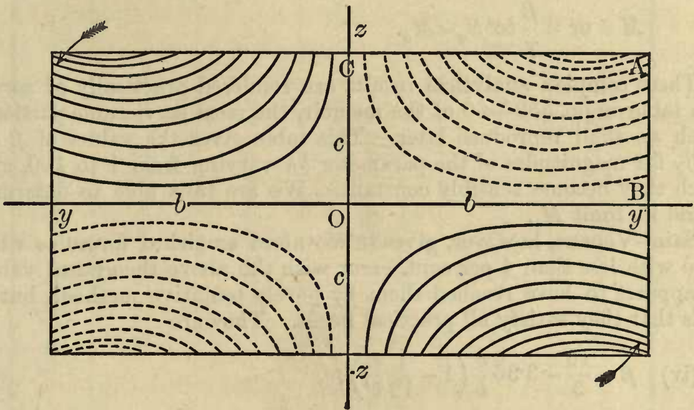

A
B

square sectioned bars it was only 5,636,625,000 on the old theory. Saint-Venant's however brings it up to 6,682,750,000, which may be considered in fair agreement with the result obtained from bars of circular section; especially when we remember the non-isotropic character which was inevitable in the iron bars of Duleau's experiments (see table p. 383). At any rate Saint-Venant's theory accounts for the greater part of the inferior resistance to torsion of square as compared with circular bars of equal sectional moment of inertia.

Some experiments on copper wires of square and circular cross-sections are tabulated on p. 386. Here the mean for the circular cross-section is $\mu = 4{,}174{,}825{,}000$; the old and the new theory give for μ the values 3,384,121,000 and 4,012,180,000; again to the advantage of the latter. The isotropy of these wires is however very questionable.

[32.] Saint-Venant deduces on pp. 387—391 the value of the numerical factor which occurs in M (see our Art. 30) by an algebraic expansion for u and a calculation after the manner of Fourier (*Théorie de la chaleur*, chap. III. art. 208, Eng. Trans. p. 137) of the indeterminate coefficients. It does not seem a very advantageous process. A remark on p. 397 as to the difference between *résistance à la rupture éloignée* and *rupture immédiate* is to the point. Saint-Venant remarks namely that experiments on the latter can throw little light on the mathematical theory of elasticity. At the same time it is regrettable that he should have retained the word *rupture* in reference even to the first limit. Some support, however, for his theory may even be derived, he thinks, from Vicat's experiments on rupture; see our Art. 731* and p. 398 of the memoir. For Vicat found that for *pierre calcaire, brique crue* and *plâtre* the moment of the forces required to break a prism of square cross-section and length at least twice the diameter was *less* than in the case of an infinitely short prism, i.e. a case where the plane section cannot be distorted. This result of Vicat is of great interest and would be well worth further experimental investigation.

[33.] We now come to the general case: *Cas d'un rapport quelconque des deux dimensions de la base* (pp. 398—413). Saint-Venant has calculated numerically all the particulars of the special case when $b/c = 2$. We reproduce the contour lines for the distorted cross-section as given by Saint-Venant on p. 400 according to the table on p. 399.



The reader will at once note the change that these lines present, and Saint-Venant on pp. 400—1 determines the value of b/c for which the change from tetra-axial to bi-axial congruency takes place.

In order to ascertain this we must find when $du/dz = 0$ at the point $y = b$, $z = 0$. For, with the tetra-axial congruency of the contour lines u is positive as we pass from $z = 0$, $y = b$ along the line $y = b$ into the first quadrant, but in the case of biaxial symmetry du/dz is negative, for u decreases or becomes negative as we pass along the same line. Our author thus obtains the equation

$$\sum_1^\infty \frac{1}{(2n-1)^2} \operatorname{sech} \frac{(2n-1)\pi c}{2b} = \left(\frac{\pi}{4}\right)^2,$$

the numerical solution of which gives $b/c = 1{\cdot}4513$.

[34.] The following general results are obtained $(b > c)$:

$$\text{(i)} \quad \begin{cases} M = \mu\tau b c^3 \beta, \\ \text{where } \beta = \left\{ \dfrac{16}{3} - 3{\cdot}361327\,\dfrac{c}{b} + \dfrac{c}{b}\left(\dfrac{4}{\pi}\right)^5 \displaystyle\sum_1^\infty \frac{1 - \tanh \dfrac{(2n-1)\pi b}{2c}}{(2n-1)^5} \right\} \end{cases}$$

(p. 401),

$$\text{(ii)} \quad \begin{cases} \text{maximum slide } \sigma = c\tau\gamma, \\ \text{where } \gamma = 2 - \left(\dfrac{4}{\pi}\right)^2 \displaystyle\sum_1^\infty \frac{1}{(2n-1)^2 \cosh \dfrac{(2n-1)\pi b}{2c}} \end{cases} \quad \text{(p. 412)},$$

and this maximum slide takes place at the centre of the longer side of the rectangular cross-section. (p. 410.)

(iii) $S_0 = \text{or} > \mu\gamma\tau c$, hence

$$M = \text{or} < \frac{\beta}{\gamma} bc^2 S_0 = M_0.$$

These complex analytical results are rendered practically of service by a table on pp. 559—60 of the memoir, the most serviceable portion of which we shall reproduce later. This table gives the values of β and of β/γ for magnitudes of the parameter b/c varying from 1 to 100, after which they become sensibly constant. We are thus able to determine M and its limit M_0.

Saint-Venant, however, gives in footnotes empirical formulae which agree with less than 4 per cent. error with the above theoretical values. He appears to have reached them by purely tentative methods, but he holds that they satisfy all practical needs. They are

(iv) $\beta = \frac{16}{3} - 3{\cdot}36 \frac{c}{b}\left(1 - \frac{1}{12}\frac{c^4}{b^4}\right).$

(v) $\frac{\beta}{\gamma} = \frac{8}{3}\Big/\left(1 + {\cdot}6\frac{c}{b}\right)$ or, $M_0 = \frac{40b^3c^3}{15b + 9c} \cdot S_0.$

{It should be noted that our $\sigma = g_x$, our $\beta = \mu$, our $\tau = \theta$, our $\mu = G$, our $S_0 = T_0$ of the memoir.}

[35.] On pp. 403—6 we have a further discussion of experiments of Duleau and Savart on the torsion of rectangular bars of iron, oak, *plâtre*, and *verre à vitre*, the paucity of the experiments, and the large variation in the values of the slide-moduli as obtained from Saint-Venant's formula do not seem to me very satisfactory. A series of experiments directly intended to test the torsion of rectangular bars for variations of the parameter c/b would undoubtedly be of considerable value.

[36.] We now reach Saint-Venant's ninth chapter which is entitled: *Torsion de prismes ayant d'autres bases que l'ellipse ou le rectangle.* It occupies pp. 414—454.

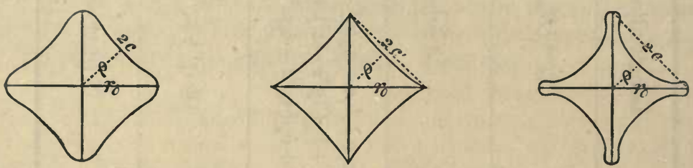
The chapter opens with an enumeration of the various forms of contour for which it is easy to integrate the equations of Art. 17. We will tabulate them on the next page.

	Contour given by : Constant =	Longitudinal shift given by : $u =$
1	$\frac{\tau}{2}(y^2+z^2)+\Sigma e^{my}(A_m \cos mz - B_m \sin mz)$	$\Sigma e^{my}(B_m \cos mz + A_m \sin mz)$
2	$\frac{\tau}{2}(y^2+z^2)+\Sigma A_m \cosh my \cos mz$	$\Sigma A_m \sinh my \sin mz$
3	$\frac{\tau}{2}(y^2+z^2)-a_1z+b_1y-a_2 2yz+b_2(y^2-z^2)-a_3(3y^2z-z^3)$ $+b_3(y^3-3yz^2)-a_4(4y^3z-4yz^3)+b_4(y^4-6y^2z^2+z^4)-\text{etc.}$	$a_0+a_1y+b_1z+a_2(y^2-z^2)+b_2 2yz+a_3(y^3-3yz^2)$ $+b_3(3y^2z-z^3)+a_4(y^4-6y^2z^2+z^4)$ $+b_4(4y^3z-4yz^3)+\text{etc.}$
4	$\frac{\tau}{2}(y^2+z^2)+\sqrt{-1}\,\phi(y+z\sqrt{-1})-\sqrt{-1}\,\psi(y-z\sqrt{-1})$	$\phi(y+z\sqrt{-1})+\psi(y-z\sqrt{-1})$
5	$\frac{\tau}{2}r^2+\Sigma(-a_nr^n \sin n\phi + b_mr^m \cos m\phi)$	$\Sigma(a_nr^n \cos n\phi + b_mr^m \sin m\phi)$

Solutions (3) and (5) are really identical. No. 4 has given rise to the solutions in terms of *conjugate functions:* see Thomson and Tait's *Natural Philosophy*, 2nd Ed. Part II. pp. 250—3.

[37.] In the present chapter Saint-Venant dismisses Nos. 1 and 2 on the ground that the resulting curves are very difficult to trace. He contents himself with two closed curves of the fourth degree and one of the eighth as given by No. 5. On pp. 421—434 he calculates and traces these curves at considerable length. The most practically valuable results are those obtained on p. 439.

We have there the following characteristic sections treated:



(*a*) The equation of the first curve is:

$$\frac{y^2+z^2}{r_0^{\,2}} - \cdot 4\,\frac{y^4-6y^2z^2+z^4}{r_0^{\,4}} = \cdot 6 \quad \text{(Square with rounded angles).}$$

$$\omega = 2\cdot 0636 r_0^{\,2}\,;\ \omega\kappa^2 = \cdot 7174 r_0^{\,4} = 1\cdot 0586\,\omega^2/2\pi\,;$$

$$M = \cdot 5873\mu\tau r_0^{\,4} = \cdot 8186\mu\tau\omega\kappa^2 = \cdot 8666\mu\tau\omega^2/2\pi.$$

(*b*) The equation to the second curve is:

$$\frac{y^2+z^2}{r_0^{\,2}} - \cdot 5\,\frac{y^4-6y^2z^2+z^4}{r_0^{\,4}} = \cdot 5 \quad \text{(Square with acute angles).}$$

$$\omega = 1\cdot 7628 r_0^{\,2}\,;\ \omega\kappa^2 = \cdot 5259 r_0^{\,4} = 1\cdot 0634\,\omega^2/2\pi\,;$$

$$M = \cdot 4088\mu\tau r_0^{\,4} = \cdot 7783\mu\tau\omega\kappa^2 = \cdot 8276\mu\tau\omega^2/2\pi.$$

(*c*) The equation to the third curve is, if $y = r\cos\phi$, $z = r\sin\phi$,

$$\frac{r^2}{r_0^{\,2}} - \frac{48}{49}\cdot\frac{16}{17}\,\frac{r^4\cos 4\phi}{r_0^{\,4}} + \frac{12}{49}\cdot\frac{16}{17}\,\frac{r^8\cos 8\phi}{r_0^{\,8}} = 1 - \frac{36}{49}\cdot\frac{16}{17}$$

(Star with four rounded points).

$$\omega = 1\cdot 2202 r_0^{\,2}\,;\ \omega\kappa^2 = \cdot 2974 r_0^{\,4} = 1\cdot 2551\,\omega^2/2\pi\,;$$

$$M = \cdot 15983\mu\tau r_0^{\,4} = \cdot 5374\mu\tau\omega\kappa^2 = \cdot 6745\mu\tau\omega^2/2\pi.$$

We add to these the results for the circle and square.

(*d*) Circle: $M = \mu\tau\omega\kappa^2 = \mu\tau\omega^2/2\pi.$

(*e*) Square: $M = \cdot 84346\mu\tau\omega\kappa^2 = \cdot 88327\mu\tau\omega^2/2\pi.$

From the above numbers we can deduce some important practical inferences, which we will do in Saint-Venant's own words.

On voit qu'il faut, de l'expression $\mu\tau\omega\kappa^2$ de l'ancienne théorie, retrancher, pour avoir M quand la section est le carré à angles arrondis et côtés légèrement concaves, une proportion des ·1814. Nous avons vu que, pour le carré rectiligne, il faut prendre $M = \cdot 84346 \mu\tau\omega\kappa^2$ ou retrancher une proportion de ·15654 seulement. La légère concavité des côtés a plus influé pour diminuer le moment de torsion (pour même moment d'inertie) que l'arrondissement des quatre angles n'a influé pour l'augmenter.

Pour le carré curviligne à côtés un peu plus concaves et angles aigus, il faut retrancher les ·2217. *Il suffit, comme l'on voit, d'une concavité assez légère des côtés de la base* (1/22 *environ*) *pour diminuer assez notablement le moment de torsion d'un prisme carré.*

Enfin, pour le prisme à côtes saillantes, il faut, de $\mu\tau\omega\kappa^2$, retrancher l'énorme proportion de ·4626, ou prendre seulement $\cdot 5374 \mu\tau\omega\kappa^2$ au lieu de $\mu\tau\omega\kappa^2$ que l'on prend pour une section circulaire, ou de $\cdot 84346 \mu\tau\omega\kappa^2$ pour une section carrée rectiligne.

Et comme on a, pour une section circulaire, $\kappa^2 = \omega/2\pi$, $M = \mu\tau\omega^2/2\pi$, l'on trouve que les prismes ayant pour bases le carré arrondi, le carré aigu et l'étoile, n'offrent respectivement que les ·867, les ·828, et les ·674 de la résistance élastique à la torsion qu'ils offriraient à égale superficie ω de la section, ou *à égale quantité de matière*, s'ils étaient à base circulaire, bien que les moments d'inertie de leurs sections soient 1^fois^ ·059, 1^fois^ ·063, 1^fois^ ·255 ceux de sections circulaires d'égale superficie.

Ainsi, les quatre saillies qui, malgré leur peu d'épaisseur, ont une influence considérable sur la grandeur du moment d'inertie n'en ont qu'une très-faible sur le moment de torsion. *Les pièces à côtes, employées si utilement contre les flexions, doivent être exclues des parties des constructions où les forces tendent à tordre, ou, du moins, il faut ne compter nullement sur une quote-part des quatre côtes ou saillies dans la résistance* (pp. 439—40).

[38.] Saint-Venant illustrates the inefficiency of projecting parts still more effectually in a footnote to Art. 105, p. 454. He takes a curve of the fourth degree whose equation is given in a footnote, p. 448, and by ascribing a particular value to one of the constants obtains two separate loops. The equation to the contour is:

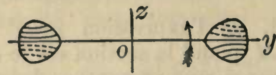
$$\frac{y^2}{b^2} + \frac{z^2}{c^2} + a\left(\frac{1}{c^2} - \frac{1}{b^2}\right)(y^2 - z^2) - a\,\frac{y^4 - 6y^2z^2 + z^4}{b^2c^2} = 1 - a\,;$$

and the longitudinal shift

$$u = -(1 - 2a)\frac{b^2 - c^2}{b^2 + c^2}\tau yz - \frac{4a\tau}{b^2 + c^2}(y^3z - yz^3).$$

The special value of the constant assumed is $c^2 = -b^2/16$. We have then a figure of the form below and the value of M is only equal to

$\cdot 01857 \mu\tau\omega\kappa^2$, or the torsion of such a pair of cylinders round an intermediate axis is *only one fifty-fourth* of that given by the old



theory:—"Cela ne doit pas étonner, si l'on considère que le glissement est nul aux points $z = 0$, $y = \pm b\sqrt{\frac{23}{32}}$ ou à très-peu près au centre de gravité de chaque orbe."

[39.] Saint-Venant on his pp. 441—9 discusses the contour-lines of the distorted cross-sections of our Art. 37. This he accomplishes by numerical tables in a footnote (pp. 441—3). Then he considers the maximum slides and *fail-points* of the same sections and finally the limiting values of the torsional couples. These values are as follows:

For section (*a*) of Art. 37 $\quad M_0 = \cdot 8269 \frac{\omega\kappa^2}{r_0} S_0 = \cdot 7094 \frac{\omega^{\frac{4}{3}}}{2\sqrt{\pi}} S_0.$

" $\quad$ (*b*) $\quad$ " $\quad M_0 = \cdot 85514 \frac{\omega\kappa^2}{r_0} S_0 = \cdot 6812 \frac{\omega^{\frac{4}{3}}}{2\sqrt{\pi}} S_0.$

" $\quad$ (*c*) $\quad$ " $\quad M_0 = \cdot 7285 \frac{\omega\kappa^2}{r_0} S_0 = \cdot 5695 \frac{\omega^{\frac{4}{3}}}{2\sqrt{\pi}} S_0.$

The reasoning by which Saint-Venant deduces the *fail-points* cannot be considered satisfactory. Indeed the statement as to the 'side of the triangle' and the deduction of the maximum slide on p. 444 are unsound. The same judgment must be passed on the process of p. 447, where the maximum slide for the section (*c*) is shewn to be on the contour. Thus Saint-Venant has *not demonstrated* his very general statement (237) on p. 448. The reader will however find little difficulty in proving the accuracy of Saint-Venant's *results* by casting the expressions on pp. 444 and 447 into other forms or by the ordinary processes of the Differential Calculus. In his edition of the *Leçons de Navier*, our author has recognised the defective reasoning of these pages and replaced them by more accurate arguments. (Cf. his § 31, pp. 308—310 and § 37, pp. 340—1: see our Art. 181 (*e*).)

[40.] In the concluding pages of this chapter Saint-Venant points out how the solutions of a number of other sections can be obtained. Thus we can take solutions like (3) of Art. 36 involving terms of the 12th and 16th degrees and so obtain curves equally symmetrical with regard to the axes of y and z.

Et en y conservant des termes du deuxième, du sixième, du dixième degré à puissances paires de y et z, tels que $b_2 r^2 \cos 2\phi = b_2 (y^2 - z^2)$, $b_6 r^6 \cos 6\phi =$ etc., l'on aurait une multitude de courbes symétriques par rapport à *chacun* des deux axes de y et z, mais *non égales dans leurs deux sens*, et ayant l'ellipse pour cas particulier (p. 449).

We have referred to an example of this in Art. 38, and another is given by Saint-Venant in a footnote; namely the curve whose equation is

$$\tau \frac{r^2}{2} + b_3 r^3 \cos 3\phi = \text{constant},$$

where

$$u = b_3 r^3 \sin 3\phi.$$

By taking $b_3 = -\frac{\tau}{6b}$ and the constant $= \frac{2}{3}\tau b^2$, the equation to the contour of the section becomes

$$\left(\frac{r}{2b}\right)^2 - \frac{2}{3}\left(\frac{r}{2b}\right)^3 \cos 3\phi = \frac{1}{3},$$

or

$$(y+b)\left(z - \frac{2b-y}{\sqrt{3}}\right)\left(z + \frac{2b-y}{\sqrt{3}}\right) = 0 \text{ ....(i)},$$

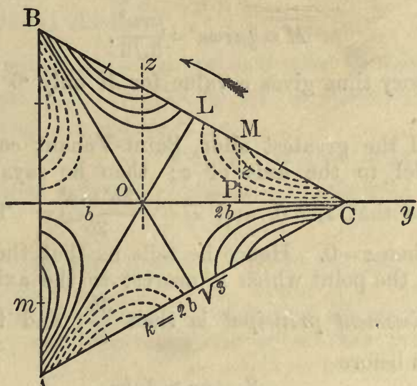
i.e. an equilateral triangle of height $3b$ and side $= 2b\sqrt{3}$, the axis of y coinciding with a median line. We reproduce Saint-Venant's entire treatment of this case as a good example of his method, and in order in one point to indicate a weakness in his reasoning.

41. We find at once that

$$u = -\frac{\tau}{6b}(3y^2 z - z^3) \text{ ....(ii)}.$$

Let c be the greatest value of u which, on the side denoted by $y + b = 0$, will be where $z = -b$; then $c = \frac{\tau b^2}{3}$, and consequently

$$\frac{u}{c} = -\frac{r^3}{2b^3} \sin 3\phi.$$



Thus the form of the surface into which the originally plane cross-section becomes changed by torsion is easily understood. In the part between Oy and the perpendicular OL, we have u negative; in the part between OL and OB we have u positive; in the part between OB and yO produced through O we have u negative; in the next piece, which is vertically opposite to the piece between Oy and OL, we have u positive; and so on.

We have as usual the equations

$$\widehat{xy} = \frac{du}{dy} - \tau z, \qquad \widehat{xz} = \frac{du}{dz} + \tau y;$$

these by (ii) of Art. 41 give

$$\widehat{xy} = -\tau\left(z + \frac{yz}{b}\right), \qquad \widehat{xz} = \tau\left(y - \frac{y^2 - z^2}{2b}\right).$$

The moment of torsion by equation (iv) of Art. 17 is

$$M = \mu\tau\left\{\int y^2 d\omega + \int z^2 d\omega + \frac{3}{2b}\int yz^2 d\omega - \frac{1}{2b}\int y^3 d\omega\right\}.$$

All these integrations are easily effected; for here if ξ denote any function of y and z, even in z, we have

$$\int \xi d\omega = 2\int\int \xi dy dz,$$

where we integrate for z from $z = 0$ to $z = \dfrac{2b - y}{\sqrt{3}}$, and for y from $y = -b$ to $y = 2b$. Thus we find that

$$\int z^2 d\omega = \tfrac{3}{2} b^4 \sqrt{3},$$
$$\int y^2 d\omega = \tfrac{3}{2} b^4 \sqrt{3},$$
$$\int yz^2 d\omega = -\tfrac{3}{5} b^5 \sqrt{3},$$
$$\int y^3 d\omega = \tfrac{3}{5} b^5 \sqrt{3}.$$

Then for the moment of inertia round the axis we have

$$\omega\kappa^2 = \int y^2 d\omega + \int z^2 d\omega = 3b^4\sqrt{3} = \frac{\omega^2}{3\sqrt{3}}, \text{ for } \omega = 3b^2\sqrt{3}.$$

Hence $$M = \tfrac{3}{5}\mu\tau\omega\kappa^2 = \frac{\mu\tau\omega^2}{5\sqrt{3}}.$$

The new theory thus gives a value for M only ·6 of that given by the old.

42. To find the greatest slide, Saint-Venant considers the side which is parallel to the axis of z; then he says that along this side $y + b = 0$, so that $\widehat{xy} = 0$, and $\widehat{xz} = -\dfrac{3b^2 - z^2}{2b}\tau$. Thus the greatest value of $\widehat{xz}$ is when $z = 0$. Hence he tells us that the *fail-point* is on the boundary at the point which is nearest to the axis. The greatest value of the *glissement principal* is then $\dfrac{3b}{2}$; and to ensure safety we must have as before

$$S_0 = \text{or} > \tfrac{3}{2}\mu b\tau.$$

Combining this with $M = \frac{3}{5}\mu\omega\kappa^2\tau$ we have at the limit

$$M_0 = \frac{2}{5}\frac{\omega\kappa^2}{b}S_0 = \frac{6}{5}S_0 b^3\sqrt{3} = \frac{2\sqrt[4]{3}}{15}S_0\omega^{\frac{3}{2}}.$$

Thus next to the circular section, the section in the form of an equilateral triangle gives the simplest results.

[The above reasoning involves the assumption that the point of maximum slide *lies on the contour* and is thus unsatisfactory. Saint-Venant has given a thorough investigation of the point in his edition of the *Leçons de Navier*, pp. 287—9.]

[43.] In conclusion we may note that Saint-Venant holds that, among the numerous curves he has considered, one can be found sufficiently close to give practically the laws of torsion for a prism of any given cross-section (pp. 451—2).

[44.] The tenth chapter of the memoir deals with those cases in which the slide-moduli are not the same in the direction of the two transverse axes taken as those of y and z. It occupies pp. 454—70.

Nous y avons aussi été déterminé par le désir de donner sous leur forme la plus simple les seules formules que l'on puisse, jusqu'à présent, appliquer à la pratique; car on n'a pas encore trouvé, par des expériences, le rapport que peuvent avoir entre eux les deux coefficients de glissement transversal μ_1, μ_2 pour diverses matières, et il faut bien les supposer ordinairement égaux (p. 454).

Although well-planned experiments on the possible inequality of μ_1, μ_2 arising either from natural structure or from some process of working are still wanting, yet the inequality in the slide-moduli is not without value as a possible explanation of several minor phenomena of physical elasticity.

[45.] The equations which we have now to solve are those numbered (vi) in our Art. 17. Let us put in those equations $y = \sqrt{\mu_1}\,y'$, $z = \sqrt{\mu_2}\,z'$; they at once reduce to

$$\left.\begin{array}{r} d^2u/dy'^2 + d^2u/dz'^2 = 0 \\ (du/dz' + \tau' y')\,dy' - (du/dy' - \tau' z')\,dz' = 0 \end{array}\right\} \dots\dots\dots\dots\dots\dots (\text{i}),$$

where $$\tau' = \sqrt{\mu_1\mu_2}\,\tau.$$

In other words our equations remain of exactly the same form provided we write $\tau' = \sqrt{\mu_1\mu_2}\,\tau$ for τ. Hence if we remember that every contour must first be projected by means of the above relation between y, z and y', z', we may make use of all the previous results and equations.

[46.] Thus in the case of the ellipse (pp. 455—8 of memoir), we must write for $\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $\frac{y'^2}{b^2/\mu_1}+\frac{z'^2}{c^2/\mu_2}=1$. Thus we obtain at once the results:

$$u=-\frac{b^2/\mu_1-c^2/\mu_2}{b^2/\mu_1+c^2/\mu_2}\tau y'z'=-\frac{b^2/\mu_1-c^2/\mu_2}{b^2/\mu_1+c^2/\mu_2}\tau yz.$$

Similarly $\widehat{xy}=-\frac{2\tau zb^2}{b^2/\mu_1+c^2/\mu_2}$, $\widehat{xz}=\frac{2\tau yc^2}{b^2/\mu_1+c^2/\mu_2}$,

and $$M=\frac{\pi b^3c^3\tau}{b^2/\mu_1+c^2/\mu_2}\text{ (see Art. 18).}$$

Saint-Venant remarks that with this inequality, the cross-section of a circular cylinder will be distorted by torsion. The elliptic prism however, for which the ratio of the semi-axes $b/c=\sqrt{\mu_1/\mu_2}$, will retain undistorted cross-sections although under torsion (p. 456). Saint-Venant in the course of the chapter again refers to relations of this kind (p. 462), but it is obvious that such are extremely unlikely to occur in practice.

It must be noted that the 'fail-limit' (*condition de non-rupture*, pp. 456—7) now takes another form, namely that of our Art. 5 (f),

$$1=\text{or}>\left(\frac{\mu_1\sigma_{xy}}{S_1}\right)^2+\left(\frac{\mu_2\sigma_{xz}}{S_2}\right)^2.$$

From this we find at once

$$(b^2/\mu_1+c^2/\mu_2)^2=\text{or}>4\tau^2\,(b^4z^2/S_1^{\,2}+c^4y^2/S_2^{\,2}).$$

We have then to find the maximum value of the right-hand side. It is easily seen to be on the contour of the cross-section, and at the extremities of the minor or major axis according as b/c is $>$ or $<$ S_1/S_2. In the first case we find that the limiting value of M is given by

$$M_1=\frac{\pi bc^2}{2}S_1.$$

[47.] Saint-Venant devotes pp. 458—460 to describing the changes which must be made in the general solutions of our Art. 36 in order to adapt them to this case of unequal slide-moduli. They follow easily from our Art. 45. On pp. 460—8 he treats at some length the case of the prism with rectangular cross-section. The results are the same as those of our Art. 27, provided we replace the ratios $\frac{c}{b}$ and $\frac{b}{c}$ where they occur in our formulae by $\frac{c}{b}\sqrt{\frac{\mu_1}{\mu_2}}$ and $\frac{b}{c}\sqrt{\frac{\mu_2}{\mu_1}}$ respectively, and the exponentials $e^{\pm\frac{(2n-1)\pi}{2}\frac{y}{c}}$ and $e^{\pm\frac{(2n-1)\pi}{2}\frac{z}{b}}$ by $e^{\pm\frac{(2n-1)\pi}{2}\frac{y}{c}\sqrt{\frac{\mu_2}{\mu_1}}}$ and $e^{\pm\frac{(2n-1)\pi}{2}\frac{z}{b}\sqrt{\frac{\mu_1}{\mu_2}}}$ respectively.

The maximum slides still occur at the middle points of the sides, but at the middle of the greater side $2b$ or the lesser side $2c$ according as $b/c >$ or $< \sqrt{\mu_1/\mu_2}$. Saint-Venant gives at the conclusion of the memoir a very useful table, which we reproduce for reference. It serves for equal slide-moduli when we simply put $\mu_1 = \mu_2$. The parameter in the first column is $\frac{b}{c}\sqrt{\frac{\mu_2}{\mu_1}}$ and for it values are taken from 1 to 100 as well as ∞. The second column

TABLE I.

Torsion of Prisms of rectangular Cross-Section.

$\frac{b}{c}\sqrt{\frac{\mu_2}{\mu_1}}$	$(M=\beta\mu_1\tau bc^3)$ β	$\left(\sigma_1=-\gamma_1 c\tau \text{ middle of side } 2b\right)$ γ_1	$\left(\sigma_2=\gamma_2 b\tau \text{ middle of side } 2c\right)$ γ_2	$\left(M_1=\frac{\beta}{\gamma_1}bc^2 S_1\right)$ β/γ_1	$\left(M_2=\frac{\beta}{\gamma_2}\frac{c^2}{b^2}\frac{\mu_1}{\mu_2}b^2 c S_2\right)$ $\frac{\beta}{\gamma_2}\frac{c^2}{b^2}\frac{\mu_1}{\mu_2}$
1	2·24923	1·35063	1·35063	1·66534	1·66534
1·05	2·35908	1·39651		1·68954	
1·1	2·46374	1·43956		1·71146	
1·15	2·56330	1·47990			
1·2	2·65788	1·51753		1·75363	
1·25	2·74772	1·55268	1·13782	1·76970	1·54556
1·3	2·83306	1·58544		1·7852	
1·35	2·91379	1·61594		1·80316	
1·4	2·99046	1·64430		1·81868	
1·45	3·06319	1·67265			
1·5	3·13217	1·69512	·97075	1·84776	1·43402
1·6	3·25977	1·73889	·91489	1·87463	1·39180
1·7	3·37486	1·77649			
1·75	3·42843	1·79325	·84098	1·91170	1·33107
1·8	3·47890	1·80877		1·92334	
1·9	3·57320	1·83643			
2	3·65891	1·86012	·73945	1·96703	1·15286
2·25	3·84194	1·90546			
2·5	3·98984	1·93614	·59347	2·06072	1·07566
2·75	4·11143	1·95687			
3	4·21307	1·97087		2·13767	
3·333			·44545		
3·5	4·37299	1·98672		2·20111	
4	4·49300	1·99395	·37121	2·25332	·757
4·5	4·58639	1·99724		2·29636	
5	4·66162	1·99874	·29700	2·33200	·628
6	4·77311	1·99974		2·38687	
6·667			·22275		
7	4·85314	1·99995		2·42663	
8	4·91317	1·99999	·18564	2·45660	
9	4·95985	2		2·47993	
10	4·99720	2	·14858	2·49860	
20	5·16527	2	·07341	2·58264	
50	5·26611	2		2·63306	
100	5·29972	2		2·64986	
∞	5·33333	2	0	2·66667	

gives the value of β, where $M=\beta\mu_1\tau bc^3$ is the value of the torsional couple. The third and fourth columns give the maximum slides by means of the coefficients γ_1 and γ_2 where $\sigma_1=-\gamma_1 c\tau$ and $\sigma_2=\gamma_2 b\tau$. The fifth and sixth columns give the maximum value M_0 of M by means of the tabulated values of β/γ_1 and $\frac{\beta}{\gamma_2}\frac{c^2}{b^2}\frac{\mu_1}{\mu_2}$, where $M_1=\left(\frac{\beta}{\gamma_1}\right)bc^2 S_1$ and $M_2=\left(\frac{\beta}{\gamma_2}\frac{c^2}{b^2}\frac{\mu_1}{\mu_2}\right)b^2cS_2$. M_0 is to be taken equal to the lesser of M_1 and M_2.

[48.] Pages 468—9 of this chapter suggest the modifications which must be made in the results obtained for prisms of other cross-sections, when μ_1 differs from μ_2; while on p. 470 we have a simple proof that in this case at corners and angles which project there is no slide, or the intersection of the lateral faces at such corners remains normal to the cross-section.

[49.] Saint-Venant's eleventh chapter deals with the torsion of hollow prisms (pp. 471—6).

In this case we have to satisfy the surface shift-equation

$$\mu_2\,(du/dz+\tau y)\,dy-\mu_1\,(du/dy-\tau z)\,dz=0 \quad \text{............(i)}$$

over two surfaces. If then we form a family of surfaces satisfying this equation and give to the arbitrary constant which appears on the right-hand side two different values we shall obtain the two boundaries of a hollow prism satisfying all the required conditions.

For example:

(*a*)
$$u=-\frac{b^2/\mu_1-c^2/\mu_2}{b^2/\mu_1+c^2/\mu_2}\,\tau yz$$

satisfies the body shift-equation. Substituting in (i) we have on integration

$$c^2y^2+b^2z^2=\text{constant}.$$

Giving the constant different values we obtain a system of similar and similarly placed ellipses. Thus we find for a hollow elliptic cylinder formed by the ellipses ($2b\times 2c$) and ($2b'\times 2c'$)

$$M=\tau\left\{\frac{\pi b^3c^3}{b^2/\mu_1+c^2/\mu_2}-\frac{\pi b'^3c'^3}{b'^2/\mu_1+c'^2/\mu_2}\right\}=\frac{\tau\pi b^3c^3}{b^2/\mu_1+c^2/\mu_2}\left\{1-\left(\frac{b'}{b}\right)^4\right\}.$$

(*b*) In the rectangular section

$$u=-\tau yz+\Sigma A_m\sinh(my/\sqrt{\mu_1})\sin(mz/\sqrt{\mu_2}),$$

where
$$A_m=\tau\frac{bc}{2}\left(\frac{4}{\pi}\right)^3\frac{c}{b}\sqrt{\frac{\mu_1}{\mu_2}}\frac{(-1)^{n-1}}{(2n-1)^3}\operatorname{sech}(mb/\sqrt{\mu_1})$$

and
$$m=\frac{(2n-1)\,\pi}{2}\frac{\sqrt{\mu_2}}{c}.$$

Substituting in (i) and integrating we find:

$$\frac{32}{\pi^3}\sum\frac{(-1)^{n-1}}{(2n-1)^3}\frac{\cosh(my/\sqrt{\mu_1})}{\cosh(mb/\sqrt{\mu_1})}\cos(mz/\sqrt{\mu_2})=1-\frac{z^2}{c^2}+C.$$

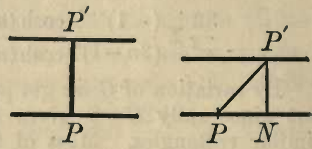
By variation of C we get possible boundary lines for hollow sections, but since only $C=0$ gives a rectangle, the boundaries will not be similar rectangles. Most of these curves would be extremely difficult to trace; for small values of C, however, we may practically assume we have a hollow cylinder whose cross-section is bounded by two nearly equal rectangles. Saint-Venant finds in curves thus obtained an analogy to the *surfaces isothermes* of Lamé.

(c) Lastly we find briefly described the method of dealing with solutions of the form (5) of our Art. 36. The curves are sketched on p. 476 for the double family given by the equation of our Art. 38. Any two of either set might serve as the basis of a hollow prism. Saint-Venant returns in the *Leçons de Navier* (pp. 306, 325–332) to this family and treats a special case of it—*Section en double spatule, analogue à celle d'un rail de chemin de fer*,—at considerable length.

[50.] I now reach Saint-Venant's twelfth chapter which is thus entitled: *Cas où il y a en même temps une torsion, une flexion, des dilatations et des glissements latéraux. Conditions de non-rupture sous leurs influences simultanées* (pp. 476—522). It deals with the all important practical question of combined strain, and may be described as the first scientific treatment of the subject: see our Arts. 1377* and 1571*. The chapter may be looked upon as an extension of the safe-stretch conditions formulated for the first time in the *Cours lithographié*, see our Art. 1567*. In the treatment of the problem to be found there it will be remembered that the slide was dealt with as constant over the cross-section; here the new results with regard to the flexural and torsional distortion of the cross-sections are applied to that extended form of the earlier formula which was cited in our Art. 5 (*d*).

[51.] Before I enter upon an analysis of Saint-Venant's results I may refer to the substance of a footnote given on pp. 477—8 of the memoir. Saint-Venant notes that under torsion the sides and fibres of a prism originally parallel become inclined and helical and so must suffer a stretch. This stretch is, however—if the product of the torsion τ and the distance of the farthest fibre from the axis be small—a small quantity of the *second order*. Wertheim in a memoir to be considered later (see our Chap. XI.) has referred to certain phenomena which he attributes to this stretch.

By a simple analysis Saint-Venant finds its absolute magnitude for a *right-circular* cylinder of radius a. Take a fibre at distance r from the axis and let us consider the element PP' of it between two planes at unit distance. Suppose owing to torsion that the two planes approach each other by a quantity η and let $P'N$ be the perpendicular from the new position of P' on the cross-section through P,



$$PP' = \sqrt{P'N^2 + PN^2} = \sqrt{(1-\eta)^2 + \tau^2 r^2}$$
$$= 1 - \eta + \frac{\tau^2 r^2}{2} \text{ nearly.}$$

Saint-Venant takes for PP' the quantity

$$(1-\eta)\sqrt{1+\tau^2 r^2}$$
$$= 1 - \eta + \frac{\tau^2 r^2}{2},$$

but I do not think he obtains the first expression very rigorously. He has practically the same value in the *Leçons de Navier* (pp. 240—1).

The traction in the fibre will now be given by

$$E d\omega \left(\frac{\tau^2 r^2}{2} - \eta\right),$$

where E is the longitudinal stretch-modulus. The quantity $-\eta$ must be determined by the condition that the total traction is zero, or

$$\int_0^a 2\pi r dr E \left(\frac{\tau^2 r^2}{2} - \eta\right) \sin P'PN = 0.$$

Since $$\sin P'PN = \frac{1-\eta}{1 - \eta + \dfrac{\tau^2 r^2}{2}} = 1 - \frac{\tau^2 r^2}{2}$$

it may be put $= 1$ in the integral.

We find $\dfrac{\tau^2 a^4}{4} = a^2 \eta$, giving $\eta = \dfrac{\tau^2 a^2}{4}$, a result which agrees with Saint-Venant's; our analysis thus proves that η is of the *second order* in τ.

Further we have for the total-moment of these tractions about the axis

$$M = \int_0^a 2\pi r dr E \left(\frac{\tau^2 r^2}{2} - \frac{\tau^2 a^2}{4}\right) \cos P'PN \times r$$
$$= E\pi\tau^3 \int_0^a r^3 dr \left(r^2 - \frac{a^2}{2}\right), \text{ since } \cos P'PN = \tau r;$$
$$= \frac{E\pi a^3}{24} (\tau a)^3.$$

If one takes account of the tractions produced by the lateral squeezes of the fibres, we shall have a similar expression with a change only in the elastic constant. Thus it appears that the effect produced by the stretch of the fibres is of the third order in the torsion and may be legitimately neglected if the torsion be small.

This point—that the stretch only varies as the cube of the torsion—was first stated by Young without proof in his *Lectures on Natural Philosophy*, Vol. I. p. 139. He thence argued that torsional resistance must be due to *detrusion* (slide) and not to stretch. When the torsion τ is considerable, then the quantity M above, due to stretch of the fibres, becomes of importance, as appears from Wertheim's experiments in the memoir referred to: see our Chap. XI.

[52.] Returning to the chief topic of the chapter under consideration we first note with Saint-Venant the linearity of the equations of elasticity, so that it is possible to combine various strains due to different forms of loading by vector-addition and so obtain the total shifts due to a combined load system: see our Art. 1568*. On pp. 479—80 Saint-Venant deduces the shifts for an elliptic prism subject at the same time to traction, flexure and torsion. Use is made of the results obtained on pp. 304 and 455 of the memoir: see our Arts. 12 and 46.

[53.] Saint-Venant now turns to equation (iii) of our Art. 5 (d) and after pointing out the difficulties of the general solution by analysis for the case of any prism (p. 482) proceeds to some more special and simple cases when the cubic can be reduced to an equation of the second degree.

Case (1). Let the elasticity be symmetrical about the axis of x, and let the solid be a prism subjected only to a uniform lateral traction, we have

$$s_y = s_z, \quad \bar{s}_y = \bar{s}_z, \quad \bar{\sigma}_{xy} = \bar{\sigma}_{xz} \text{ and } \sigma_{yz} = 0.$$

Hence, if $\sigma_x = \sqrt{\sigma_{xz}^2 + \sigma_{xy}^2}$, we find

$$\left(\frac{s}{\bar{s}} - \frac{s_x}{\bar{s}_x}\right)\left(\frac{s}{\bar{s}} - \frac{s_y}{\bar{s}_y}\right) = \frac{\sigma_x^2}{\bar{\sigma}_x^2},$$

or

$$\frac{s}{\bar{s}} = \frac{1}{2}\left(\frac{s_x}{\bar{s}_x} + \frac{s_y}{\bar{s}_y}\right) \pm \sqrt{\frac{1}{4}\left(\frac{s_x}{\bar{s}_x} - \frac{s_y}{\bar{s}_y}\right)^2 + \left(\frac{\sigma_x}{\bar{\sigma}_x}\right)^2}.$$

In this equation we may put $\bar{s}_x = T_1/E_1$, $\bar{s}_y = T_2/E_2$, $\bar{\sigma}_x = S/\mu$, $\bar{s}_x/\bar{s}_y = \dfrac{E_2 T_1}{E_1 T_2} = \eta_1/\eta$ and $s_y = -\eta s_x$, where $\eta =$ ratio of lateral squeeze to longitudinal stretch. Thus we find as safe-stretch limit

$$\text{(i)} \ldots 1 = \text{maximum of } \frac{1-\eta_1}{2}\frac{E_1}{T_1}s_x + \sqrt{\left(\frac{1+\eta_1}{2}\frac{E_1}{T_1}s_x\right)^2 + \left(\frac{\mu\sigma_x}{S}\right)^2}.$$

We take the positive sign of the radical, because if $\sigma_x = 0$ we should have the alternative between $\frac{E_1}{T_1}s_x$ and $-\eta_1\frac{E_1}{T_1}s_x = \frac{E_2}{T_2}s_y$, and the former will be considered the greater (Saint-Venant, p. 484).

Case (2). A like equation is obtained, if, without supposing an axis of elasticity, two out of the three slide components vanish at a *fail-point*.

Case (3). This is a case of approximation, Saint-Venant supposes σ_{yz} to be zero; but $-s_y/\bar{s}_y$ and $-s_z/\bar{s}_z$ without being equal to differ but slightly, and he then takes them equal to $\eta_2\frac{s_x}{\bar{s}_x}$ '*une certaine moyenne entre ces deux rapports.*' Thus he replaces $(s/\bar{s} - s_y/\bar{s}_y)\ (s/\bar{s} - s_z/\bar{s}_z)$ by $(s/\bar{s} + \eta_2 s_x/\bar{s}_x)^2$ and divides out all the terms by the same factor. We thus reach the equation

$$\left(\frac{s}{\bar{s}} - \frac{s_x}{\bar{s}_x}\right)\left(\frac{s}{\bar{s}} + \eta_2\frac{s_x}{\bar{s}_x}\right) - \frac{\sigma_{xy}^2}{\bar{\sigma}_{xy}^2} - \frac{\sigma_{xz}^2}{\bar{\sigma}_{xz}^2} = 0,$$

and obtain for the safe-stretch condition

(ii) ... 1 = maximum of

$$\frac{1-\eta_2}{2}\frac{E_1}{T_1}s_x + \sqrt{\left(\frac{1+\eta_2}{2}\frac{E_1}{T_1}s_x\right)^2 + \left(\frac{\mu_1\sigma_{xy}}{S_1}\right)^2 + \left(\frac{\mu_2\sigma_{xz}}{S_2}\right)^2}.$$

Here η_2 is given, I think, most satisfactorily by the arithmetic mean

$$\tfrac{1}{2}\left(\frac{s_y}{\bar{s}_y} + \frac{s_z}{\bar{s}_z}\right) = -\eta_2\frac{s_x}{\bar{s}_x}.$$

Now if $s_y = -\eta s_x$, and $s_z = -\eta' s_x$,

$$\begin{aligned}\eta_2 &= \tfrac{1}{2}\left(\eta\frac{\bar{s}_x}{\bar{s}_y} + \eta'\frac{\bar{s}_x}{\bar{s}_z}\right)\\ &= 2\left(\eta\frac{1}{\bar{\sigma}_{xy}^2} + \eta'\frac{1}{\bar{\sigma}_{xz}^2}\right)\bar{s}_x^2 : \text{ see our Art. 5 } (d),\\ &= 2\left\{\eta\left(\frac{\mu_1}{S_1}\right)^2 + \eta'\left(\frac{\mu_2}{S_2}\right)^2\right\}\frac{T_1^2}{E_1^2}.\end{aligned}$$

This result gives a constant value for η_2 and appears to agree with Saint-Venant's note on *Clebsch*, p. 275. I do not think the value given for $\epsilon''_1 (= \text{our } \eta_2)$ on p. 485 of the memoir is quite satisfactory.

It will be noted that in all three cases the resulting quadratic is practically of the same form and the condition may for all three be thrown into a somewhat different shape, namely, transposing and squaring we find

$$\text{(iii)} \ldots \left(1 - \frac{E_1 s_x}{T_1}\right)\left(1 + \eta_2\frac{E_1}{T_1}s_x\right) - \left(\frac{\mu_1\sigma_{xy}}{S_1}\right)^2 - \left(\frac{\mu_2\sigma_{xz}}{S_2}\right)^2 = \text{or} > 0.$$

On p. 486 Saint-Venant gives the value of the moduli in terms of the 21 coefficients, and points out the changes which arise when we assume bi- or uni-constant isotropy. On pp. 487—8 we have a direct deduction of the formula of Case (2) on the lines of the *Cours lithographié*: see our Art. 1571*.

[54.] On pp. 488—491 Saint-Venant points out the method by which a general solution for a prism can be worked out. Let the axes of z and y be the principal axes of inertia of the cross-section and P_x, P_y, P_z the load-components parallel to the axes at one terminal and M_x, M_y, M_z the moments round the corresponding axes. Let σ'_{xy}, σ'_{xz} be the slides at any point on the section ω due to the flexure or to M_y, M_z; let σ''_{xy}, σ''_{xz} be the slide-components due to the torsional couple M_x, then

$$s_x = \frac{P_x}{E_1 \omega} + \frac{M_y z}{E_1 \omega \kappa_y^2} + \frac{M_z y}{E_1 \omega \kappa_z^2}.$$

Further the σ' and σ'' components of slide will be known as soon as the section is known and their *sums* must pair and pair be substituted in equation (ii) or (iii) of Art. 53 for σ_{xy} and σ_{xz}.

The equations of equilibrium,

$$\text{(iv)} \quad \left.\begin{aligned} P_y &= \mu_1 \int_0^\omega \sigma'_{xy}\, d\omega \\ P_z &= \mu_2 \int_0^\omega \sigma'_{xz}\, d\omega \end{aligned}\right\}, \qquad M_x = \mu_2 \int_0^\omega \sigma''_{xz}\, y\, d\omega - \mu_1 \int_0^\omega \sigma''_{xy}\, z\, d\omega,$$

will determine the constants in terms of the applied forces.

[55.] In section 125 (pp. 492—4) Saint-Venant treats the exceptional case of a cross-section constrained to remain plane.

Telles sont celles qui sont soumises à ce que M. Vicat appelle un *encastrement complet*, c'est-à-dire qui ne sont pas seulement contenues, mais *scellées* ou *soudées* avec une matière plus rigide; ou bien celles qui se trouvent serrées et sollicitées latéralement *dans leur plan même* par des forces tendant à trancher, comme il arrive aux sections des rivets dans le plan de contact des tôles qu'ils assemblent, ou aux bases des prismes tordus *de longueur nulle* comme dit le même illustre ingénieur (p. 492).

Other such sections occur from the symmetry of load distribution etc.

For such non-distorted sections, we can suppose the 'fibres' formerly perpendicular to become equally inclined, or the slide due to flexure constant, and that due to torsion to follow the old law of Coulomb, i.e.

$$\text{(v)} \quad \begin{cases} \mu_1 \sigma'_{xy} = P_y/\omega, & \mu_2 \sigma'_{xz} = P_z/\omega, \\ \sigma''_{xy} = -\tau z, & \sigma''_{xz} = \tau y, \end{cases}$$

whence by means of equation (iv) of our Art. 54, we can easily express the slides in terms of M_x, P_y and P_z.

The expressions (v) of course are only true for these exceptional sections, which can never occur in pure torsion as sections of danger, while in practical cases of flexure combined with torsion or slide they are frequently found to be specially strengthened (e.g. built-in ends).

[56.] We will now enumerate the examples Saint-Venant gives of the above condition of safety.

Case (1). Consider a rectangular prism (cross-section $2b \times 2c$) subjected only to a force P parallel to the axis of z (or side $2c$). Let the built-in terminal of the prism be so fixed that it can be distorted by flexure. Then if the length of the prism be a, and $2c$ be much greater than $2b$, we have

$$\mu_2 \sigma_{xz} = \frac{3}{2}\frac{P}{\omega}\left(1 - \frac{z^2}{c^2}\right), \qquad P_x = P_y = 0,$$

$$\sigma_{xy} = 0, \qquad Es_x = Paz \Big/ \frac{\omega c^2}{3};$$

so that, *granting uniconstant isotropy*, $S = \frac{4}{5}T$, $\eta = \frac{1}{4}$, and thus the equation (i) of our Art. 53 becomes

$$1 = \text{maximum of } \frac{3Pa}{T\omega c}\left[\frac{3}{8}\frac{z}{c} + \frac{5}{8}\sqrt{\frac{z^2}{c^2} + \left(\frac{c}{a}\right)^2\left(1 - \frac{z^2}{c^2}\right)^2}\,\right].$$

Saint-Venant gives a table of the values of the quantity between square brackets for values of $z/c = 0$ to 1, and for values of $2c/a$ (depth to length) from 3 to 6. From this table the following results may be drawn. So long as $2c/a < 3{\cdot}05$ the *fail-point* lies on the surface of the prism where $z/c = 1$, or at that point *where there is no slide.* If then the ratio of depth to length be $< 3{\cdot}05$, the prism's resistance is just that of flexure without consideration of slide. If on the other hand $2c/a > 3{\cdot}05$ the maximum passes abruptly to the points for which $z/c = {\cdot}2$ about, and approaches more and more to those for which $z = 0$. But this latter point lies on the *neutral-axis*, or it must be *slide* and not *flexure* which produces the failure. When $2c/a = 3{\cdot}2$ we may calculate the resistance either from flexure or transverse slide, but after $2c/a = 4$, it is the slide alone which is of importance. Similar conclusions Saint-Venant tells us may be obtained for a circular section (radius r); in this case the fail-point passes abruptly when $2r/a = 4{\cdot}3$ from $z = r$ to $z = {\cdot}2r$ about.

The reader who bears in mind Vicat's attack upon the mathematical theory of elasticity (see our Arts. 732*—733*) will find that the above remarks satisfactorily explain Vicat's experimental results.

Case (2). This is that of a prism (length $2a$, section $2b \times 2c$) terminally supported and centrally loaded. Here the section of greatest strain suffers no distortion. If the load P be in the direction of the axis of z, we have by equation (v) of our Art. 55, $\sigma_{xy}=0$ and $\sigma_{xz}=P/(\omega\mu)$. Whence supposing uni-constant isotropy we find:

$$1=\frac{3}{8}\frac{3Pa}{4Tbc^2}+\sqrt{\left(\frac{5}{8}\cdot\frac{3Pa}{4Tbc^2}\right)^2+\left(\frac{P}{4Sbc}\right)^2}.$$

Suppose b' and c' to be the values to be given to b and c that the prism might safely withstand a couple Pa producing flexure only, and b'', c'' to be the values to be given to b and c that it might safely withstand a shearing force P applied to the undistorted section. Then we easily find

$$1=\frac{3Pa}{4Tb'c'^2}, \text{ and } 1=\frac{P}{4Sb''c''}.$$

Hence:

$$1=\frac{3}{8}\frac{b'}{b}\left(\frac{c'}{c}\right)^2+\sqrt{\left(\frac{5}{8}\frac{b'c'^2}{bc^2}\right)^2+\left(\frac{b''c''}{bc}\right)^2}$$

gives the limiting safe values of b and c for the strain in question. Saint-Venant puts first $c'=c''=c$ and so gets

$$b=\tfrac{3}{8}b'+\sqrt{(\tfrac{5}{8}b')^2+b''^2},$$

whence he deduces and tabulates the values of b/b' and b/b'' for various values of b''/b' and b'/b'' respectively, and also the value of

$$\frac{2c}{2a}\left(=\frac{3S}{T}\frac{b''}{b'}, \text{ or}=\frac{12}{5}\frac{b''}{b'} \text{ for isotropy}\right).$$

From his table it appears that when

$\dfrac{2c}{2a}=\dfrac{\text{depth}}{\text{length}}=$ or $>\frac{1}{2}$ the slide begins to influence sensibly the result,

$\dfrac{2c}{2a}=$ or <10 the flexure begins to influence sensibly the result.

Between $2c/2a=\frac{1}{2}$ and 10 we are compelled to take both into account.

Case (3). This is the treatment of a cylinder on a circular base subjected at the same time to flexure, torsion and extension. Saint-Venant neglects the flexural slides and ultimately the extension. He obtains an equation similar in character to that of the preceding case and tabulates the values of the radius of safety in terms of the radius of safety in the case of flexure alone for different values of the elastic constant η_1. He remarks (p. 503) that it is not necessary to consider values of $\eta_1>\frac{1}{2}$ for then a stretch would not produce a positive dilatation, '*ce qui n'est point supposable.*' This remark is omitted in the *Leçons de Navier* where a number of values of $\eta_1>\frac{1}{2}$ are dealt with. I may add that the problem is far more completely treated in that work (pp. 414—21). Saint-Venant's tables shew that the results obtained are for values of η_1 between 1/5 and 1/3 very much the same, or we may adopt generally without fear of error the uni-constant hypothesis $\eta_1=1/4$. This hypothesis Saint-Venant tells us is amply verified by the experiments of M. Gouin (see page 486 of the memoir).

I shall have something to say of these experiments when dealing with Morin's *Résistance des matériaux*, 1853: see our Chap. XI.

Case (4). This case gives the calculation of the 'solid of equal resistance' for a bar built-in at one end and acted upon at the other by a non-central load perpendicular to its axis, i.e. combined flexure and torsion. Saint-Venant supposes uni-constant isotropy and neglects the flexural slides. His final equation is

$$2\frac{T\pi r^3}{P} = 3x + 5\sqrt{x^2+k^2}.$$

Here P is the load acting on an arm k, and r is the sectional radius at distance x from the loaded terminal. (p. 504.)

Case (5). An axle terminally supported has weight Π and carries two heavy wheels (ϖ and ϖ') upon which act forces, whose moments about the axle are equal and whose directions are perpendicular to the axle. We have thus another case of combined flexure and torsion, which is dealt with as before.

[57.] The next case treated by Saint-Venant is of greater complexity; it occupies pp. 507—18 of the memoir. It is the investigation of combined flexural and torsional strain in rectangular prisms ($2b \times 2c$), and possesses considerable theoretical interest. In practice also the non-central loading of beams of rectangular section must be a not infrequent occurrence.

Case (6). Saint-Venant in his treatment does not suppose the elasticity round the prismatic axis to be isotropic, but takes the general case of two slide-moduli, supposing, however, that $b\sqrt{\mu_2} > c\sqrt{\mu_1}$.

He neglects also the flexural slide-components. Let the torsional slide-components be given by $\sigma_1 = -\gamma_y c\tau$ and $\sigma_2 = \gamma_z b\tau$ for $z/c = 1$ and $y/b = 1$ respectively. τ must be eliminated by means of the relation $M'' = \beta\mu_1\tau bc^3$. If ϕ be the angle the plane of the flexural load makes with the plane through the prismatic axis and the axis of y, and M' the flexural moment at section x, we easily obtain for the stretch s_x the value

$$\begin{aligned} s_x &= \frac{3M'}{4Ebc}\left(\frac{z\cos\phi}{c^2} + \frac{y\sin\phi}{b^2}\right) \\ &= (\text{for } z=c)\ \frac{3M'}{4bc^2E}\left(\cos\phi + \frac{c}{b}\frac{y}{b}\sin\phi\right) \\ &= (\text{for } y=b)\ \frac{3M'}{4bc^2E}\left(\frac{z}{c}\cos\phi + \frac{c}{b}\sin\phi\right). \end{aligned}$$

Let us substitute these values in equation (ii) of our Art. 53. Taking these expressions alternately for the sides $2b$ and $2c$ we obtain:

$$\begin{aligned} 1 = \text{maximum}\ &\frac{1-\eta_2}{2T}\frac{3M'}{4bc^2}\left(\cos\phi + \frac{c}{b}\frac{y}{b}\sin\phi\right) \\ &+ \sqrt{\left[\frac{1+\eta_2}{2T}\frac{3M'}{4bc^2}\left(\cos\phi + \frac{c}{b}\frac{y}{b}\sin\phi\right)\right]^2 + \left(\frac{\gamma_y}{S_1}\frac{M''}{\beta bc^2}\right)^2}, \end{aligned}$$

$$1 = \text{maximum}\ \frac{1-\eta_2}{2T}\frac{3M'}{4bc^2}\left(\frac{z}{c}\cos\phi + \frac{c}{b}\sin\phi\right)$$
$$+\sqrt{\left[\frac{1+\eta_2}{2T}\frac{3M'}{4bc^2}\left(\frac{z}{c}\cos\phi+\frac{c}{b}\sin\phi\right)\right]^2 + \left(\frac{\gamma_z}{S_2}\frac{b\mu_2}{c\mu_1}\frac{M''}{\beta bc^2}\right)^2}.$$

By means of the Table II. below and Table I. on our p. 39 all the terms of these expressions can be calculated; for γ_y/γ_1 and γ_z/γ_2 are given for values of $\dfrac{b\sqrt{\mu_2}}{c\sqrt{\mu_1}}$ and also for values of y/b and z/c respectively. Hence so soon as ϕ and the section of danger, i.e. where M' is greatest, are known we can solve the problem by equating to unity the greater of the two maxima written down above and so determine bc^2 for the section.

Saint-Venant by using b', c', b'', c'' with similar meanings to those of our Art. 56, *Case* (2), throws the equation into a somewhat different form.

If the section for which M' is greatest be so built-in or symmetrically situated that no distortion is possible the values of the slides must be those of equations (v) of our Art. 55 and not σ_1, σ_2 as taken above.

TABLE II.

Slides at points of the contour of the Cross-Section of a Prism on rectangular base subjected to Torsion.

$\sigma_1 = -\gamma_y c\tau$ (for $z=c$, or along the sides $2b$)					$\sigma_2=\gamma_z b\tau$ (for $y=b$, or along the sides $2c$)			
	Value of ratio γ_y/γ_1					Value of ratio γ_z/γ_2		
For y/b	$\frac{b\sqrt{\mu_2}}{c\sqrt{\mu_1}}=1$	$=1{\cdot}5$	$=2$	$=4$	For z/c	$\frac{b\sqrt{\mu_2}}{c\sqrt{\mu_1}}=1$	$=2$	$=4$
0	1·0000	1·0000	1·0000	1·0000	0	1·0000	1·0000	1·0000
·1	·9932	·9949	·9962	·9991	·1	·9932	·9932	·9933
·2	·9750	·9795	·9846	·9973	·2	·9750	·9729	·9729
·3	·9429	·9526	·9639	·9928	·3	·9429	·9384	·9383
·4	·8963	·9127	·9321	·9842	·4	·8963	·8887	·8885
·5	·8333	·8572	·8857	·9678	·5	·8333	·8224	·8220
·6	·7510	·7820	·8196	·9371	·6	·7510	·7369	·7363
·7	·6447	·6811	·7260	·8793	·7	·6447	·6282	·6278
·8	·5063	·5441	·5916	·7695	·8	·5063	·4892	·4885
·9	·3185	·3497	·3896	·5540	·9	·3185	·3044	·3040
1	·0000	·0000	·0000	·0000	1·0	·0000	·0000	·0000

This Table gives γ_y, γ_z in terms of the principal slides γ_1, γ_2 at the centre of the corresponding sides $2b$ and $2c$; the values of γ_1, γ_2 are given in Table I. p. 39.

[58.] Saint-Venant treats with numerical tables the following special cases:

(1) $\phi = 0$ and $c < b$ (pp. 511—2).

(2) c so much less than b that $c/b.\tan\phi$ may be neglected as compared with 1, i.e. the case of a 'plate' (pp. 511—2).

(3) Prism on square base, when $\tan\phi = 0$, $=\frac{1}{2}$, $=1$, and = anything whatever when there is a non-distorted section for section of least safety (pp. 512—4). The fail-points are also determined.

(4) Prism on rectangular base for which $b = 2c$, when $\tan\phi = 0$, $=\frac{1}{2}$, $=1$, $=2$, $=\infty$, and = anything whatever when there is a non-distorted section for that of least safety (pp. 514—518). The fail-points are also determined.

[59.] On pp. 518—22 we have the treatment of a prism on elliptic base subjected at the same time to flexure and torsion. Saint-Venant only works this out numerically for the case of uni-constant isotropy and when $\tan\phi = \infty$.

It is found that after a certain value of the ratio of torsional to flexural couple, the fail-point leaves the end of the major axis (through which the flexural load-plane passes[1]) and traverses the quadrant of the ellipse till it reaches the end of the minor axis (p. 522).

[60.] We now turn to Saint-Venant's final chapter (pp. 522—558). This consists of three parts: § 135 *Résumé général*; § 136 *Récapitulation des formules et règles pratiques* and § 137 *Exemples d'applications numériques.*

In the first article there is little to be noted. A reference is made on p. 528 to the models of M. Bardin shewing the *gauchissement* of the cross-section to which we have previously referred. Saint-Venant also mentions the visible distortion of the cross-sections obtained by marking them on a prism of caoutchouc and then subjecting it to torsion.

In the general recapitulation of formulae we have some results not in the body of the memoir, as on p. 536 (d_1) where the flexural slides for the prism whose base is the curve $\left(\frac{y}{b}\right)^4 + \left(\frac{z}{c}\right)^2 = 1$ are cited from the memoir on flexure: see our Art. 90. So again on p. 546 for the flexural slides of other cross-sections. The best *résumé*, however, of formulae as well as numbers for both flexure and torsion is undoubtedly to be found in Saint-Venant's *Leçons de Navier* to which we shall refer later. The last section § 137 contains some instructive numerical examples of Saint-Venant's treatment of combined strain.

[1] Saint-Venant terms this *sollicité de champ*. When the load-plane is perpendicular to this the prism is *sollicité à plat*.

The memoir concludes with the tables for rectangular prisms which we have in part reproduced on pp. 39 and 49.

[61.] We here bring to a close our review of this great memoir. Since Poisson's fundamental essay of 1828 (see our Art. 434*) no other single memoir has really been so epoch-making in the science of elasticity. It is indeed not a memoir, but a classical treatise on those branches of elasticity which are of first-class technical importance. Written by an engineer who has kept ever before him practical needs, it is none the less replete with investigations and methods of the greatest theoretical interest. Many of its suggestions we shall find have been worked out in fuller detail by Saint-Venant himself, not a few remain to this day unexhausted mines demanding further research.

Section II.

Memoirs of 1854 *to* 1864.

Flexure, Distribution of Elasticity, etc.

[62.] *Comptes rendus*, T. XXXIX. pp. 1027—1031, 1854. *Mémoire sur la flexion des prismes élastiques, sur les glissements qui l'accompagnent lorsqu'elle ne s'opère pas uniformément ou en arc de cercle, et sur la forme courbe affectée alors par leurs sections transversales primitivement planes.* This is a *résumé* of the results of the later memoir on flexure (see our Arts. 69 and 93). It cites the general equations for flexure, and the particular results for the case of a rectangular cross-section.

[63.] *L'Institut*, Vol. 22, 1854, pp. 61—63. *Solution du problème du choc transversal et de la résistance vive des barres élastiques appuyées aux extrémités.* This is an account of Saint-Venant's memoir presented to the *Société Philomathique*. It contains only matter given in the *Comptes rendus*, and afterwards more completely in the annotated *Clebsch*: see our Art. 104.

[64.] In the same volume of the same Journal, pp. 220—1, are particulars of the memoir on the Flexure of Prisms communicated to the *Société Philomathique.*

[65.] In the same volume of this Journal, pp. 396—398, is another communication of Saint-Venant's to the *Société Philomathique* (July 8, 1854). This deals with the formulae for the flexure of prisms and for their strength, when the cross-section does not possess inertial isotropy. It gives the general equations and treats specially the case of a rectangular cross-section: see the *Leçons de Navier*, pp. 52—58 and our Arts. 1581*, 14 and 171.

A final paragraph to the paper points out that the resistance to torsion varies more nearly inversely than directly as the axial moment of inertia: see our Art. 290.

[66.] On pp. 428—31 of the same volume of the same Journal Saint-Venant communicates to the *Société Philomathique* (July 8 and October 21, 1854) the results obtained from the stretch-condition of strength. These results were afterwards published in the memoir on Torsion: see our Arts. 53 *et seq.*

[67.] Volume 23 of the same Journal, pp. 248—50. Further results of the memoir on *Torsion* communicated to the *Société Philomathique* (April 12 and May 12, 1855), notably the case of a prism on an equal-sided triangular base: see our Arts 40—2.

[68.] The same volume of the same Journal, pp. 440—442. *Diverses considérations sur l'élasticité des corps, sur les actions entre leurs molécules, sur leurs mouvements vibratoires atomiques, et sur leur dilatation par la chaleur.* An account of a memoir presented October 20, 1855, to the *Société Philomathique* containing general remarks on the rari-constant theory of intermolecular action. The expression for the velocity of sound on p. 441 *b* should be $\sqrt{\dfrac{3G-p}{\rho}}$ and not $\sqrt{\dfrac{3G+p}{\rho}}$: see *L'Institut*, Vol. 24, p. 215. Saint-Venant refers to the labours of Newton, Ampère and others on this subject: see our Art. 102. He points out that in order to explain heat by translational vibrations, the second differential of the function which expresses the law of intermolecular force must be *positive*: see our Arts. 268 and 273.

The method, however, of dealing with the velocity of sound by means of an initial stress in an isotropic medium is unsatisfactory. This was recognised by Saint-Venant himself, and he cancelled the entire paragraphs on p. 441, beginning *Newton va même* and *Quelque différents,* of 42 and 10 lines respectively: see *Comptes rendus,* 1876, Vol. 82, p. 34.

[69.] *Mémoire sur la flexion des prismes, sur les glissements transversaux et longitudinaux qui l'accompagnent lorsqu'elle ne s'opère pas uniformément ou en arc de cercle, et sur la forme courbe affectée alors par leurs sections transversales primitivement planes. Journal de Mathématiques de Liouville,* Deuxième Série, T. I. 1856, pp. 89—189.

This is Saint-Venant's classical memoir on flexure; extracts from it will be found in the *Comptes rendus,* T. XXXIX. 1854, p. 1027 and T. XLI. 1855, p. 143.

Certain portions are reproduced in the *Leçons de Navier,* pp. 389—414, but the analytical work does not seem yet to have passed into the text-books.

[70.] Sections 1, 2 (pp. 89—98) are occupied with a history of the old theories and an account of the Bernoulli-Eulerian hypothesis as generally accepted at the date of the memoir. Saint-Venant refers to the labours of Galilei (see our Art. 3*), Mariotte (Art. 10*), Hooke (Art. 7*), James Bernoulli (Art. 18*), Coulomb (Art. 117*), Leibniz (Art. 11*), Duleau (Art. 227*), Barlow (Art. 189*), Hodgkinson (Art. 232*), Tredgold (Art. 197*), Girard (Art. 127*), Navier (Art. 254*), Young (Art. 134*), Robison (Art. 146*), Dupin (Art. 162*) for the theory of beams, and to those of Cauchy, Poisson, Lamé and Clapeyron for the general theory of elasticity. His remarks are reproduced at greater length in the *Historique Abrégé,* and as the reader of our first volume is already acquainted with the researches of these scientists we pass over these pages of the memoir.

In the second section Saint-Venant points out the falseness of the Bernoulli-Eulerian theory, and refers to the corrections and criticisms of Vicat, Persy and himself: see our Arts. 721*, 726*, 811* and 1571*.

As we have already pointed out Saint-Venant in the memoir on *Torsion* had given the outlines of the true theory of flexure: see our Arts. 9—13.

[71.] The third section (pp. 98—101) is entitled: *Objet et sommaire de ce memoire.* Saint-Venant here indicates that he intends to use the semi-inverse method (see our Art. 3) to test how far the Bernoulli-Eulerian formulae:

$$\begin{cases} \text{Traction} = Ez/\rho, \\ \text{Bending moment} = E\omega\kappa^2/\rho, \\ \int z d\omega = 0, \end{cases}$$

(see our Arts. 20*, 65*, 75*, etc.)

are correct, when consideration is paid to the influence of slide. There is also a succinct account of the contents of Sections 4—32 of the memoir.

[72.] Sections 4—12 (pp. 101—120) contain an elementary sketch of the general theory of elasticity. Saint-Venant wrote three other such sketches, namely (i) in the memoir on *Torsion* (see our Art. 4); (ii) in the *Leçons de Navier* (see our Art. 190); and (iii) for Moigno's *Statique* (see our Arts. 224—9). This sketch falls between (i) and (ii). It adopts rari-constancy and bases it upon intermolecular action being central and a function of central distance only. This rari-constancy Saint-Venant holds to be without doubt true for bodies of 'confused crystallisation' such as are used for the materials of construction (p. 108). At the same time for the sake of the 'weaker brethren,' and as it does not increase the difficulty of solving the elastic equations, he adopts multi-constant formulae.

[73.] As a specimen of the mode of treatment, we reproduce his proof of the equality of the cross-stretch and direct-slide coefficients, i.e. in our notation $|xxyy| = |xyxy|$[1].

We have to shew that the coefficient of s_y in $\widehat{xx}$ = the coefficient of σ_{xy} in $\widehat{xy}$.

Suppose all the strain-components zero except s_y and σ_{xy} and these to be constant for all points of the body. Suppose the central distance of two molecules m', m'' to have length r, and projections x, y, z on the coordinate axes before strain. After strain x and z remain unchanged, but y will be increased by ys_y owing to the stretch and $x\sigma_{xy}$ owing to

[1] See the footnote to our Art. 116.

the slide. Thus the distance r between the molecules will be increased by the quantity

$$\delta r = (y s_y + x\sigma_{xy}) \frac{y}{r}.$$

A mutual action

$$f(r) \cdot (y s_y + x\sigma_{xy}) \frac{y}{r}$$

will thus be developed between the molecules by the displacement, where $f(r)$ is some function of r.

If these molecules m', m'' form part of two groups situated at either side of an elementary area ω taken perpendicular to the axis of x, we shall have $\omega \,.\, \widehat{xx}$ and $\omega \,.\, \widehat{xy}$ for the stresses obtained by resolving such mutual actions as the above along the axes of x and y respectively and summing them for all actions which cross the area ω. (See our Art. 1563*.)

Thus we have

$$\omega \,.\, \widehat{xx} = \Sigma f(r)\,(y s_y + x\sigma_{xy}) \frac{y}{r} \cdot \frac{x}{r},$$

$$\omega \,.\, \widehat{xy} = \Sigma f(r)\,(y s_y + x\sigma_{xy}) \frac{y}{r} \cdot \frac{y}{r},$$

or,

$$\widehat{xx} = \frac{s_y}{\omega} \Sigma \frac{f(r)}{r^2} xy^2 + \frac{\sigma_{xy}}{\omega} \Sigma \frac{f(r)}{r^2} x^2 y,$$

$$\widehat{xy} = \frac{s_y}{\omega} \Sigma \frac{f(r)}{r^2} y^3 + \frac{\sigma_{xy}}{\omega} \Sigma \frac{f(r)}{r^2} xy^2.$$

The form of these expressions thus proves the identity of the cross-stretch and direct-slide coefficients on the rari-constant hypothesis.

[74.] In Section 12 (pp. 117—120) Saint-Venant applies the general formulae of elasticity to the simple case of a prism under pure traction. He then deduces the stretch-modulus in terms of the elastic constants for various kinds of elastic bodies.

In a footnote to p. 120 he supposes the body to have weight and to be vertically stretched. He obtains with the notation of our Art. 1070* the following results:

$$\left.\begin{matrix} u \\ v \end{matrix}\right\} = -\begin{Bmatrix} \eta \\ \eta' \end{Bmatrix} \frac{1}{E}\left(\frac{F'}{\omega} - \frac{Wz}{\omega l}\right)\begin{Bmatrix} x \\ y \end{Bmatrix},$$

$$w = \frac{1}{E}\left(\frac{F'z}{\omega} - \frac{W}{\omega}\frac{z^2}{2l} - \frac{W}{\omega}\frac{\eta y^2 + \eta' z^2}{2l}\right).$$

These results agree with those of our Art. 1070*, if we take $\eta = \eta'$, or suppose isotropy in the cross-section. Here η, η' are the stretch-squeeze ratios in the directions z, x and z, y respectively.

I had not noticed this footnote when commenting in the first volume on Lamé's treatment of the problem.

[75.] Section 13 (pp. 121—123) deals with Poisson and Cauchy's method of treating the problem of flexure by expanding the stresses as positive integral algebraic functions of the co-ordinates of the point on the cross-section referred to axes in the cross-section: see our Arts. 466* and 618* (footnote). This method Saint-Venant admits had served for the departure of his own researches (p. 99), and he deals more gently with it here (p. 124) than he does in his later work. The assumption of the possibility of the expansion in a convergent series is a very dangerous one, and leads in the case of torsion to very erroneous results: see our Arts. 1626* and 191 (or *Leçons de Navier*, footnote pp. 621—7).

[76.] In §§ 14—17 (pp. 125—36) Saint-Venant gives the general solution of the problem of flexure, carefully stating his assumptions and once integrating his equations. He reduces the solution to the determination of a single function F, which can be chosen to suit a great variety of cross-sections. I will reproduce as briefly as possible the matter of these sections.

[77.] Taking a portion of a weightless prism between two cross-sections Saint-Venant proposes to determine its state of equilibrium after it has been subjected to flexure on the following suppositions:

(i) The character of a certain portion of the shifts and strains is assumed; namely, the axis of the prism, or the right line joining the centroids of the cross-sections, is supposed to become a plane curve (*elastic line* here one with the *neutral line*), and further the stretches of the longitudinal 'fibres' vary in a uniform manner with their distances from each other measured parallel to the plane of the elastic line.

Let x be the direction of the line of centroids before flexure and let the origin be its fixed extremity (see (iii)), and let xz be the plane of flexure (or of the elastic line), then the above condition is analytically represented by

$$s_x = Cz + C' \qquad\qquad (1),$$

where C and C' are constants for the cross-section.

(ii) The character of a certain portion of the stresses is assumed; namely, it is supposed that the fibres exercise no mutual *traction* upon each other, or that their mutual action is solely of the nature of shear. Further, on the terminal cross-sections there is supposed to be no tractive loading.

These assumptions may be expressed analytically by

$$\widehat{yy} = \widehat{zz} = \widehat{yz} = 0 \text{ ....} (2),$$

$$\int \widehat{xx}\, d\omega = 0 \text{ for a terminal cross-section................}(3).$$

Further, it is supposed that although the mode of application and distribution of the load is unknown, yet the resultant load and its moment (M) for each cross-section ω at distance x from the origin are known.

It follows that

$$M = \int \widehat{xx}\, z d\omega \text{ for each section ....} (4).$$

Further, to simplify the equations of unnecessary elements all motion of rotation, or translation of the prism as a whole, all stretching of the central axis or torsion of the prism are excluded. The latter elements by the principle of superposition of strains can afterwards be added.

(iii) One extremity of the central axis, the central elementary area of the cross-section at that extremity and an elementary strip along the trace of the plane of flexure on the cross-section remain fixed.

Analytically this gives us the conditions:

$$u = v = w = 0,\ du/dz = 0, \text{ when } x = y = z = 0 \text{............}(5),$$

$$v = 0,\ dv/dz = 0, \text{ when } y = z = 0 \text{ for all values of } x \text{.........}(6).$$

[78.] Let us adopt the following additional notation: l, $\omega\kappa^2$ and ρ are the length, cross-sectional moment of inertia $(= \int z^2 d\omega)$ and radius of curvature at any point of elastic line of the prism. Let us further suppose that the material is such that the cross-sections of the prism are planes of elastic symmetry, it follows easily that the stress-strain relations will be of the form

$$\left.\begin{aligned}
\widehat{xx} &= a s_x + f' s_y + e'' s_z + h\sigma_{yz} \\
\widehat{yy} &= f'' s_x + b s_y + d' s_z + k\sigma_{yz} \\
\widehat{zz} &= e' s_x + d'' s_y + c s_z + n\sigma_{yz} \\
\widehat{yz} &= h' s_x + k' s_y + n' s_z + d\sigma_{yz} \\
\widehat{zx} &= e\sigma_{zx} + h''\sigma_{xy} \\
\widehat{xy} &= h'''\sigma_{zx} + f\sigma_{xy}
\end{aligned}\right\} \text{ ....} (7).$$

See the annotated *Clebsch*, pp. 75, 6.

Since $\widehat{yy} = \widehat{zz} = \widehat{yz} = 0$ we can determine from the first four equations $\widehat{xx}$, s_y, s_z and σ_{yz} in terms of s_x, we may thus write:

$$\widehat{xx} = E s_x, \quad s_y = -\eta_1 s_x, \quad s_z = -\eta_2 s_x, \quad \sigma_{yz} = \epsilon s_x \text{............} (8).$$

[79.] Considering the portion of the prism between the cross-section ω at distance x and the cross-section at the origin we have by (3) and (4):—

$$0 = \int \widehat{xx}\, d\omega = \int E\,(Cz + C')\, d\omega,$$

$$M = \int \widehat{xx}\, z d\omega = \int E\,(Cz + C')\, z d\omega,$$

whence $$C' = 0 \text{ and } C = M/E\omega\kappa^2 \text{........................}(9).$$

It follows that $$\widehat{xx} = zM/\omega\kappa^2 \text{..............................}(10).$$

If we now turn to the body stress-equations we find they reduce to

$$\left.\begin{aligned} \frac{d\widehat{xy}}{dy} + \frac{d\widehat{xz}}{dz} &= -\frac{z}{\omega\kappa^2}\frac{dM}{dx} \\ \frac{d\widehat{xy}}{dx} = 0, \quad \frac{d\widehat{xz}}{dx} &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(11),$$

while the surface stress-equations reduce to the single one

$$\widehat{xz}\,dy - \widehat{xy}\,dz = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

The last two equations of (11) lead us by means of the last two of (7) to the conditions[1]

$$\frac{d\sigma_{xy}}{dx} = 0, \quad \frac{d\sigma_{zx}}{dx} = 0,$$

or, to

$$\frac{d^2u}{dxdy} + \frac{d^2v}{dx^2} = 0, \quad \frac{d^2u}{dxdz} + \frac{d^2w}{dx^2} = 0.$$

Hence, putting for $du/dx = s_x$ its value $zM/E\omega\kappa^2$, we have, *since M is supposed a function of x only*,

$$\frac{d^2v}{dx^2} = 0, \quad -\frac{d^2w}{dx^2} = M/E\omega\kappa^2 \ldots\ldots\ldots\ldots\ldots\ldots(13).$$

The first equation tells us that there is no curvature in the direction of y after flexure, the second that the curvature $\left(1/\rho = -\frac{d^2w}{dx^2}\right.$ for small shifts$\left.\right)$ in direction of z is equal to $M/E\omega\kappa^2$.

We thus obtain

$$M = E\omega\kappa^2/\rho, \quad s_x = z/\rho, \quad \widehat{xx} = Ez/\rho \ldots\ldots\ldots\ldots\ldots(14),$$

the formulae of the Bernoulli-Eulerian theory, here deduced without its invalid assumptions (i.e. that the cross-sections remain plane and normal to the strained fibres).

[80.] The first equation of (11) shews that if M is variable or in other words the curvature changes, the stresses $\widehat{xy}$, $\widehat{xz}$ and therefore the slides σ_{xy}, σ_{xz} cannot be zero, or it involves the contradiction of the Bernoulli-Eulerian assumptions.

Further differentiating the same equation with regard to x, we deduce by the second and third equations of (11) the result

$$\frac{d^2M}{dx^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15),$$

or M must be of the linear form in x,

$$= P(a - x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

[1] Provided the relation $e/h'' = h''/f$ does not hold between the elastic constants.

if we suppose Pa to be the value of M when $x=0$. In many cases $a=l$, the length of the prism.

This result (obtained on p. 130 of the memoir) is extremely important, and does not seem to me to have been sufficiently regarded. I remark that it is obtained without any consideration of the surface condition (12). *It thus follows that the assumptions* $s_x = Cz + C'$, $\widehat{zz} = \widehat{yy} = \widehat{yz} = 0$ *are not legitimate, if* M *is other than a linear function of the length of the prism. In other words all the important practical cases of continuous loading are excluded from Saint-Venant's theory of flexure, and it remains yet to be shewn that for such cases the Bernoulli-Eulerian hypothesis of* (14) *gives even an approximation to the truth.*

[81.] With regard to the quantity P of the previous Article, we obviously have $-P$ equal to the resultant, in the direction of z of the load, or to the total shear across each section, that is

$$\int_0^\omega \widehat{xz}\, d\omega = -P \text{.............................} (17)$$

for all sections.

Thus we see that Saint-Venant's theory, even without the limitation of equation (12), excludes the possibility of any discontinuous change in the shear, or the transverse load. He supposes the resultant of the whole external load to act either at the extreme section ($x=l$) or beyond it in the central axis produced. This again narrows down very much the number of practical cases for which the Bernoulli-Eulerian equations have been shewn to be applicable.

[82.] Saint-Venant now proceeds to a first integration of his equations and deduces the following results (p. 131):

$$\left.\begin{aligned} &u = P\frac{2ax - x^2}{2E\omega\kappa^2}z + F(y, z), \quad v = -P\frac{a-x}{2E\omega\kappa^2}(2\eta_1 yz - \epsilon z^2), \\ &w = \sigma_0 x + P\frac{a-x}{2E\omega\kappa^2}(\eta_1 y^2 - \eta_2 z^2) - P\frac{3ax^2 - x^3}{6E\omega\kappa^2} \end{aligned}\right\} \text{...(18)},$$

where σ_0 is a constant, representing the value of σ_{xz} at the origin, and $F(y, z)$ is a function to be determined by the conditions

$$\left.\begin{aligned} &F = 0, \quad dF/dz = 0, \text{ when } y = z = 0; \\ &f\frac{d^2F}{dy^2} + (h'' + h''')\frac{d^2F}{dydz} + e\frac{d^2F}{dz^2} \\ &\qquad\qquad = P\frac{E - \eta_1 f - \eta_2 e + \epsilon h''}{E\omega\kappa^2}z + \eta_1\frac{h''' - h''}{E\omega\kappa^2}Py, \\ &\text{for all points of the cross-section; and,} \\ &\left(h'' - f\frac{dz}{dy}\right)\left(\frac{dF}{dy} + P\frac{2\eta_1 yz - \epsilon z^2}{2E\omega\kappa^2}\right) + \left(e - h'''\frac{dz}{dy}\right) \\ &\qquad\qquad \times\left(\frac{dF}{dz} + \sigma_0 + P\frac{\eta_2 z^2 - \eta_1 y^2}{2E\omega\kappa^2}\right) = 0 \\ &\text{for all points of the contour of the cross-section.} \end{aligned}\right\} \text{...(19)}.$$

These results follow by simple analytical work if we start with the value of u obtained from the equation $s_x = z/\rho = P(a-x)z/E\omega\kappa^2$ and then proceed to those of v and w given by the second and third equations of (8), the values so found being made to satisfy (11), etc.

[83.] Saint-Venant, however, does not deal in his special examples with this general case of elastic distribution; he assumes the material to have planes of elastic symmetry perpendicular to y and z, as well as perpendicular to x. We then have $h'' = h''' = h = k = n = h' = k' = n' = 0$, and clearly $\epsilon = 0$.

Further,

$$\widehat{xy} = f\sigma_{xy}, \quad \widehat{xz} = e\sigma_{xz}, \quad \sigma_{yz} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(20).$$

The equations (18) and (19) now become, if we take[1]

$$f = \mu_1, \qquad e = \mu_2,$$

$$\frac{\eta_1 f}{E} = \gamma_1, \quad \frac{\eta_2 e}{E} = \gamma_2.$$

$$\left.\begin{aligned} u &= P\frac{2ax - x^2}{2E\omega\kappa^2}z + F(y,z), \qquad v = -\gamma_1 P\frac{(a-x)}{\mu_1\omega\kappa^2}yz, \\ w &= \sigma_0 x + \frac{P(a-x)}{2\omega\kappa^2}\left(\frac{\gamma_1 y^2}{\mu_1} - \frac{\gamma_2 z^2}{\mu_2}\right) - P\frac{3ax^2 - x^3}{6E\omega\kappa^2} \end{aligned}\right\} \ldots\ldots(18'),$$

$$\left.\begin{aligned} &\mu_1\frac{d^2F}{dy^2} + \mu_2\frac{d^2F}{dz^2} = P\frac{1-\gamma_1-\gamma_2}{\omega\kappa^2}z \text{ throughout the section;} \\ &F = 0, \quad dF/dz = 0, \text{ when } y = z = 0; \\ &-\mu_1\left\{\frac{dF}{dy} + \gamma_1\frac{Pyz}{\mu_1\omega\kappa^2}\right\}dz + \mu_2\left\{\frac{dF}{dz} + \sigma_0 + \frac{P}{2\omega\kappa^2}\left(\frac{\gamma_2 z^2}{\mu_2} - \frac{\gamma_1 y^2}{\mu_1}\right)\right\}dy = 0 \end{aligned}\right\} \ldots(19').$$

over the contour of the cross-section.

[84.] The last section of general treatment (pp. 133—6) gives formulae for various quantities used for the special cases afterwards dealt with. Thus we note:

First, the values of the stresses:

$$\left.\begin{aligned} \widehat{xx} &= P\frac{(a-x)}{\omega\kappa^2}z, \qquad \widehat{xy} = \mu_1\left(\frac{dF}{dy} + \gamma_1\frac{Pyz}{\mu_1\omega\kappa^2}\right), \\ \widehat{xz} &= \mu_2\left\{\frac{dF}{dz} + \sigma_0 + \frac{P}{2\omega\kappa^2}\left(\frac{\gamma_2 z^2}{\mu_2} + \frac{\gamma_1 y^2}{\mu_1}\right)\right\} \end{aligned}\right\} \ldots\ldots(20').$$

It follows that

$$\sigma_{xz} = \widehat{xz}/\mu_2, \text{ or } = \sigma_0 \text{ for } y = z = 0,$$

that is the inclinations of all the cross-sections at their centres to the axis is the same and equals σ_0.

[1] I have altered Saint-Venant's notation to correspond with that of our History, he puts for our $\begin{Bmatrix} \mu_1, & \mu_2, & \gamma_1, & \gamma_2, & \omega\kappa^2, & \eta_1, & \eta_2, & \epsilon, & \sigma_0 \\ G, & G', & \eta, & \eta', & I, & \epsilon, & \epsilon', & \epsilon'', & g_0 \end{Bmatrix}$.

Secondly, the equation to the curved surface taken by the cross-section, on neglecting small quantities of the second order, is shewn to be

$$x' = \sigma_0 z' + F(y', z') \quad\text{.........................(21)},$$

where the origin is the centroid of the cross-section, the axis of x' is the tangent there to the elastic line, that of y' is parallel to y and the plane $y'z'$ is the tangent plane to the cross-section at the origin.

It is obvious that x' is not a function of x, or the cross-sections all assume the same distorted form. Hence we see why it is that the different fibres are stretched precisely as they would be, were the cross-sections to remain plane.

Thirdly, the total deflection δ (*la flèche de flexion*) is obtained by putting $y = z = 0$ and $x = l$ in the value of w in (18′), or,

$$\delta = -\sigma_0 l + \frac{P(3al^2 - l^3)}{6E\omega\kappa^2} \quad\text{......................(22)}.$$

Saint-Venant assumes the resultant load to be applied at the terminal, or that $a = l$, thus still further limiting his solution. In this case

$$\delta = -\sigma_0 l + Pl^3/3E\omega\kappa^2 \quad\text{......................(22′)}.$$

[85.] The next twelve sections (18—29), pp. 136—68, deal with the determination of σ_0 and F for various forms of cross-section.

In the first place Saint-Venant assumes F is to be a positive integral algebraic function of y, z. In this case it must be of the form

$$F(y,z) = A_0 y + A\left(y^2 - \frac{\mu_1}{\mu_2}z^2\right) + A'yz + B'\left(y^2 z - \frac{\mu_1}{3\mu_2}z^3\right) + B''\left(yz^2 - \frac{\mu_2}{3\mu_1}y^3\right)$$
$$+ P\frac{1-\gamma_1-\gamma_2}{6\mu_2\omega\kappa^2}z^3 + C''\left(y^4 - 6\frac{\mu_1}{\mu_2}y^2z^2 + \frac{\mu_1^2}{\mu_2^2}z^4\right) + C'''\left(yz^3 - \frac{\mu_2}{\mu_1}y^3z\right) + \ldots \text{(23)},$$

in order to satisfy the first of equations (19′).

If this value be substituted in the third equation of (19′) we obtain the differential equation to the corresponding contour-curve.

[86.] Saint-Venant deals however only with the special case, in which the terms in y^2z and z^3 are alone retained. He puts

$$m = 1 - \gamma_1 - \frac{2\mu_1\omega\kappa^2}{P}B',$$

and thus throws F into the form

$$F(y, z) = P\frac{m-\gamma_2}{6\mu_2\omega\kappa^2}z^3 + P\frac{1-m-\gamma_1}{2\mu_1\omega\kappa^2}y^2z \quad\text{...........(24)}.$$

After some reductions and an integration he finds for the contour from the third equation of (19′):

$$Cy^{\frac{m}{1-m}} + \frac{\mu_2}{\mu_1}\,\frac{1-2\gamma_1-m}{3m-2}\,y^2 + z^2 = -\frac{2\mu_2\omega\kappa^2}{mP}\,\sigma_0 \quad\text{............(25)},$$

where C is a constant.

If $C=0$ this represents a family of ellipses. If C be finite and we give various values to m we have curves symmetrical with regard to the axis of y, and symmetrical or not with regard to the axis of z according as $m/(1-m)$ is even or odd. Equation (25) can be thrown into a somewhat different form by assuming c to be the semi-axis of the curve in the direction of $\pm z$, and b the semi-axis in the direction y. Thus $y=0$, $z=\pm c$, but for $z=0$, $y=+b$ always, $=-b$ also if $m/(1-m)$ be treated as even.

In putting $y=0$, $z^2=c^2$ we find,

$$\sigma_0 = -\frac{mPc^2}{2\mu_2\omega\kappa^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26).$$

Equation (25) now becomes:

$$\left(1 - \frac{1-2\gamma_1-m}{3m-2}\frac{\mu_2 b^2}{\mu_1 c^2}\right)\left(\frac{y}{b}\right)^{\frac{m}{1-m}} + \frac{1-2\gamma_1-m}{3m-2}\frac{\mu_2 b^2}{\mu_1 c^2}\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Saint-Venant now proceeds (pp. 138—143) to discuss the various forms that can be taken by this system of curves. This discussion seems to me perhaps a little too brief. Thus, he says: Supposing $m/(1-m)$ to be treated as even, then it is sufficient and necessary in order that the curve may be closed,—and so capable of serving as a contour for a cross-section,—that z/c have a real value when $y/b=1-\chi$, χ being an extremely small positive quantity. This leads him to the condition that m must lie between

$$\frac{1-2\gamma_1}{1+\mu_1 c^2/(\mu_2 b^2)} \text{ and } 1.$$

[87.] We may note the following cases:

The *ellipse* ($2b \times 2c$) is obtained (not by putting $m/(1-m)=2$ which leads to a logarithmic curve owing to the appearance of indeterminate forms, but) by making the coefficient of $y^{m/(1-m)}$ vanish. Thus we have

$$m = \frac{2\mu_1 c^2 + (1-2\gamma_1)\mu_2 b^2}{3\mu_1 c^2 + \mu_2 b^2}.$$

The *circle* (radius b) is obtained by putting $b=c$ or

$$m = \frac{2\mu_1 + (1-2\gamma_1)\mu_2}{3\mu_1 + \mu_2}.$$

The *false ellipse*, $\frac{y^4}{b^4} + \frac{z^2}{c^2} = 1$, is obtained by putting $m/(1-m)=4$ in the case of isotropic material for which uni-constancy holds, or

$$\mu_2 = \mu_1,\ E/\mu_1 = 5/2 \text{ and } \gamma_1 = 1/10.$$

More generally we must take $m = 2\gamma_1 - 1$, for a similar curve in bodies with tri-planar elastic symmetry.

[88.] On pp. 139—40 Saint-Venant deals with and figures the various curves which arise in the case of isotropy, when m is given different values, especially for the cases $c=b$ and $c=2b$.

On pp. 141—3 he refers to the case of $m/(1-m)$ being odd, and shews that not only are the limits for m narrower than in the previous case, but that the ratio b/c must remain within certain limits determined by those for m.

The case of $m=5/7$ and $\mu_1=\mu_2$ is fully treated and it is shewn that the equations represent for four values of b/c:

des *ovales* ou courbes ovoïdes *dont un des bouts est plus gros que l'autre.* Le *petit bout* dégénère en pointe pour la première et pour la dernière.

L'axe des z ne passe qu'exceptionnellement par le centre de gravité des sections terminées par ces contours non symétriques ; mais peu importe, car comme les fibres restent toutes dans les plans, tout ce qui précède est également vrai si l'on prend pour axe des x l'une quelconque des fibres qui ne varieront pas de longueur. (p. 143.)

[89.] We will next write down in a form corresponding to equation (25), the values of the three stresses; these we easily obtain from equations (20). They are:

$$\left.\begin{aligned} \widehat{xx} = \frac{P(a-x)z}{\omega\kappa^2}, \quad \widehat{xy} = \frac{P(1-m)yz}{\omega\kappa^2} \\ \widehat{xz} = -\frac{mP(c^2-z^2)}{2\omega\kappa^2} + \frac{\mu_2}{\mu_1}\frac{P(1-2\gamma_1-m)}{2\omega\kappa^2}y^2 \end{aligned}\right\} \text{.........} \quad (27).$$

As one terminal cross-section usually corresponds to $x=l=a$, we see that $\widehat{xx}=0$ across it, or the total external force exhibits itself as a shearing load, the resultant of which $-P$ is distributed according to a paraboloidal law.

Saint-Venant adds to these results that for the total deflection δ from equations (22) and (26); thus we have (see his p. 148):

$$\delta = \frac{Pl^3}{3E\omega\kappa^2}\left(1 + \frac{3m}{2}\frac{E}{\mu_2}\frac{c^2}{l^2}\right) \text{..................} \quad (28).$$

The form of the distorted cross-section deduced from equation (21) is:

$$x' = -\frac{P}{\omega\kappa^2}\left(\frac{m}{2\mu_2}c^2z - \frac{m-\gamma_2}{6\mu_2}z^3 - \frac{1-\gamma_1-m}{2\mu_1}y^2z\right) \text{......} \quad (29).$$

If $x'_0 = \dfrac{2m+\gamma_2}{3}\dfrac{Pc^3}{2\mu_2\omega\kappa^2}$ be the value of x' when $y=0$, $z=-c$, this may be written:

$$\frac{x'}{x'_0} = -\frac{3m}{2m+\gamma_2}\frac{z}{c} + \frac{m-\gamma_2}{2m+\gamma_2}\left(\frac{z}{c}\right)^3 + 3\frac{\mu_2}{\mu_1}\frac{1-\gamma_1-m}{2m+\gamma_2}\left(\frac{y}{c}\right)^2\frac{z}{c} \text{...} (29').$$

[90.] Saint-Venant specialises the results of the previous Article on pp. 144—148 for definite values of m. Thus he takes the case of the *false ellipse* for a uni-constant isotropic material ($m = 4/5$, $\gamma_1 = \gamma_2 = 1/10$); of the curve $m = 9/10$ (or 18/20, considered as even), also for a uni-constant isotropic material—this curve approaches a rectangle of which the angles have been rounded off and the top and bottom hollowed out; and of $m = 1$ ($= 2/2$), the contour is here a quadrilateral formed by four curved lines. Then he proceeds to cases which have for practical purposes more definite contours, namely :

(i) *The ellipse.* Here, if $q = \mu_1 c^2/\mu_2 b^2$, we have :

$$m = \frac{1 + 2q - 2\gamma_1}{1 + 3q};$$

$$\sigma_0 = -\frac{1 + 2q - 2\gamma_1}{1 + 3q}\frac{2P}{\mu_2\omega}, \left[= -\frac{5c^2 + 2b^2}{3c^2 + b^2}\frac{4P}{5\mu\omega}, \text{ for uni-constant isotropy;}\right]$$

$$\widehat{xy} = \frac{4P}{\omega}\frac{q + 2\gamma_1}{1 + 3q}\frac{yz}{c^2}, \left[= \frac{4P}{5\omega}\frac{5c^2 + b^2}{3c^2 + b^2}\frac{yz}{c^2}, \text{ for uni-constant isotropy;}\right]$$

$$\widehat{xz} = -\frac{2P}{\omega}\frac{1 + 2q - 2\gamma_1}{1 + 3q}\left(1 - \frac{z^2}{c^2}\right) + \frac{2P}{\omega}\frac{1 - 6\gamma_1}{1 + 3q}\frac{y^2}{b^2},$$

$$\left[= -\frac{4P}{5\omega}\frac{5c^2 + 2b^2}{3c^2 + b^2}\left(1 - \frac{z^2}{c^2}\right) + \frac{4P}{5\omega}\frac{y^2}{3c^2 + b^2}, \text{ for uni-constant isotropy;}\right]$$

$$\delta = \frac{4Pl^3}{3E\omega c^2}\left\{1 + \frac{3(1 + 2q - 2\gamma_1)}{2(1 + 3q)}\frac{E}{\mu_2}\frac{c^2}{l^2}\right\},$$

$$\left[= \frac{4Pl^3}{3E\omega c^2}\left\{1 + \frac{15c^2 + 6b^2}{6c^2 + 2b^2}\frac{c^2}{l^2}\right\}, \text{ for uni-constant isotropy.}\right]$$

(ii) *The circle.* We have only to put $b = c$ in the above results.

[91.] We may note that the term to be added to the deflection owing to shear is generally about $3\left(\frac{c}{l}\right)^2$ of that due to bending, if we deal with a uni-constant isotropic material (i.e. for circle $\frac{21}{8}\left(\frac{c}{l}\right)^2$, for false ellipse $3\left(\frac{c}{l}\right)^2$, for rectangle with flattened angles $\frac{27}{8}\left(\frac{c}{l}\right)^2$, etc.). This represents the amount neglected in the ordinary theory. If in practice we may safely neglect an error of 1/100 in the deflection, it follows that the ordinary theory will give sufficiently close practical results so long as the length of the beam is 8 or 9 times its diameter.

[92.] On pp. 148—156 Saint-Venant goes through some most interesting work to trace the form of the distorted cross-sections. He traces these surfaces by means of level or contour lines for different ratios of x'/x'_0 [see equation (29')], that is by the trace of the surfaces on planes parallel to the tangent plane at the origin.

The form of these families of curves may be roughly described as follows:

The critical member ($x'=0$) of the family is an ellipse (or in special cases a circle) and its diameter (the neutral axis). The critical member divides the family into two—for x'/x'_0 a positive fraction, we have a loop below the neutral axis and a 'snake' passing outside and *above* the critical ellipse with the neutral axis for its asymptote;—for x'/x'_0 a negative fraction we have curves congruent to these only the loop is above and the 'snake' below the neutral axis.

The contour of the section itself falls almost entirely within the critical ellipse and so gives a surface cutting the loops, the 'snakes' only apply for the distorted cross-section ideally produced.

The traces of the section made by planes parallel to the plane of flexure are cubical parabolas and are hatched in Saint-Venant's figures. It appears from them that the slide σ_{xz} has its maximum value at the centre. Saint-Venant draws attention to a noteworthy point on p. 152: Since b does not occur in the equation (29′) the contour-lines are the same for all sections having the same m, c and x'_0. The constancy of x'_0 involves $P/\omega\kappa^2$ remaining the same, except in the case of the *false-ellipse* where the term involving $\frac{y}{c}$ disappears from the equation to the contour; thus such ellipses are all orthogonal projections of each other.

We have reproduced three figures giving the form of the distorted sections on the frontispiece to this volume.

Only in Fig. (i) the 'snakes,' which are contour-lines falling outside the real section, are given. The contour-lines for elevations above the tangent plane are given by whole lines, those depressed below it by dotted lines. The traces by planes parallel to the plane of flexure are shaded. The figure corresponds to a circular cross-section when the material has uni-constant isotropy.

It gives very approximately the surface for elliptic cross-sections when b is $< 1\cdot5c$.

In Fig. (ii) we have the contour-lines for a *false ellipse*.

In Fig. (iii) for the rectangle with rounded angles and hollowed top and bottom referred to in our Art. 90 ($m=9/10$). We see that the contour-lines become *straight*.

In calculating and plotting out both Figs. (ii) and (iii) Saint-Venant has supposed uni-constant isotropy.

It may be remarked that the conception of these surfaces is much assisted by plaster-models, which exist for the case of the circular and square cross-sections (see below Art. 111).

[93.] Saint-Venant now passes to the discussion of the flexure of a beam of rectangular cross-section. This occupies pp. 156—168.

By the assumption

$$F(y, z) = \chi(y, z) + \frac{P(1-\gamma_2)z^3}{6\mu_2\omega\kappa^2} - \frac{\gamma_1 P}{2\mu_1\omega\kappa^2} y^2 z \quad \text{.........(30)},$$

Saint-Venant reduces the equations of condition (19′) for $F(y, z)$ to

$$\left.\begin{aligned}
&\mu_1 \frac{d^2\chi}{dy^2} + \mu_2 \frac{d^2\chi}{dz^2} = 0 \text{ for all values of } y \text{ and } z,\\
&\chi(-y, z) = \chi(y, z) \text{ everywhere},\\
&\chi = 0 \text{ and } d\chi/dz = 0 \text{ for } y = z = 0,\\
&\frac{d\chi}{dz} = -\sigma_0 - \frac{Pc^2}{2\mu_2\omega\kappa^2} + \frac{\gamma_1 P}{\mu_1\omega\kappa^2} y^2 \text{ for } z = \pm c \text{ and } y \text{ between } \pm b,\\
&\frac{d\chi}{dy} = 0 \text{ for } y = \pm b \text{ and } z \text{ between } \pm c.
\end{aligned}\right\} \text{...(31).}$$

Here $2b$ and $2c$ are the horizontal and vertical (flexure plane) sides of the rectangle.

The first equation of (31) is satisfied by taking

$$\chi = \Sigma e^{qz}\left\{A_q \cos\sqrt{\frac{\mu_2}{\mu_1}}\, qy + A'_q \sin\sqrt{\frac{\mu_2}{\mu_1}}\, qy\right\} \text{.........(32).}$$

The sines must however disappear in virtue of the second equation, and since $\chi = 0$ when $y = z = 0$, we must have $A_q = -A_{-q}$, or,

$$\chi = \Sigma A_q (e^{qz} - e^{-qz}) \cos\sqrt{\frac{\mu_2}{\mu_1}}\, qy.$$

The condition $d\chi/dz = 0$ for $y = 0$, $z = 0$, shews us that a certain relation must hold among the coefficients A_q; it will serve later to determine σ_0.

The condition $d\chi/dy = 0$ for $y = \pm b$ will be satisfied if

$$q = \frac{n\pi}{b}\sqrt{\frac{\mu_1}{\mu_2}},$$

n being any whole number, and obviously it will be sufficient to deal only with positive whole numbers. For $n = 0$, we must introduce a term $A_0(e^{0\cdot z} - e^{-0\cdot z})$ which gives us a quantity Kz.

Hence finally we may write:

$$\chi = Kz + \sum_1^\infty 2A_n \sinh \frac{n\pi z}{b}\sqrt{\frac{\mu_1}{\mu_2}} \cdot \cos n\pi y/b.$$

The fifth condition of (31) then gives us the following equation to determine A_n by Fourier's method

$$K+\sum_1^\infty 2A_n \frac{n\pi}{b}\sqrt{\frac{\mu_1}{\mu_2}}\cosh\frac{n\pi c}{b}\sqrt{\frac{\mu_1}{\mu_2}}\cos n\pi y/b$$

$$= -\sigma_0 - \frac{Pc^2}{2\mu_2\omega\kappa^2} + \frac{\gamma_1 P}{\mu_1\omega\kappa^2}y^2 \quad \text{...(33).}$$

Saint-Venant indicates in a foot-note (p. 159) that the form (32) is the most general form which will satisfy all the conditions of the problem.

[94.] Equation (33) easily gives us the following results:

$$K = -\sigma_0 - \frac{Pc^2}{2\mu_2\omega\kappa^2} + \frac{\gamma_1 Pb^2}{3\mu_1\omega\kappa^2},$$

$$2A_n = -\frac{4b^3}{\pi^3}\sqrt{\frac{\mu_2}{\mu_1}}\frac{\gamma_1 P}{\mu_1\omega\kappa^2}\frac{(-1)^{n-1}}{n^3}\operatorname{sech}\left(\frac{n\pi c}{b}\sqrt{\frac{\mu_1}{\mu_2}}\right).$$

We are thus able to write down the complete value of χ, namely:

$$\chi = \left(-\sigma_0 - \frac{Pc^2}{2\mu_2\omega\kappa^2} + \frac{\gamma_1 Pb^2}{3\mu_1\omega\kappa^2}\right)z$$

$$-\frac{\gamma_1 Pb^3}{\mu_1\omega\kappa^2}\sqrt{\frac{\mu_2}{\mu_1}}\frac{4}{\pi^3}\sum_1^\infty\frac{(-1)^{n-1}}{n^3}\frac{\sinh\left(\frac{n\pi z}{b}\sqrt{\frac{\mu_1}{\mu_2}}\right)}{\cosh\left(\frac{n\pi c}{b}\sqrt{\frac{\mu_1}{\mu_2}}\right)}\cos\frac{n\pi y}{b} \quad \text{...(34).}$$

In order finally to fulfil the condition $\frac{d\chi}{dz}=0$ for $y=z=0$ we must take

$$\sigma_0 = -\frac{Pc^2}{2\mu_2\omega\kappa^2}\left\{1-\frac{\mu_2}{\mu_1}\cdot\frac{2\gamma_1 b^2}{3c^2}\left[1-\frac{12}{\pi^2}\sum_1^\infty\frac{(-1)^{n-1}}{n^2}\operatorname{sech}\frac{n\pi c}{b}\sqrt{\frac{\mu_1}{\mu_2}}\right]\right\} \quad \text{...(35).}$$

We have thus the complete determination of all the constants of the problem.

[95.] In the following pp. 162—3, Saint-Venant deduces from (18'), (20), (30), (34) and (35) the values of the three shifts and the three stresses; we tabulate them for reference.

$$u = P\frac{2lx-x^2}{2E\omega\kappa^2}z$$

$$+(1-\gamma_2)\frac{Pz^3}{6\mu_2\omega\kappa^2} - \gamma_1\frac{Py^2z}{2\mu_1\omega\kappa^2} + \gamma_1\frac{Pb^2z}{\mu_1\omega\kappa^2}\frac{4}{\pi^2}\sum_1^\infty\frac{(-1)^{n-1}}{n^2}\operatorname{sech}\left(\frac{n\pi c}{b}\sqrt{\frac{\mu_1}{\mu_2}}\right)$$

$$-\gamma_1\frac{Pb^3}{\mu_1\omega\kappa^2}\sqrt{\frac{\mu_2}{\mu_1}}\cdot\frac{4}{\pi^3}\sum_1^\infty\frac{(-1)^{n-1}}{n^3}\frac{\sinh\left(\frac{n\pi z}{b}\sqrt{\frac{\mu_1}{\mu_2}}\right)\cos\frac{n\pi y}{b}}{\cosh\left(\frac{n\pi c}{b}\sqrt{\frac{\mu_1}{\mu_2}}\right)},$$

$$v = -\gamma_1 \frac{P(l-x)}{\mu_1 \omega \kappa^2} yz,$$

$$w = -\frac{Pc^2}{2\mu_2 \omega\kappa^2}\left\{1 - \gamma_1 \frac{\mu_2}{\mu_1}\frac{2b^2}{3c^2}\left[1 - \frac{12}{\pi^2}\sum_1^\infty \frac{(-1)^{n-1}}{n^2} \operatorname{sech}\left(\frac{n\pi c}{b}\sqrt{\frac{\mu_1}{\mu_2}}\right)\right]\right\} x$$
$$- \frac{P}{2E\omega\kappa^2}\frac{3lx^2 - x^3}{3} + P\frac{l-x}{2\omega\kappa^2}\left(\frac{\gamma_1 y^2}{\mu_1} - \frac{\gamma_2 z^2}{\mu_2}\right),$$

$$\widehat{xy} = \frac{\gamma_1 P}{\omega\kappa^2}\frac{4b^2}{\pi^2}\sqrt{\frac{\mu_2}{\mu_1}}\sum_1^\infty \frac{(-1)^{n-1}}{n^2}\frac{\sinh\left(\frac{n\pi z}{b}\sqrt{\frac{\mu_1}{\mu_2}}\right)\sin\frac{n\pi y}{b}}{\cosh\left(\frac{n\pi c}{b}\sqrt{\frac{\mu_1}{\mu_2}}\right)},$$

$$\widehat{xz} = -\frac{Pc^2}{2\omega\kappa^2}\left(1 - \frac{z^2}{c^2}\right)$$
$$+ \gamma_1 \frac{Pb^2}{3\omega\kappa^2}\frac{\mu_2}{\mu_1}\left\{1 - \frac{3y^2}{b^2} - \frac{12}{\pi^2}\sum_1^\infty \frac{(-1)^{n-1}}{n^2}\frac{\cosh\left(\frac{n\pi z}{b}\sqrt{\frac{\mu_1}{\mu_2}}\right)\cos\frac{n\pi y}{b}}{\cosh\left(\frac{n\pi c}{b}\sqrt{\frac{\mu_1}{\mu_2}}\right)}\right\}.$$

Saint-Venant verifies these results by shewing that they satisfy the boundary equation $\widehat{xz}\,dy - \widehat{xy}\,dz = 0$ and the load-conditions $\int \widehat{xz}\,d\omega = -P$, $\int \widehat{xy}\,d\omega = 0$.

[96.] The next two sections 28 and 29 (pp. 164—8) are occupied with numerical, graphical and simpler algebraic expressions for the quantities which occur in the previous sections.

For σ_0 Saint-Venant obtains the following results *when* $\gamma_1 = \frac{1}{10}$:

When $\frac{c}{b}\sqrt{\frac{\mu_1}{\mu_2}} =$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	1·25	1·5	2	2·5	3
$-\sigma_0 = \frac{Pc^2}{2\mu_2\omega\kappa^2} \times$	·67624	·84918	·90729	·94031	·96177	·97101	·98341	·98934	·99259

It is shewn that for all values of $c\sqrt{\mu_1} > b\sqrt{\mu_2}$ the sum-term in equation (35) may be omitted, or we can write

$$-\sigma_0 = \frac{3P}{2\mu_2\omega} - \frac{\gamma_1 b^2}{c^2}\frac{P}{\mu_1\omega}.$$

Further the deflection δ is then given by :

$$\delta = \frac{Pl^3}{3E\omega\kappa^2}\left(1 + \frac{3E}{2\mu_2}\frac{c^2}{l^2} - \eta_1\frac{b^2}{l^2}\right)$$

since $$\gamma_1 = \eta_1 \frac{\mu_1}{E}.$$

For the case of isotropy: $\eta_1 = \frac{1}{4}$, $E/\mu_1 = 5/2$, or

$$\delta = \frac{Pl^3}{3E\omega\kappa^2}\left(1 + \frac{15c^2 - b^2}{4l^2}\right).$$

The Bernoulli-Eulerian theory takes no account of the second term within the brackets.

[97.] Saint-Venant devotes his next few pages to a calculation of the value of x' which gives (see our equation (21)) the form of the distorted surface. He treats especially the case of $b = c$, and uni-constant isotropy (i.e. $\gamma_1 = \gamma_2 = 1/10$, $\mu_1 = \mu_2$).

I have reproduced his diagram of the contour-lines, as Fig. iv. of the frontispiece; the hatched lines as before denoting sections of the surface by planes parallel to that of flexure. The contour-lines are drawn for $x'_0 = 0$ to ± 1 by steps of $\cdot 2$.

The trigonometrical terms in x' have little importance when $b < c$, so that in that case we can practically take

$$x' = -\frac{Pc^3}{2\mu_2 \omega \kappa^2}\left\{\frac{z}{c} - \frac{1-\gamma_2}{3}\left(\frac{z}{c}\right)^3\right\}$$

$$= -x_0'\frac{3}{2+\gamma_2}\left\{\frac{z}{c} - \frac{1-\gamma_2}{3}\left(\frac{z}{c}\right)^3\right\}.$$

This is equivalent to neglecting terms in the expression for x' involving the factor b/c. It is obvious that the contour-lines now become straight lines.

The above value of x' is obtained by Saint-Venant from very simple considerations in a foot-note on pp. 184—5. It had already been given in the memoir on *Torsion* (see our Art. 12) without the term γ_2 (circa 1/10); a similar proof of the formula is given in the *Leçons de Navier*: see our Art. 183 (*a*).

[98.] Saint-Venant's thirtieth section (pp. 168—171) is entitled: *Sections de forme quelconque.* This amounts to little more than the statement that, a solution having been found for the equations (19) with regard to certain cross-sections we may infer that a solution exists for all cross-sections. The inference is strengthened by reference to a corresponding problem in the conduction of heat.

[99.] Section 31 (pp. 171—187) is termed: *Démonstration directe et sans analyse des formules connues de la flexion des prismes due à leurs seules dilatations longitudinales.* This investigation can be easily followed by those who have grasped the analytical calculations, but it seems to me very doubtful if it would be of value for elementary teaching (e.g. of engineering students). Saint-Venant did not reproduce it in his *Leçons de Navier.*

[100.] The final section of the memoir (§ 32, pp. 187—9) is entitled: *Conclusion. Observation générale pour le cas où le mode d'application et de distribution des forces extérieures vers les extrémités est différent de celui qui rend tout à fait exactes les formules auxquelles conduit la méthode mixte.*

This reiterates the principle of the practical equivalence in elastic effect of two surface distributions of load which are statically equivalent: see our Arts. 8 and 9.

101. *Sur les conséquences de la théorie de l'élasticité en ce qui regarde la théorie de la lumière. L'Institut,* Vol. 24, 1856, 32—34. The article adopts the view that much remains to be done to render the theory of Physical Optics satisfactory; it supports the views of Cauchy, especially with regard to the existence of a *third* ray as obtained by him in his discussion of what is termed *double* refraction. The article concludes thus:

Quoi qu'il puisse être de ces explications, que nous devons nous borner à soumettre aux physiciens et aux physiologistes, et bien que l'on puisse continuer sans doute de regarder le mouvement de la lumière dans les cristaux comme représenté *approximativement* par la surface d'onde du quatrième degré de Fresnel, nous pensons qu'il convient de ne plus passer sous silence les composantes longitudinales des vibrations pour éluder quelques difficultés dont elles sont le sujet, et que, pour rendre la théorie de la lumière exempte d'inexactitude logique, et provoquer pour l'avenir des recherches qui seront peut-être suivies d'importantes découvertes, il y a lieu de ne plus présenter les vibrations de l'éther, dans les milieux biréfringents, comme étant tout à fait parallèles aux divers plans tangents à la surface des ondes lumineuses qui s'y propagent.

102. *Sur la vitesse du son. L'Institut,* Vol. 24, 1856, 212—216. Newton obtained a certain expression for the velocity of sound which gives a result much smaller than that found by experiment. Laplace modified the formula, and thus obtained a result agreeing with experiment: see our Arts. 310* and 68. Saint-Venant is not satisfied with any investigation which has been given, even with the aid of the formulae of the theory of elasticity. He says

On voit toujours, par ce qui précède, qu'il reste encore bien des choses à savoir sur la théorie du son, objet des recherches d'hommes tels que Newton, Lagrange, Euler, Laplace, Poisson et Dulong; qu'on ne doit pas s'étonner de trouver des différences entre les résultats de l'observation et ceux de la formule de vitesse la plus généralement adoptée jusqu'ici $\sqrt{\frac{c}{c'}\frac{p}{\rho}}$, ni se hâter de déduire de cette formule, probablement fausse, des valeurs du rapport c/c', comme l'ont fait plusieurs physiciens éminents; enfin que ce qu'il paraîtrait y avoir de mieux à faire dans l'enseignement, jusqu'à éclaircissement, serait de démontrer la formule newtonienne et d'énoncer simplement les raisons qui rendent son résultat trop faible (pp. 115—6).

Saint-Venant's article contains valuable references to preceding writers on the subject. See too *Die Fortschritte der Physik im Jahre* 1856, pp. 159—164.

103. *Sur la résistance des solides. L'Institut*, Vol. 24, 1856, pp. 457—459. This article relates to the moments of inertia and the situation of the principal axes of plane figures; the results given are useful in connexion with the resistance of beams to flexure, and are accompanied by various numerical calculations. Two formulae are given with respect to the moment of inertia of a triangle which may have been new at the time, but which now are particular cases of a known general proposition, namely that the moment of inertia of a triangle of mass M about any axis is the same as that of three particles of mass $\frac{1}{12}M$ at the angular points, and a particle of mass $\frac{3}{4}M$ at the centre of gravity. From this may be easily deduced another formula which Saint-Venant gives: the moment of inertia of a trapezium of mass M about one of the non-parallel sides is $\frac{1}{6}M\ (y^2+y'^2)$, where y and y' are the perpendiculars from the two opposite angles on this side. Again we have a formula respecting the *product of inertia* for a right-angled triangle. Let M be the mass, and a, b the lengths of the sides. Then if the origin be at the angular point, and the axes coincide with the sides, the value as found by an obvious integration is $\frac{1}{12}Mab$. Hence if the origin be at the centre of gravity and the axes parallel to the sides, the value is $\frac{1}{12}Mab-\frac{1}{9}Mab$, that is $-\frac{1}{36}Mab$. This will hold also if the origin is on either of the straight lines through the centre of gravity parallel to the sides, the axes remaining always parallel to their original position.

[104.] *Sur l'Impulsion transversale et la Résistance vive des barres élastiques appuyées aux extrémités. Comptes rendus*, T. XLV.

1857, pp. 204—8. This memoir was presented on August 10, 1857. It was referred to Poncelet, Lamé, Bertrand and Hermite. An extract by the author is given in the *Comptes rendus.* Some of the results of this memoir were communicated to the *Société Philomathique,* November 5, 1853 and January 21, 1854, and partially published in *L'Institut,* T. 22, 1854, pp. 61—3, under the title: *Solution du problème du choc transversal et de la résistance vive des barres élastiques appuyées aux extrémités.* This is a special case of the resilience problem experimentally investigated by Hodgkinson and theoretically by Cox : see our Arts. 939*, 942*, 999* and 1434—7*. Saint-Venant, however, does not like Cox neglect the vibrations of the bar, or assume that its form will be that of the elastic line for a beam which centrally loaded has the same central deflection. In the *Comptes rendus,* Saint-Venant gives some account of the history of both transverse and longitudinal impact problems, but Cox's memoir seems to have escaped him.

The following result is given in the *Comptes rendus*, p. 206 :

$$y = V\tau\Sigma\frac{4}{m^3}\frac{\dfrac{\sin mx/l}{\cos m}-\dfrac{\sinh mx/l}{\cosh m}}{\sec^2 m-\operatorname{sech}^2 m+\dfrac{2}{m^2}\dfrac{P}{Q}}\sin(m^2t/\tau),$$

where the Σ refers to all the real and positive roots m of the equation

$$m(\tan m-\tanh m)=2P/Q,$$

and the following is the notation used :

$2l$ = length of bar, P its weight, Q that of body striking the bar horizontally with velocity V at its mid-section, y is the horizontal displacement at distance x from one end and $\tau=\sqrt{Pl^3/(2gE\omega\kappa^2)}$.

[105.] Saint-Venant makes the following remark :

Du calcul tant numérique que graphique d'une suite de ces valeurs du déplacement y, on peut déduire la suite des formes très-variées prises par la barre heurtée ; ce qui permet de modeler un relief en plâtre donnant la surface que décrirait cette barre supposée emportée transversalement d'un mouvement rapide, perpendiculaire au sens où elle oscille. Cette surface est très-ondulée à cause des oscillations provenant des second et troisième termes surtout de la série Σ (p. 206).

This surface in plaster of Paris was actually prepared under Saint-Venant's directions ; and I have found a copy of it very useful for lecture purposes.

When P/Q does not exceed 3, the deflection obtained is very approximately that given by Cox in his memoir: see our Art. 1437*. It is not directly upon the *deflection*, however, but upon the *greatest curvature* that the maximum resistance of the bar depends, and this when $P/Q = 2$ is about 1·5 as great when obtained from the true transcendental formula as when obtained from statical considerations in Cox's manner. (See also *Notice* II. p. 20, under 2°.)

[106.] If the transverse blow be vertical, we must add to the above value of y the statical deflection and replace $V\tau \sin(m^2t/\tau)$ by the expression $V\tau \sin(m^2t/\tau) - (g\tau^2/m^2)\cos(m^2t/\tau)$.

[107.] Saint-Venant compares his results with the numbers obtained by Hodgkinson: see our Arts. 1409*—10*. He finds that the values of the stretch-modulus so obtained agree among themselves, but differ from the statical values obtained from pure traction-experiments. He attributes this to thermal differences, such as had been considered by Duhamel and Wertheim: see our Arts. 889* and 1301*. On p. 207 there is a brief reference to some results for longitudinal impact.

[108.] The memoir itself appears never to have been published but its results together with many extensions and developments are given in the *Note finale du* § 61 of the annotated *Clebsch* pp. 490—596. Just *thirty years* after their discovery! We shall consider them in detail when dealing with that work, as the problem is an extremely important one in the theory of structures. See in particular *Notice* I. pp. 36—41 and *Notice* II. pp. 19—20.

[109.] *Établissement élémentaire des Formules de la torsion des prismes élastiques. Comptes rendus*, T. XLVI. pp. 34—8, 1858. The formulae in question are those of our Art. 17 but they are obtained only for the torsion of isotropic bodies. Saint-Venant's object is to deduce the results of the memoir on Torsion in an elementary fashion for the use of technical schools and practical men. The method does not seem to me entirely clear and satisfactory, and it is not at once obvious why the reasoning only applies to an *isotropic* body. Special proofs of various portions of the theory of elasticity may be now and then of service, but it cannot be denied that they, by tending to obscure the broad lines and general principles of the subject, may do more harm than good to the student.

The fairly elementary treatment of the *Leçons de Navier* seems to me more advantageous (pp. 245—250). The treatment of the present paper is also reproduced in § 7 (pp. 250—2) of the same work.

[110.] *L'Institut,* Vol. 26, 1858, pp. 178—9. Further results on Torsion communicated to the *Société Philomathique* (April 24 and May 15, 1858) and afterwards incorporated in the *Leçons de Navier* (pp. 305—6, 273—4). They relate to cross-sections in the form of doubly symmetrical quartic curves and to torsion about an external axis: see our Arts. 49 (*c*), 182 (*b*), 181 (*d*), and 182 (*a*).

[111.] Vol. 27, 1860, of same Journal, pp. 21—2. Saint-Venant presents to the *Société Philomathique* the model *de la surface décrite par une corde vibrante transportée d'un mouvement rapide perpendiculaire à son plan de vibration.* Copies of this as well as some other of Saint-Venant's models may still be obtained of M. Delagrave in Paris and are of considerable value for class-lectures on the vibration of elastic bodies.

[112.] Vol. 28, 1861, of same Journal, pp. 294—5. This gives an account of a paper of Saint-Venant's read before the *Société Philomathique* (July 28, 1860). In this he deduces the *conditions of compatibility,* or the six differential relations of the types:

$$2\frac{d^2 s_x}{dy\,dz} = \frac{d}{dx}\left(\frac{d\sigma_{xz}}{dy} + \frac{d\sigma_{xy}}{dz} - \frac{d\sigma_{yz}}{dx}\right)$$

$$\frac{d^2\sigma_{yz}}{dy\,dz} = \frac{d^2 s_y}{dz^2} + \frac{d^2 s_z}{dy^2}$$

which must be satisfied by the strain-components. These conditions enable us in many cases to dispense with the consideration of the shifts. A proof of these conditions by Boussinesq will be found in the *Journal de Liouville,* Vol. 16, 1871, pp. 132—4. At the same meeting Saint-Venant extended his results on torsion to: (1) prisms on any base with at each point only one plane of symmetry perpendicular to the sides, (2) prisms on an elliptic base with or without any plane of symmetry whatever; see our Art. 190 (*d*).

[113.] *Sur le Nombre des Coefficients inégaux des formules donnant les composantes des pressions dans l'intérieur des solides élastiques. Comptes rendus*, T. LIII. 1861, pp. 1107—1112. This paper gives very meagrely the outlines of Appendix V. to the *Leçons de Navier*: see our Arts. 192 to 195. Cf. also Moigno's *Statique*, Art. 270 and Stokes' *Report on Double-Refraction*, p. 260.

[114.] *Sur les divers genres d'homogénéité des corps solides et principalement sur l'homogénéité semi-polaire ou cylindrique, et sur les homogénéités polaire ou sphériconique et sphérique.* This paper was read to the Academy on May 21, 1860 and published in Liouville's *Journal de Mathématiques*, 1865, pp. 297—349. An abstract appeared in the *Comptes rendus*, T. L. 1860, pp. 930—4. See also *Notice* II. p. 23 and Moigno's *Statique*, p. 668.

This memoir is important as the first attempt to explain various results of experiment inconsistent with uni-constant isotropy by an extended conception of homogeneity applied to aeolotropic bodies. Cauchy had defined *homogeneity* as consisting in the elasticity of a body being the same *for the same directions* at all points. Saint-Venant alters the latter words and thus defines homogeneity:

> *Un corps est homogène lorsque l'un quelconque de ses éléments imperceptibles est identique à tout élément du même corps pris ailleurs ayant même volume et même forme, mais orienté d'une certaine manière qui peut changer d'un endroit à l'autre.* Il l'est même encore lorsque cette identité de deux éléments, pris n'importe où et convenablement orientés, souffre exception pour certains points isolés ou *ombilicaux* (tels que sont ceux de l'intersection commune des plans des cercles de longitude de la sphère dont on vient de parler...).
>
> Le mode d'orientation des éléments, ou la direction relative de leurs lignes homologues, détermine le genre de l'homogénéité, genre dont chacun admet, comme nous verrons au no. 3, des *sous-genres* où les orientations possibles en chaque point sont multiples. (p. 299.)

Let us take any two lines of the elastic system at right angles and arrange all lines homologous to the first along the normals to a given surface, the second system of lines may then be arranged according to any law we please, e.g. as tangents to any system of curves we please to draw on the surface. If the given surface be of the nth order, we have an *n-ic distribution of elastic homogeneity*; the curves on the surface to which the second system of homologous lines are tangents determine the *sous-genre* or *sub-class*.

[115.] The following paragraphs describe the quadric distributions of elasticity with which Saint-Venant proposes to deal.

After describing the *amorphic* body or body of *confused-crystallisation*, such as a rolled metal plate, the elasticity of which varies in length, breadth and depth,—Saint-Venant continues:

Qu'on enroule en tuyau cylindrique cette plaque homogène rectangulaire non isotrope supposée mince, en dirigeant, par exemple, les génératrices dans le sens de sa longueur. *Elle ne cessera pas d'être homogène;* mais l'égalité d'élasticité aux divers points n'aura pas lieu pour les directions parallèles entre elles. Il y aura égale élasticité suivant les rayons qui vont tous couper perpendiculairement l'axe du cylindre : ce sera l'élasticité dans le sens de l'épaisseur. Il y aura égale élasticité suivant les diverses tangentes aux cercles ayant leur centre sur cet axe. Il n'y aura que les élasticités égales suivant la longueur qui auront conservé des directions parallèles entre elles. (p. 298.)

We shall term this a cylindrical distribution of elastic homogeneity.

The following describes a spherical distribution:

Qu'on imagine maintenant une sphère solide pleine ou creuse, ou un corps de forme quelconque divisible en couches sphériques concentriques. Si la résistance ou la réaction élastique, pour mêmes déplacements de ses points, est partout égale dans le sens des rayons, et partout égale aussi dans certains sens perpendiculaires entre eux et aux rayons, ceux par exemple où se comptent les latitudes et les longitudes pour un équateur donné, la matière est homogène, mais *polairement*, ou d'une manière que nous pouvons appeler *sphériconique* vu le rôle qu'y jouent les *cônes de latitude* ayant un axe déterminé, le même pour tous. (p. 298.)

Such distributions of elasticity are, Saint-Venant asserts,—and I hold him to be entirely right—the true explanation of the anomalies which occur in experiments on a variety of cast, rolled and forged bodies. Even granted that isotropy is bi-constant, it is certainly not scientific to seek by means of two constants to account for the divergency between uni-constant formulae and experimental results on wires, plates, or cylindrical and spherical bodies. Physically it is obvious that the *working* of such bodies really produces in them varied distributions of elastic homogeneity, which bi-constant formulae only serve to mask. The 'isotropic boilers' treated of by Lamé (see our Art. 1038*) or his 'isotropic piezometers' (see our Art. 1358*) have practically no existence (see our Arts. 332* and 1357*), and all elasticians can adopt Saint-

Venant's formulae with entire approval although they may not accept his view of the equations of uni-constant isotropy:

Formules qui sont les conséquences obligées et rigoureuses de la loi des actions moléculaires *que tout le monde invoque ouvertement ou tacitement*, et même sans laquelle tout établissement de formules mathématiques d'élasticité est illusoire. (p. 300.)

[116.] Saint-Venant on pp. 301—3 makes some remarks on the elastic coefficients, and on the subject of multi-constancy; for the purpose of the memoir, however, he adopts the 21 constants of Green[1].

If the stress be given by formulae of the type

$$p_{xx} = |xxxx|\, s_x + |xxyy|\, s_y + |xxzz|\, s_z + |xxyz|\, \sigma_{yz} + |xxzx|\, \sigma_{zx} + |xxxy|\, \sigma_{xy},$$
$$p_{yz} = |yzxx|\, s_x + |yzyy|\, s_y + |yzzz|\, s_z + |yzyz|\, \sigma_{yz} + |yzzx|\, \sigma_{zx} + |yzxy|\, \sigma_{xy},$$

then the coefficients can only be treated as constants when we suppose the axes-system to vary in direction from point to point of the material. This granted, the above expressions for the stresses will be given in terms of constant coefficients.

[117.] In section 3 (pp. 303—6) after some general remarks as to homogeneity and its various sub-classes, Saint-Venant supposes the distribution of elasticity to be symmetrical with regard to

[1] He refers to Rankine's terminology, which we may here throw into a form brief enough for convenience:

$|xxxx|$ = direct stretch coefficient = the coefficient of direct elasticity of Rankine.
$|xxyy|$ = cross stretch coefficient = the coefficient of lateral elasticity of Rankine.
$|xyxy|$ = direct slide coefficient = the coefficient of tangential elasticity of Rankine.

$$\left.\begin{array}{l} |xyyz| = \text{cross slide coefficient} \\ |xxxz| = \text{direct slide-stretch coefficient} \\ |xxyz| = \text{cross slide-stretch coefficient} \\ |xyxx| = \text{direct stretch-slide coefficient} \\ |xyzz| = \text{cross stretch-slide coefficient} \end{array}\right\} = \text{coefficients of asymmetrical elasticity of Rankine.}$$

All elasticians agree that the slide-stretch coefficients whether direct or cross are equal to the corresponding stretch-slide coefficients; further that the cross stretch and cross slide coefficients are equal for the pair of faces involved in the cross. This amounts to saying that we may interchange the first and second pairs of subscripts. We have thus the fifteen relations of Green. For a body with three planes of elastic symmetry all the *asymmetrical* coefficients vanish. The rari-constant elasticians assert that the cross stretch coefficients are equal to the direct slide coefficients, when the cross is made for the two directions involved in the slide (i.e. $|xxyy| = |xyxy|$), and further that the cross slide-stretch coefficients are equal to the cross slide coefficients when the direction of the stretch is involved in both the slides which are crossed (i.e. $|xxyz| = |xyxz|$). This gives the six additional relations of Poisson, or we may interchange between the first and second pair of subscripts.

three planes, or all the asymmetrical coefficients to vanish. In this case the types of traction and shear are:

$$(a)\quad \begin{array}{ll} \widehat{xx} = as_x + f's_y + e's_z & \widehat{yz} = d\sigma_{yz}, \\ \widehat{yy} = f's_x + bs_y + d's_z & \widehat{zx} = e\sigma_{zx}, \\ \widehat{zz} = e's_x + d's_y + cs_z & \widehat{xy} = f\sigma_{xy}. \end{array}$$

(See our Art. 78.)

(b) If the normal to the distribution-surface be the axis of x and the elasticity be isotropic in the tangent plane, we have also:

$$b = c,\ e = f,\ e' = f' \text{ and } b = 2d + d'.$$

(c) If the material be *amorphic*, there is an *ellipsoidal* distribution of direct-stretch coefficients (see our Arts. 139 and 142), and we have

$$2d + d' = \sqrt{bc},\ 2e + e' = \sqrt{ca},\ 2f + f' = \sqrt{ab}.$$

(d) In the case of *rari-constant* elasticity, the dashed and undashed letters are equal. Thus for the amorphic body we have:]

$$\begin{array}{ll} \widehat{xx} = 3\dfrac{ef}{d}s_x + fs_y + es_z & \widehat{yz} = d\sigma_{yz}, \\ \widehat{yy} = fs_x + 3\dfrac{fd}{e}s_y + ds_z & \widehat{zx} = e\sigma_{zx}, \\ \widehat{zz} = es_x + ds_y + 3\dfrac{de}{f}s_z & \widehat{xy} = f\sigma_{xy}. \end{array}$$

(See, however, our Art. 313.)

[118.] Before we can apply these formulae to any given distribution of elasticity determined by curvilinear coordinates, it is necessary to find:

(1) Expressions for the above strain-components (s_x, s_y, s_z, σ_{yz}, σ_{zx}, σ_{xy}) corresponding to the elements of the three rectangular surface normals or intersection-traces in terms of the curvilinear coordinates.

(2) To express the body-stress equations in terms of curvilinear coordinates. Saint-Venant indicates in § 4 (pp. 306—12) two methods of attacking this problem, and compares them with Lamé's method (in the *Leçons*, 1852, § 77) which he terms "un procédé en quelque sorte *mixte*." The analysis of the problem does not probably admit of much simplification, and for practical purposes the general results of Lamé's treatise on Curvilinear Coordinates may well be assumed: see our Arts. 1150*—3*.

In § 5 (pp. 312—18) and in § 9 (pp. 333—9) Saint-Venant obtains expressions for the strains and the body-stress equations in terms of cylindrical and spherical coordinates respectively. These agree with those of Lamé[1]: see our Arts. 1087* and 1093*. The relations between stress and strain are then given by the formulae of the preceding article.

[119.] The novelty of the present memoir consists in the solution of the elastic equations for cylindrical and spherical shells subjected internally and externally to uniform tractive loads, when the material of these shells is amorphic and has cylindrical or spherical distribution of elasticity. By means of the solutions given, we see that the difficulties encountered by Regnault and others can be more naturally met by presupposing aeolotropy, than by assuming bi-constant isotropy.

[120.] Saint-Venant takes first (§ 7) the case of a long cylindrical shell subjected to internal tractive load $-p_0$ and external $-p_1$. As in Lamé's problem, we may suppose it closed by flat ends in such a manner that the transverse sections are not distorted. Supposing $dw/dz = \gamma$, we easily deduce (see footnote) the equation $\dfrac{d\widehat{rr}}{dr} + \dfrac{\widehat{rr} - \widehat{\phi\phi}}{r} = 0$, or substituting the stress-values from formulae (a) of Art. 117 expressed in terms of the strains given in the footnote we have, if $a_r = du/dr$

$$a\,(u_{rr} + u_r/r) - bu/r^2 + (e' - d')\,/\,(\gamma r) = 0.$$

[1] As in this volume we shall have frequent occasion to refer to these formulae I tabulate them here for reference—the notation will readily explain itself:

	Cylinder		Sphere (ϕ = co-latitude)
Body Stress-Equations	$\dfrac{d\widehat{rr}}{dr} + \dfrac{d\widehat{r\phi}}{rd\phi} + \dfrac{d\widehat{rz}}{dz} + \dfrac{\widehat{rr} - \widehat{\phi\phi}}{r} + \rho R = 0$ $\dfrac{d\widehat{\phi r}}{dr} + \dfrac{d\widehat{\phi\phi}}{rd\phi} + \dfrac{d\widehat{\phi z}}{dz} + \dfrac{2\widehat{r\phi}}{r} + \rho\Phi = 0$ $\dfrac{d\widehat{zr}}{dr} + \dfrac{d\widehat{z\phi}}{rd\phi} + \dfrac{d\widehat{zz}}{dz} + \dfrac{\widehat{zr}}{r} + \rho Z = 0$		$\dfrac{d\widehat{rr}}{dr} + \dfrac{d\,(\widehat{r\phi}\cos\phi)}{r\cos\phi d\phi} + \dfrac{d\widehat{r\psi}}{r\cos\phi d\psi} + \dfrac{2^{rr} - \widehat{\phi\phi} - \widehat{\psi\psi}}{r} + \rho R = 0$ $\dfrac{d\widehat{\phi r}}{dr} + \dfrac{d\,(\widehat{\phi\phi}\cos\phi)}{r\cos\phi d\phi} + \dfrac{d\widehat{\phi\psi}}{r\cos\phi d\psi} + \dfrac{2\widehat{\phi r} + \widehat{\psi\psi}\tan\phi}{r} + \rho\Phi = 0$ $\dfrac{d\widehat{\psi r}}{dr} + \dfrac{d(\widehat{\psi\phi}\cos\phi)}{r\cos\phi d\phi} + \dfrac{d\widehat{\psi\psi}}{r\cos\phi d\psi} + \dfrac{3\widehat{\psi r} - \widehat{\psi\phi}\tan\phi}{r} + \rho\Psi = 0$
s_r	u_r	s_r	u_r
s_ϕ	$v_\phi/r + u/r$	s_ϕ	$v_\phi/r + u/r$
s_z	w_z	s_ψ	$w_\psi/(r\cos\phi) + u/r - v/r\,.\tan\phi$
$\sigma_{\phi z}$	$v_z + w_\phi/r$	$\sigma_{\phi\psi}$	$v_\psi/(r\cos\phi) + w_\phi/r + w/r\,.\tan\phi$
σ_{zr}	$w_r + u_z$	$\sigma_{\psi r}$	$w_r + u_\psi/(r\cos\phi) - w/r$
$\sigma_{r\phi}$	$u_\phi/r + v_r - v/r$	$\sigma_{r\phi}$	$u_\phi/r + v_r - v/r$

Hence we find for the shifts

$$u = Cr^{\sqrt{\frac{b}{a}}} + C'r^{-\sqrt{\frac{b}{a}}} + \frac{d'-e'}{a-b}\gamma r, \quad v=0, \quad w=\gamma z.$$

The stresses and strains can be at once deduced; they will contain constant terms in γ and powers of r of order $\pm\sqrt{\frac{b}{a}}-1$. The constants C, C' and γ are to be determined from the surface conditions and the relation $p_0\pi r_0^2 - p_1\pi r_1^2 = 2\pi\int_{r_0}^{r_1}\widehat{zz}rdr$ for total terminal tractive stress.

[121.] Saint-Venant considers various special cases:

(1) $r_1 - r_0$ is a small thickness ϵ. (pp. 324—5.)

(2) $a=b$. Here the solution changes its form, we have (p. 326):

$$u = C_1 r + C_1'/r + \frac{d'-e'}{2a}\gamma r \log r.$$

If $d'=e'$ the solution becomes that found by Lamé and Clapeyron, and applied by Lamé to Regnault's piezometers: see our Arts. 1012* and 1358*.

(3) When there is an ellipsoidal distribution of elasticity and rari-constancy is assumed, i.e. when $a=3ef/d$, $b=3fd/e$, $c=3de/f$. In this case $$u = Cr^{d/e} + C'r^{-d/e} - d\gamma r/\{3f(d/e+1)\}.$$

The values of the stresses are then easily determined, as well as those of C, C' and γ (p. 329).

The results contain *three independent elastic constants,* and they differ in the *form* of the r-index from those found for the case of isotropy. Hence we can explain by means of them as well as or better than by biconstant formulae the divergencies remarked by Regnault in his piezometer experiments.

[122.] A result is given on p. 331, which is worth citing. The constants d, e, f of the ellipsoidal distribution are not easy to determine by direct experiment. Let E_r, E_ϕ, E_z however be the three stretch moduli in directions r, ϕ, z, then we easily find that:

$$d = \tfrac{2}{5}\sqrt{E_\phi E_z}, \quad e = \tfrac{2}{5}\sqrt{E_z E_r}, \quad f = \tfrac{2}{5}\sqrt{E_r E_\phi}.$$

From equations 50 (p. 332) Saint-Venant might have deduced the criterion for failure arising first by lateral or first by longitudinal stretch. These equations are:

$$s_z = \frac{1}{4}\,\frac{2\sqrt{E_\phi}-\sqrt{E_z}}{E_z\sqrt{E_\phi}}\,\frac{p_0-p_1}{\epsilon}\,r', \qquad s_\phi = \frac{1}{8}\,\frac{8\sqrt{E_z}-\sqrt{E_\phi}}{E_\phi\sqrt{E_z}}\,\frac{p_0-p_1}{\epsilon}\,r',$$

where $r_0 = r' - \frac{\epsilon}{2}$ and $r_1 = r' + \frac{\epsilon}{2}$, so that $r' = \frac{r_0+r_1}{2}$, $r_1 - r_0 = \epsilon$.

So long as $E_z > \cdot 4195\ E_\phi$, s_ϕ is $> s_z$ and failure will occur by lateral stretch. If the absolute strengths R_z and R_ϕ were, as some writers have supposed, proportional to the moduli, and rupture took place in the same manner as failure of linear elasticity, we should say the cylinder would burst across a cross-section or open up longitudinally according as the longitudinal absolute strength R_z was < or > than $\cdot 4195$ times the transverse absolute strength R_ϕ.

A footnote on pp. 331—3 criticises with hardly sufficient severity a memoir of Virgile to which we shall refer later.

[123.] Saint-Venant (pp. 339—47) obtains similar results for the case of a spherical shell. He seeks first to find a solution of the equations (footnote p. 79 and stress-strain relations (a) of Art. 117) by taking $v = 0$, $w = 0$ and $u_\phi = 0$. This gives three equations to be satisfied which are inconsistent unless a certain relation is satisfied by the constants. Now $v = w = 0$ must for the case of uniform internal and external tractive loads be a necessary condition for change in size without distortion. Hence the equation (74) arrived at by Saint-Venant must be the condition for such a strain; it is:

$$\text{(i)} \quad \left(\frac{b-c}{e'-f'}\right)^2 + \frac{b-c}{e'-f'} = \frac{b+c+2d'-e'-f'}{a} \qquad \text{(p. 340).}$$

In this case the solution is simply

$$\text{(ii)} \quad u = Cr^{\frac{b-c}{e'-f'}}.$$

The condition (i) is however not sufficient; we find also from the surface equations that we must have

$$p_0/p_1 = (r_0/r_1)^{\frac{b-c}{e'-f'}-1} \qquad \text{(p. 342).}$$

It will be seen that without elastic isotropy in the tangent plane, it is only very special surface loads which will not produce distortion.

[124.] In § 11 (pp. 342—8) the problem of isotropy for all directions in the tangent plane is dealt with. In this case $e' = f'$, $b = c$, and stresses and constants are easily obtained by aid of the solution:

$$v = w = 0, \quad u = Cr^{n-\frac{1}{2}} + C'r^{-n-\frac{1}{2}},$$

where

$$n = \tfrac{1}{2}\sqrt{1 + 8\,\frac{b+d'-e'}{a}},$$

the body-shift equations being now reduced to the single one:

$$au_{rr} + 2au_r/r - 2\,(b + d' - e')\,u/r^2 = 0.$$

By evaluating the constants Saint-Venant obtains the following expression for u:

$$u = \frac{1}{r_1^{2n} - r_0^{2n}}\left\{\frac{p_0 r_0^{n+\frac{3}{2}} - p_1 r_1^{n+\frac{3}{2}}}{(n-\frac{1}{2})\,a + 2e'}\, r^{n-\frac{1}{2}} + \frac{p_0 r_0^{\frac{3}{2}-n} - p_1 r_1^{\frac{3}{2}-n}}{(n+\frac{1}{2})\,a - 2e'}\,(r_0 r_1)^{2n} r^{-n-\frac{1}{2}}\right\},$$

which gives the lateral stretches $s_\phi = s_\psi = u/r$ at once.

The important point in the piezometer problem is the dilatation of the spherical cavity. This is equal to $\dfrac{3u_0}{r_0}$

$$= \frac{3r_0^{n-\frac{3}{2}}}{r_1^{2n} - r_0^{2n}} \left\{ \frac{p_0 r_0^{n+\frac{3}{2}} - p_1 r_1^{n+\frac{3}{2}}}{(n-\frac{1}{2})\,a + 2e'} + \frac{p_0 r_0^{\frac{3}{2}-n} - p_1 r_1^{\frac{3}{2}-n}}{(n+\frac{1}{2})\,a - 2e'}\, r_1^{2n} \right\}.$$

We see that it involves *three* elastic coefficients, and is thus, even as an *empirical* formula, better adapted to satisfy *numerically* Regnault's experiments than Lamé's bi-constant isotropic formula obtained by putting $d' = e'$, $b = a$ and $n = 3/2$.[1] On the other hand it is physically more plausible. The constants reduce to *two*, if we suppose the body *amorphic* and of *rari-constant* elasticity ellipsoidally distributed. If we take $r_0 = r' - \epsilon/2$, $r_1 = r' + \epsilon/2$, we easily find for the mid-sphere of radius r':

$$s_\phi = s_\psi = \frac{a}{a(b + d') - 2e'^2} \frac{(p_0 - p)\,r'}{2\epsilon},$$

or in the case just mentioned

$$s_\phi = s_\psi = \frac{3}{20d} \frac{(p_0 - p_1)\,r'}{\epsilon}.$$

But $E_\phi = \dfrac{5}{2}\dfrac{fd}{e}, = \dfrac{5d}{2}$ by Art. 117 (6) if there be tangential isotropy. Hence finally:

$$s_\phi = s_\psi = \frac{3}{8E_\phi} \frac{(p_0 - p_1)\,r'}{\epsilon}.$$

[125.] The final section of the memoir is entitled: *Vase cylindrique terminé par deux calottes sphériques* (pp. 347—9). This treats a problem similar to that dealt with by Lamé in his *Note* of 1850: see our Art. 1038*. The mean lateral expansion of the spherical ends is made to take the same value as that of the cylindrical body by equating the expressions for s_ϕ obtained in our Arts. 122 and 124. Saint-Venant thus reaches a more general rule than that given by Lamé as a result of bi-constant isotropy. We have:

$$\tfrac{1}{8}\,\frac{8\sqrt{E_z} - \sqrt{E_\phi}}{E_\phi \sqrt{E_z}} \frac{r'}{\epsilon} = \tfrac{3}{8}\,\frac{1}{E_{\phi 1}} \frac{r_1'}{\epsilon_1}$$

where the subscript $_1$ refers to the spherical portions of the surface.

Hence

$$\frac{\epsilon}{\epsilon_1} = \frac{r'}{r_1'} \times \frac{8/E_\phi - 1/\sqrt{E_z E_\phi}}{3/E_{\phi 1}}.$$

In the case of the two portions being of the same *isotropic* material, we have $E_\phi = E_z = E_{\phi 1}$, or

$$\frac{\epsilon}{\epsilon_1} = \frac{7}{3} \frac{r'}{r'}.$$

[1] In Lamé's notation $a = \lambda + 2\mu$ and $e' = \lambda$: see our Art. 1093*.

This agrees with Lamé's result: see our Vol. I. p. 564. If the thicknesses are equal, the radii ought to be as 3 : 7:

ce qui est la règle indiquée par M. Lamé pour les fonds sphériques compensateurs, élevant en quelque sorte, dit-il, le système des chaudières cylindriques au rang des formes naturelles ou des solides d'égale résistance. (p. 349.)

[126.] *Sur la distribution des élasticités autour de chaque point d'un solide ou d'un milieu de contexture quelconque, particulièrement lorsqu'il est amorphe sans être isotrope; Comptes rendus,* T. LVI. 1863, pp. 475—479, p. 804. This is an abstract of the memoir published in Liouville's *Journal* in 1863: see the following article.

[127.] *Mémoire sur la distribution des élasticités autour de chaque point d'un solide ou d'un milieu de contexture quelconque, particulièrement lorsqu'il est amorphe sans être isotrope.* This memoir was presented to the Academy, March 16, 1863, and some account of it appeared in the *Comptes rendus,* see preceding article. It is printed at length in Liouville's *Journal de mathématiques,* Vol. VIII. 1863, pp. 257—95 and 353—430.

[128.] The opening pages of the memoir (257—9) as well as the concluding (425—30) entitled respectively: *Objet* and *Résumé et conclusions pratiques,* give an account of the purpose and results of the memoir. As these will sufficiently appear in our treatment of the intervening five sections (four, according to Saint-Venant, but III occurs twice by mistake), we shall not reproduce here any part of these preliminary and final remarks.

[129.] The second section is entitled: *Formules diverses où entrent les coefficients dont l'élasticité dépend. Établissement, de plusieurs manières, d'une partie souvent omise, où figurent six constantes complémentaires, qui sont les composantes des pressions pouvant exister antérieurement aux déplacements des points* (pp. 260—286).

The aim of this section may be thus expressed: Let there be an initial system of stress given by $\widehat{xx}_0$, $\widehat{yy}_0$, $\widehat{zz}_0$, $\widehat{yz}_0$, $\widehat{zx}_0$, $\widehat{xy}_0$, and let the elastic nature of the body be given by thirty-six constants $|xxxx|$, $|xyxy|$, $|xyyy|$, etc. Green has decisively determined that these thirty-six can be reduced to twenty-one by the law of

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energy: see the footnote to our Art. 117. It is desirable to obtain a proof of the elastic formulae due to Cauchy without appealing to the principle of inter-molecular action being central and a function *only* of the distance.

Subscript letters attached to the shifts u, v, w denoting fluxions, the formulae are given by the types:

$$\left.\begin{aligned} \widehat{xx} &= \widehat{xx}_0 (1 + u_x - v_y - w_z) + 2\widehat{xy}_0\, u_y + 2\widehat{zx}_0\, u_z + \widehat{xx}_1 \\ \widehat{yz} &= \widehat{yz}_0 (1 - u_x) + \widehat{yy}_0\, w_y + \widehat{zz}_0\, v_z + \widehat{zx}_0\, v_x + \widehat{xy}_0\, w_x + \widehat{yz}_1 \end{aligned}\right\} \text{........(i)},$$

where

$$\left.\begin{aligned} \widehat{xx}_1 &= |xxxx|\, s_x + |xxyy|\, s_y + |xxzz|\, s_z + |xxyz|\, \sigma_{yz} + |xxzx|\, \sigma_{zx} + |xxxy|\, \sigma_{xy} \\ \widehat{yz}_1 &= |yzxx|\, s_x + |yzyy|\, s_y + |yzzz|\, s_z + |yzyz|\, \sigma_{yz} + |yzzx|\, \sigma_{zx} + |yzxy|\, \sigma_{xy} \end{aligned}\right\} \text{(ii)},$$

while the type of resulting body-shift equation is:

$$\left.\begin{aligned} -\rho X = {} & \widehat{xx}_0 u_{xx} + \widehat{yy}_0 u_{yy} + \widehat{zz}_0 u_{zz} + 2\widehat{yz}_0 u_{yz} + 2\widehat{zx}_0 u_{zx} + 2\widehat{xy}_0 u_{xy} \\ & + |xxxx|\, u_{xx} + |xyxy|\, u_{yy} + |xzxz|\, u_{zz} \\ & + 2\,|zxxy|\, u_{yz} + 2\,|xxzx|\, u_{zx} + 2\,|xxxy|\, u_{xy} \\ & + |xxxy|\, v_{xx} + |xyyy|\, v_{yy} + |zxyz|\, v_{zz} \\ & + \{|xyyz| + |zxyy|\}\, v_{yz} + \{|xxyz| + |zxxy|\}\, v_{zx} + \{|xxyy| + |xyxy|\}\, v_{xy} \\ & + |xxzx|\, w_{xx} + |xyyz|\, w_{yy} + |zxzz|\, w_{zz} \\ & + \{|zxyz| + |xyzz|\}\, w_{yz} + \{|xxzz| + |zxzx|\}\, w_{zx} + \{|xxyz| + |xyzx|\}\, w_{xy} \end{aligned}\right\} \text{(iii)}.$$

These results representing the most general equations of elasticity for small strains were originally given by Cauchy, as is implied in our Arts. 615*, 616*, 662*, 666*. He obtained them by calculating the stresses as the sums of intermolecular actions on the rari-constant hypothesis. Saint-Venant in this section proposes to deduce them from the principle of energy (by Green's method) in a manner which will satisfy multi-constant elasticians.

[130.] The proof attempted by Saint-Venant is not legitimate, because in the expression he takes for the work the linear term

$$\widehat{xx}_0\, s_x + \widehat{yy}_0\, s_y + \widehat{zz}_0\, s_z + \widehat{yz}_0\, \sigma_{yz} + \widehat{zx}_0\, \sigma_{zx} + \widehat{xy}_0\, \sigma_{xy}$$

occurs where s_x, s_y, s_z, σ_{yz}, σ_{zx}, σ_{xy} are stretches and slides. Assuming this term correct, which it is not, these ought to be expressed to the second power of the shift-fluxions as in our Art. 1622*, for we want the work to the second power. This Saint-Venant does not do, but treats the strains s and σ as if they were the quantities ϵ_x, ϵ_y, ϵ_z, η_{yz}, η_{zx}, η_{xy} of our Art. 1619*. This mistake was pointed out by Brill and Boussinesq, and is acknowledged by Saint-Venant in a memoir of 1871: see our Art. 237. The formulae (i)—(iii) of the preceding article can thus only be considered as valid, when we accept the rari-constant hypothesis and deduce them after the manner of Cauchy. We shall see this point more clearly when dealing with the memoir of 1871. Green gets over the difficulty by expanding his work-function in powers of the ϵ's and η's; he thus gets a linear term, whose constants vanish with the initial stresses, but are not determined as functions of the initial stresses, still less does he show what functions, if any, the remaining constants are of the initial stresses.

[131.] In the course of this section Saint-Venant gives a proof of Cauchy's formulae (i) to (iii) above on the rari-constant hypothesis (footnote, pp. 273—5); he refers to the memoir of C. Neumann (*Zur Theorie der Elasticität,* Crelle, LVII, 1860, p. 281: see our Chap. XI.), where a similar method to his own is used for the case of isotropy (footnote, pp. 275—80), and to the memoir of Haughton (see our Art. 1505*) for a treatment which generalised leads to the same formulae on the rari-constant theory (p. 280 and footnote). Finally we may refer to his footnote (pp. 284—5) for a process by which the body-shift equations (iii) are deduced by means of the rari-constant hypothesis, without a previous investigation of the stresses[1].

[132.] The third section of the memoir (pp. 286—95) is entitled: *Formule symbolique générale fournissant, en fonction des coefficients d'élasticité pour des axes donnés, ceux qui sont relatifs à d'autres axes aussi donnés et rectangulaires, et, aussi, les coefficients qui doivent entrer dans l'expression d'une composante quelconque de pression même oblique.*

Saint-Venant adopts a symbolic representation of the stresses, strains and coefficients in order to express the relations among them. He thus describes this method:

On abrége singulièrement le calcul et l'on arrive à quelque chose de fort simple au moyen de notations symboliques comme celles que plusieurs auteurs anglais appellent *Sylvestrian umbrae,* parce que M. Sylvester, qui les a employées avec succès, appelle *ombres de quantités* (shadows of quantities) ces sortes de notations dont se sont servis précédemment, au reste, Cauchy et d'autres analystes (p. 290).

[1] There is a wrong reference to Rankine's paper (p. 269, footnote), it should be Vol. VI. (p. 63), not Vol. V., of the *Camb. and Dublin Math. Journal.*

There is a footnote referring to Sylvester's papers in *Camb. and Dublin Math. Journal*, Vol. VII. 1852, p. 76, and *Phil. Trans.* 1853, p. 543.

[133.] Suppose *symbolically* $\iota_{jklm} = \iota_{jk}\iota_{lm} = \iota_j\iota_k\iota_j\iota_m$ to represent $|jklm|$, where j, k, l, m are any of the letters xyz, $x'y'z'$ etc. Further $\widehat{r}\ \widehat{r'}$ to represent the stress $\widehat{rr'}$, and $\epsilon_r\epsilon_r$ or ϵ^2_r to represent s_r, and finally $2\epsilon_r\epsilon_{r'}$ to represent $\sigma_{rr'}$. Let $c_{rr'}$ denote the cosine of the angle between the directions r, r'.

We are now able to reproduce in symbolic form the following well-known typical relations:

$$\widehat{rr'} = \widehat{xx}\, c_{rx}\, c_{r'x} + \widehat{yy}\, c_{ry}\, c_{r'y} + \widehat{zz}\, c_{rz}\, c_{r'z}$$
$$+ \widehat{yz}\,(c_{ry}\, c_{r'z} + c_{rz}\, c_{r'y}) + \widehat{zx}\,(c_{rz}\, c_{r'x} + c_{rx}\, c_{r'z}) + \widehat{xy}\,(c_{rx}\, c_{r'y} + c_{ry}\, c_{r'x}) \ldots\text{(iv)},$$

$$s_x = s_{x'}\, c^2_{xx'} + s_{y'}\, c^2_{xy'} + s_{z'}\, c^2_{xz'} + \sigma_{y'z'}\, c_{xy'}\, c_{xz'} + \sigma_{z'x'}\, c_{xz'}\, c_{xx'} + \sigma_{x'y'}\, c_{xx'}\, c_{xy'} \ldots\text{(v)},$$

$$\sigma_{yz} = 2s_{x'}\, c_{yx'}\, c_{zx'} + 2s_{y'}\, c_{yy'}\, c_{zy'} + 2s_{z'}\, c_{yz'}\, c_{zz'}$$
$$+ \sigma_{y'z'}\,(c_{yy'}\, c_{zz'} + c_{yz'}\, c_{zy'}) + \sigma_{z'x'}\,(c_{yz'}\, c_{zx'} + c_{yx'}\, c_{zz'}) + \sigma_{x'y'}\,(c_{yx'}\, c_{zy'} + c_{yy'}\, c_{zx'}) \ldots\text{(vi)}.$$

See our Arts. 659* and 663*.

(The last two are most readily obtained from the stretch-quadric of Art. 612* for axes $x'y'z'$, namely:

$$s_{x'}\, x'^2 + s_{y'}\, y'^2 + s_{z'}\, z'^2 + \sigma_{y'z'}\, y'z' + \sigma_{z'x'}\, z'x' + \sigma_{x'y'}\, x'y' = \pm 1.$$

Substitute for x' its equivalent $xc_{xx'} + yc_{yx'} + zc_{zx'}$ and similar quantities for x' and y', then the coefficients of x^2 and yz will be s_x and σ_{yz} as given above.)

The symbolical forms are:

$$\widehat{xx} \text{ or } \widehat{yz} = \iota_{xx} \text{ or } \iota_{yz} \times (\iota_x\epsilon_x + \iota_y\epsilon_y + \iota_z\epsilon_z)^2 \ldots\ldots\ldots\ldots\text{(vii)},$$

whence it follows from (iv) that

$$\widehat{rr'} = (\iota_x c_{rx} + \iota_y c_{ry} + \iota_z c_{rz})\,(\iota_x c_{r'x} + \iota_y c_{r'y} + \iota_z c_{r'z}) \times (\iota_x\epsilon_x + \iota_y\epsilon_y + \iota_z\epsilon_z)^2 \ldots\text{(viii)}.$$

Further we have from (v):

$$\epsilon_x\epsilon_x = (\epsilon_{x'}c_{xx'} + \epsilon_{y'}c_{xy'} + \epsilon_{z'}c_{xz'})^2$$
$$\epsilon_y\epsilon_z = (\epsilon_{x'}c_{yx'} + \epsilon_{y'}c_{yy'} + \epsilon_{z'}c_{yz'})\,(\epsilon_{x'}c_{zx'} + \epsilon_{y'}c_{zy'} + \epsilon_{z'}c_{zz'}),$$

whence we can take

$$\epsilon_j = \epsilon_{x'}c_{jx'} + \epsilon_{y'}c_{jy'} + \epsilon_{z'}c_{jz'} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ix)}.$$

Put $j = x$, y, z successively and substitute for ϵ_x, ϵ_y, ϵ_z in (viii), we have

$$\widehat{rr'} = (\iota_x c_{rx} + \iota_y c_{ry} + \iota_z c_{rz})\,(\iota_x c_{r'x} + \iota_y c_{r'y} + \iota_z c_{r'z}) \times$$
$$\{(\iota_x c_{xx'} + \iota_y c_{yx'} + \iota_z c_{zx'})\,\epsilon_{x'} + (\iota_x c_{xy'} + \iota_y c_{yy'} + \iota_z c_{zy'})\,\epsilon_{y'}$$
$$+ (\iota_x c_{xz'} + \iota_y c_{yz'} + \iota_z c_{zz'})\,\epsilon_{z'}\}^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(x)}.$$

But we may obviously also express $\widehat{rr'}$ in the form

$$\widehat{rr'} = |rr'x'x'|\; s_{x'} + \ldots\ldots + \ldots\ldots$$
$$+ |rr'y'z'|\; \sigma_{y'z'} + \ldots\ldots + \ldots\ldots$$
$$= \iota_{r'}\iota_{r'}\,(\iota_{x'}\,\epsilon_{x'} + \iota_{y'}\,\epsilon_{y'} + \iota_{z'}\,\epsilon_{z'})^2 \ldots\ldots\ldots\ldots\ldots\ldots\text{(xi)}.$$

Comparing (x) and (xi) which must on development give the same result, we see that it is necessary to take:

$$\iota_n = \iota_x c_{nx} + \iota_y c_{ny} + \iota_z c_{nz} \text{.....................(xii)},$$

or,

$$|rr'x'y'| = \iota_r \iota_{r'} . \iota_{x'} \iota_{y'} \text{.........................(xiii)},$$

where ι_n is given by (xii), and x', y' are any two of the three new axial directions, (x', y', z'), r, r' any two directions we please, and n any arbitrary direction.

Thus we have any coefficient of one set of axes expressed in terms of those obtained for another set. The product $\iota_r \iota_{r'} . \iota_{x'} \iota_{y'}$ ought to be made in the order indicated, except that the first pair and the last pair may have their members interchanged *in themselves*. If r, r' are both axial directions (i.e. chosen from x', y', z') then the first pair may be interchanged as a whole with the last pair. If we accept the rari-constant hypothesis, however, for axial directions all interchanges of the order of the ι's will be permissible.

[134.] Saint-Venant notes one or two other symbolical results. Thus, if ϕ be Green's work-function and we suppose no initial stresses:

$$\phi = \tfrac{1}{2} \left[(\iota_x \epsilon_x + \iota_y \epsilon_y + \iota_z \epsilon_z)^2\right]^2 \text{...................(xiv)}.$$

Further the types of stress and of the general body-shift equations (i) to (iii) of our Art. 129 become on the rari-constant hypothesis:

$$\begin{Bmatrix} \widehat{xx} \\ \widehat{yz} \end{Bmatrix} = \begin{Bmatrix} \widehat{xx}_0 \\ \widehat{yz}_0 \end{Bmatrix} (1 - u_x - v_y - w_z)$$

$$+ \left(\widehat{x}_0 \frac{d}{dx} + \widehat{y}_0 \frac{d}{dy} + \widehat{z}_0 \frac{d}{dz}\right) \begin{Bmatrix} 2\widehat{x}_0 u \\ \widehat{y}_0 w + \widehat{z}_0 v \end{Bmatrix}$$

$$+ \begin{Bmatrix} \iota_{xx} \\ \iota_{yz} \end{Bmatrix} (\iota_x \epsilon_x + \iota_y \epsilon_y + \iota_z \epsilon_z)^2 \text{............................(xv)},$$

$$-\rho X = \left(\widehat{x}_0 \frac{d}{dx} + \widehat{y}_0 \frac{d}{dy} + \widehat{z}_0 \frac{d}{dz}\right)^2 u$$

$$+ \iota_x \left(\iota_x \frac{d}{dx} + \iota_y \frac{d}{dy} + \iota_z \frac{d}{dz}\right) \left(\iota_x \frac{d}{dx} + \iota_y \frac{d}{dy} + \iota_z \frac{d}{dz}\right) (\iota_x u + \iota_y v + \iota_z w) \text{ (xvi)}.$$

[135.] The next section, III *bis* (pp. 353—380), contains some very interesting and important matter. It is entitled: *Surfaces donnant la distribution des élasticités autour d'un même point.—Maxima et minima—Distribution ellipsoïdale des élasticités directes.—Solides ou milieux amorphes.—Intégrabilité des équations.*

Some of the results had already been given by Rankine in his memoir: *On Axes of Elasticity and Crystalline Forms, Phil. Trans.* 1856, pp. 261—85, but there is much that is new and the method is very good.

[136.] The relation (xiii) gives for $|rrrr|$ the value

$$|rrrr| = [(\iota_x c_{rx} + \iota_y c_{ry} + \iota_z c_{rz})^2]^2 \qquad\text{(xvii)},$$

or the direct stretch coefficient in direction r, (c_{rx}, c_{ry}, c_{rz}), in terms of the system of elastic coefficients for the axes x, y, z.

If we put

$$x = c_{rx}/\sqrt[4]{|rrrr|}, \quad y = c_{ry}/\sqrt[4]{|rrrr|}, \quad z = c_{rz}/\sqrt[4]{|rrrr|},$$

and substitute, we obtain the surface

$$1 = \{(\iota_x x + \iota_y y + \iota_z z)^2\}^2,$$

which expanded gives us Rankine's *tasinomic* quartic :

$$\left.\begin{aligned}1 = |xxxx|\, x^4 &+ |yyyy|\, y^4 + |zzzz|\, z^4 \\ &+ 2\,\{|yyzz| + 2\,|yzyz|\}\, y^2z^2 + 2\,\{|zzxx| + 2\,|zxzx|\}\, z^2x^2 + 2\,\{|xxyy| \\ &\qquad + 2\,|xyxy|\}\, x^2y^2 \\ &+ 4\,\{|xxyz| + 2\,|zxxy|\}\, x^2yz + 4\,\{|yyzx| + 2\,|xyyz|\}\, y^2zx \\ &\qquad + 4\,\{|zzxy| + 2\,|yzzx|\}\, z^2xy \\ &+ 4\,|yyyz|\, y^3z + 4\,|zzzy|\, z^3y + 4\,|zzzx|\, z^3x + 4\,|xxxz|\, x^3z \\ &\qquad + 4\,|xxxy|\, x^3y + 4\,|yyyx|\, y^3x\end{aligned}\right\} \quad\text{(xviii)}.$$

This equation with its fifteen *homotatic* coefficients was first given by Haughton in his memoir of 1846. These 15 coefficients are the 15 coefficients of *rari-constancy* multiplied by the numbers 1, 6, 12 or 4, so that the expressions for the work, stresses etc., can on that hypothesis be given in terms of the coefficients of this equation.

Its fundamental property is that the direct-stretch coefficient in any direction varies inversely as the fourth power of the corresponding ray.

[137.] Paragraphs 10 and 11 together with the footnote pp. 359—62 reproduce results of Rankine and Haughton with regard to the nature of the elastic coefficients. Thus it is pointed out :

(i) That there are sixteen directions real or imaginary for which $|rrrr|$ is a maximum or minimum. These directions cut the tasinomic surface at right-angles, and possess the peculiarity that any stretch in their direction produces a *traction* only across a plane normal to their direction (pp. 356—7).

(ii) That if we take

$$|x'x'x'x'| + |x'x'y'y'| + |x'x'z'z'| = S_{x'},$$

or $S_{x'}$ equal to the sum of direct- and cross-stretch coefficients for the direction x', then

$$S_{x'} = (\iota_x c_{x'x} + \iota_y c_{x'y} + \iota_z c_{x'z})^2\,(\iota_{xx} + \iota_{yy} + \iota_{zz}).$$

Thus $S_{x'}$ varies inversely as the square of the ray of the ellipsoidal surface:

$$1 = (\iota_{xx} + \iota_{yy} + \iota_{zz})\,(\iota_x x + \iota_y y + \iota_z z)^2,$$

which developed gives us:

$$1 = S_x x^2 + S_y y^2 + S_z z^2 + 2R_{yz}yz + 2R_{zx}zx + 2R_{xy}xy,$$

where $$R_{nm} = \iota_{xxnm} + \iota_{yynm} + \iota_{zznm} \quad\text{..................(xix).}$$

This is the ellipsoid discovered by Haughton in 1846 and termed by Rankine *orthotatic*. It shews us that by a suitable change of axes we can put $R_{yz} = R_{zx} = R_{xy} = 0$, which give three inter-constant relations, and so reduce the 21 (or 15) elastic constants to 18 (or 12).

(iii) That if an equal stretch s be given in the three orthotatic directions {i.e. those of the axes of the ellipsoid (xix)} this stretch system will produce no shear, for if x_1, y_1, z_1 be these orthotatic directions:

$$\widehat{y_1z_1} = |y_1z_1x_1x_1|\,s + |y_1z_1y_1y_1|\,s + |y_1z_1z_1z_1|\,s \quad \text{(from Equation (ii) of Art. 129).}$$
$$= R_{y_1z_1}\,s = 0.$$

The orthotatic directions are thus those for which the sum of the corresponding (direct and cross) slide-stretch coefficients vanish.

(iv) That a body may possess orthotatic isotropy, or $R_{x'y'} = 0$ for all rectangular systems x', y', z'. The orthotatic surface now becomes a sphere or $S_x = S_y = S_z$. Such a body however does not possess complete elastic isotropy.

(v) That there exists a surface which measures the difference D between a cross-stretch and direct-slide coefficient, i.e.

$$D = |y'y'z'z'| - |y'z'y'z'|.$$

This is Rankine's *heterotatic* surface, and is given by

$$\left.\begin{aligned} D = \{|yyzz| - |yzyz|\}\,c^2_{xx'} + \{|zzxx| - |zxzx|\}\,c^2_{yx'} + \{|xxyy| - |xyxy|\}\,c^2_{zx'} \\ + 2\,\{|xxyz| - |zxxy|\}\,c_{yx'}c_{zx'} + 2\,\{|yyzx| - |xyyz|\}\,c_{zx'}c_{xx'} \\ + 2\,\{|xxyz| - |yzzx|\}\,c_{xx'}c_{yx'} \end{aligned}\right\}\text{(xx).}$$

The thorough-going rari-constant elastician will fail to observe the existence of this surface, at least the Ossa of his multi-constant colleague will appear to him a wart.

(vi) Finally that there exist nine axes at each point of a body for which

$$|y_1y_1y_1z_1| = |z_1z_1z_1y_1|,$$

or the two direct-slide-stretch coefficients are equal. These directions Rankine terms *metatatic*. The condition for the metatatic isotropy of a body, or for metatatism in all pairs of rectangular directions, is

$$|y'y'z'z'| + 2\,|y'z'y'z'| = \tfrac{1}{2}\,\{|y'y'y'y'| + |z'z'z'z'|\} \quad\text{...............(xxi).}$$

Such a body, however, is not elastically isotropic[1].

[1] I have here introduced some portion of Rankine's work as given with great clearness by Saint-Venant in order that it may be the more easy to refer to these results in later articles.

[138.] Saint-Venant in the twelfth paragraph of this section of his memoir (pp. 360—5) treats the case in which the elastic material has three rectangular planes of symmetry. This reduces the 21 coefficients to nine, for all the stretch-slide coefficients and cross-slide coefficients (i.e. Rankine's *asymmetrical elasticities*) must now vanish.

$$\text{Let}\quad \left.\begin{array}{lll} a, b, c & \text{be the} & \text{direct-stretch} \\ d, e, f & ,, & \text{direct-slide} \\ d', e', f' & ,, & \text{cross-stretch} \end{array}\right\} \text{coefficients.}$$

Then the tasinomic surface (xviii) becomes:

$$1 = ax^4 + by^4 + cz^4 + 2\,(2d + d')\,y^2z^2 + 2\,(2e + e')\,z^2x^2 + 2\,(2f + f')\,x^2y^2 \ldots \text{(xxii)}.$$

The maximum-minimum values of $|rrrr|$ are now sought and are found to lie in the three axial directions x, y, z, and in pairs of others lying in each plane yz, zx, xy, or 9 in all. The first three solutions are always real; the second six will be imaginary, since the ratio of their direction-cosines become imaginary, when

$$\left.\begin{array}{l} 2d + d' \\ 2e + e' \\ 2f + f' \end{array}\right\} \text{lie between} \left\{\begin{array}{l} b \text{ and } c \\ c \text{ and } a \\ a \text{ and } b \end{array}\right\} \text{respectively} \ldots\ldots \text{(xxiii)}.$$

Saint-Venant remarks that the conditions (xxiii) are those for the *gradual variation in one sense* of the stretch-coefficients in the three principal planes of elastic symmetry—a physical characteristic, he holds, probably possessed by all natural bodies.

[139.] In the following section we have the statement of the conditions for *ellipsoidal* elasticity, i.e. that the first three quantities of (xxiii) be respectively equal: (i) to the arithmetic, or (ii) to the geometric mean of the corresponding second three quantities of (xxiii). In either case the direct-stretch coefficient $|rrrr|$ can be represented by the ray of an ellipsoid. In the first case the direct-stretch coefficient varies as the inverse square of the ray of the ellipsoid:

$$1 = ax^2 + by^2 + cz^2;$$

and in the second case as the inverse fourth power of the ray of the ellipsoid:

$$1 = x^2\sqrt{a} + y^2\sqrt{b} + z^2\sqrt{c}.$$

The practical application of this ellipsoidal distribution has been discussed by Saint-Venant in the annotated *Clebsch:* see our analysis of that work in Arts. 307 to 313.

[140.] The next two paragraphs (pp. 367—72) are occupied with an extension of Lamé's solution of the equations of elastic equilibrium by means of *potential functions:* see our Arts. 1061*—3*.

On the rari-constant hypothesis we should have $d = d'$, $e = e'$ and $f = f'$. As a sop to Cerberus Saint-Venant assumes that

$$d'/d = e'/e = f'/f = i \qquad \text{(xxiv)}.$$

We may, however, doubt whether Cerberus would accept this sop; for, while supposing the constants unequal, it yet assumes their inequality isotropic in character. If multi-constancy really does exist, the relations (xxiv) are still probably very approximately satisfied for many bodies: see our Arts. 149 and 310.

Writing

$$a/(2+i) = \mathrm{a}^2, \quad b/(2+i) = \mathrm{b}^2, \quad c/(2+i) = \mathrm{c}^2,$$

and supposing ellipsoidal distribution of the second kind, Saint-Venant finds

$$f = f'/i = \mathrm{ab}, \quad d = d'/i = \mathrm{bc}, \quad e = e'/i = \mathrm{ca}.$$

This enables him to reduce his body-shift equations to the type

$$\mathrm{a}u_{xx} + \mathrm{b}u_{yy} + \mathrm{c}u_{zz} + (1+i)\frac{d\phi}{dx} = 0,$$

where

$$\phi = \mathrm{a}u_x + \mathrm{b}v_y + \mathrm{c}w_z \qquad \text{(xxv)}.$$

A very straightforward analysis then leads him to the result:

$$\phi = \iiint f(\alpha, \beta, \gamma)\left\{\frac{(x-\alpha)^2}{\mathrm{a}} + \frac{(y-\beta)^2}{\mathrm{b}} + \frac{(z-\gamma)^2}{\mathrm{c}}\right\}^{-\frac{1}{2}} d\alpha\, d\beta\, d\gamma \qquad \text{(xxvi)}.$$

He also obtains (p. 371) the shift-type:

$$u = \iiint \chi_1(\alpha, \beta, \gamma)\left\{\frac{(x-\alpha)^2}{\mathrm{a}} + \frac{(y-\beta)^2}{\mathrm{b}} + \frac{(z-\gamma)^2}{\mathrm{c}}\right\}^{\frac{1}{2}} d\alpha\, d\beta\, d\gamma \qquad \text{(xxvii)},$$

where v and w will have other arbitrary functions χ_2, χ_3.

These arbitrary functions χ_1, χ_2, χ_3 do not seem to me so arbitrary as the reader might assume from Saint-Venant's words. We have so to choose χ_1, χ_2, χ_3 that the value of ϕ obtained from (xxv) by means of (xxvii) shall be the same as that obtained for ϕ from (xxvi).

It appears to me that u, v, w ought to be the x-, y-, z-fluxions respectively of a quantity

$$\psi = \tfrac{1}{2}\iiint f(\alpha, \beta, \gamma)\sqrt{\frac{(x-\alpha)^2}{\mathrm{a}} + \frac{(y-\beta)^2}{\mathrm{b}} + \frac{(z-\gamma)^2}{\mathrm{c}}}\, d\alpha\, d\beta\, d\gamma \qquad \text{(xxviii)}.$$

In addition we might add to them certain expressions arising from the twists and giving a zero value for ϕ.

[141.] In the following paragraph Saint-Venant shows that the ellipsoidal conditions of the type $(2d + d') = \sqrt{bc}$ are necessary if a solution in terms of *direct and inverse potentials* is obtainable (pp. 372—4).

[142.] Hitherto the set of ellipsoidal conditions of the type

$$2d + d' = \sqrt{bc}$$

has been seen as one only of the number which satisfies the relations (xxiii). Saint-Venant now attempts to give it a far more important and special physical meaning. Namely, he proceeds to show that these relations hold exactly or very closely for bodies which originally isotropic have afterwards received a permanent strain unequal in different directions. He describes the bodies in question in the following terms:

En effet, dans les corps à cristallisation confuse tels que les métaux, etc., employés dans les constructions, où les molécules affectent indistinctement toutes les orientations, si les élasticités sont égales dans trois directions rectangulaires, elles doivent l'être en tous sens, car on ne voit aucune raison pour qu'elles soient plus grandes ou moindres dans les autres directions. Si les élasticités y sont inégales, cela ne peut tenir qu'à des rapprochements moléculaires plus grands dans certains sens que dans d'autres, par suite du forgeage, de l'étirage, du laminage, etc., ou des circonstances de la solidification. Calculons les grandeurs nouvelles que doivent prendre les coefficients d'élasticité dans un corps primitivement isotrope ainsi modifié (p. 374).

Bodies with 'confused crystallisation' Saint-Venant terms *amorphic solids,* and he now proceeds to show that within certain limits of aeolotropy, they possess an ellipsoidal distribution of elasticity. He assumes that the bodies have *rari-constant* elasticity.

[143.] Let s, s', s'' be the *principal stretches* of the permanent set given to the body, let ρ_0, r_0, x_0, y_0, z_0 be the density, distance between two elements, and its projections on the directions of the principal stretches before the isotropy is altered. Then if ρ, r, x, y, z be the value of these quantities after aeolotropy is produced, we have

$$\left.\begin{array}{c} x = x_0(1+s), \quad y = y_0(1+s'), \quad z = z_0(1+s'') \\ \rho = \rho_0 / \{\overline{1+s} \,.\, \overline{1+s'} \,.\, \overline{1+s''}\} \end{array}\right\} \text{.......} \quad \text{(xxix).}$$

Let $f(r)$ be the law of intermolecular action, and $F(r) = \dfrac{1}{r}\dfrac{d}{dr}\left\{\dfrac{f(r)}{r}\right\}$, then we have, m being the mass of a molecule:

$$\left\{\begin{array}{c} |xxxx| \\ |yyyy| \\ |xyxy| \end{array}\right\} = \frac{\rho}{2} \Sigma\, mF(r) \left\{\begin{array}{c} x^4 \\ y^4 \\ x^2y^2 \end{array}\right\} \text{................} \quad \text{(xxx).}$$

These results flow at once from the definition of stress on the rari-constant hypothesis and had been given by Cauchy in 1829 (see for example the annotated *Leçons de Navier*, p. 570, footnote and our Art. 615*).

Further if $r - r_0$ be small, we have:

$$F(r) = F(r_0) + (r - r_0) F'(r_0),$$

$$r - r_0 = \frac{x_0^2}{r_0} s + \frac{y_0^2}{r_0} s' + \frac{z_0^2}{r_0} s''.$$

In the case of primitive isotropy we have

$$\left.\begin{aligned}
&\frac{\rho_0}{2} \Sigma\, m\, F(r_0)\, x_0^4 = \frac{\rho_0}{2} \Sigma\, m\, F(r_0)\, y_0^4 = c_4, \text{ say,}\\
&\frac{\rho_0}{2} \Sigma\, m \frac{F'(r_0)}{r_0} x_0^6 = \frac{\rho_0}{2} \Sigma\, m \frac{F'(r_0)}{r_0} y_0^6 = c_6, \text{ say,}\\
&\frac{\rho_0}{2} \Sigma\, m \frac{F'(r_0)}{r_0} \{x_0^4 y_0^2, \text{ or } x_0^4 z_0^2, \text{ or } y_0^4 z_0^2, \text{ or } y_0^4 x_0^2\} \text{ are all equal}\\
&\qquad\qquad\qquad\qquad\qquad = c_{4,2}, \text{ say.}
\end{aligned}\right\} \text{(xxxi).}$$

We will also put

$$\frac{\rho_0}{2} \Sigma\, m\, F(r_0)\, x_0^2 y_0^2 = c_{2,2},$$

$$\frac{\rho_0}{2} \Sigma\, m \frac{F'(r_0)}{r_0} x_0^2 y_0^2 z_0^2 = c_{2,2,2}.$$

Now substitute from (xxix) in (xxx) and using these values, we find

$$\left.\begin{aligned}
|xxxx| &= \frac{(1+s)^3}{(1+s')(1+s'')} c_4 + c_6 s + c_{4,2} s' + c_{4,2} s'',\\
|yyyy| &= \frac{(1+s')^3}{(1+s)(1+s'')} c_4 + c_{4,2} s + c_6 s' + c_{4,2} s'',\\
|xyxy| &= \frac{(1+s)(1+s')}{1+s''} c_{2,2} + c_{4,2} s + c_{4,2} s' + c_{2,2,2} s''.
\end{aligned}\right\} \ldots\text{(xxxii).}$$

Now there are certain relations holding between the constants c, which are easily found thus: Change the axis of x by linear transformation:

$$x_0' = \alpha x_0 + \beta y_0 + \gamma z_0, \text{ where } \alpha^2 + \beta^2 + \gamma^2 = 1,$$

then from the initial isotropy we have

$$\Sigma\, m\, \chi(r_0)\, x_0^4 = \Sigma\, m\, \chi(r_0)\, x_0'^4,$$

and

$$\Sigma\, m\, \chi(r_0)\, x_0^6 = \Sigma\, m\, \chi(r_0)\, x_0'^6,$$

where $\chi(r_0)$ is any function of r_0 and α, β, γ may be any direction-cosines we please; it follows that:

$$\left.\begin{aligned}
(\alpha^2+\beta^2+\gamma^2)^2\, \Sigma\, m\, \chi(r_0)\, x_0^4 &= \Sigma\, m\, \chi(r_0)\, (\alpha x_0 + \beta y_0 + \gamma z_0)^4\\
(\alpha^2+\beta^2+\gamma^2)^3\, \Sigma\, m\, \chi(r_0)\, x_0^6 &= \Sigma\, m\, \chi(r_0)\, (\alpha x_0 + \beta y_0 + \gamma z_0)^6
\end{aligned}\right\} \ldots\text{(xxxiii).}$$

These must be identities as they are true for all values of α, β, γ.

Hence we may equate like powers of α, β, γ on both sides. In the first relation by equating the coefficients of β^4 and again of $\alpha^2\beta^2$ we deduce the first of relations (xxxi) and also

$$\Sigma\, m\chi(r_0)\, x_0^{\,4} = 3\,\Sigma\, m\chi(r_0)\, x_0^{\,2} y_0^{\,2},$$

or

$$c_4 = 3\, c_{2,2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxxiv)}.$$

In the second relation by equating the coefficients of $\alpha^4\beta^2$, $\alpha^4\gamma^2$, $\beta^4\gamma^2$ and $\beta^4\alpha^2$ we obtain the third of relations (xxxi), as well as the new one

$$c_6 = 5c_{4,2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxxv)}.$$

By equating the coefficients of β^6 we reach the second of relations (xxxi); and by equating those of $\alpha^2\beta^2\gamma^2$ the new one:

$$c_6 = 15c_{2,2,2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxxvi)}.$$

From these relations[1] among the c's we have by multiplying out the first two expressions of (xxxii) *and neglecting the products of* s, s', s'',

$$|xxxx| \times |yyyy| = 9\left\{\frac{(1+s)^2(1+s')^2}{(1+s'')^2}\, c_{2,2}^{\,2} + c_{2,2}\, c_{2,2,2}\,(6s + 6s' + 2s'')\right\}$$
$$= 9\,|xyxy|^2.$$

This is the required type of relation on the hypotheses of *rari-constancy* and *small* permanent strain.

[144.] With regard to the latter assumption Saint-Venant remarks that the terms neglected can only produce very small errors:

> ...si l'on considère que les écrouissages et la trempe, qui changent très-sensiblement la ténacité et les coefficients d'élasticité, altèrent à peine la densité des corps. On peut d'ailleurs s'assurer, par un calcul, que les portions ainsi négligées de l'expression de $3|xyxy|$ sont constamment comprises entre les portions correspondantes de celles de $|xxxx|$ et $|yyyy|$, en sorte qu'en supposant même qu'elles altèrent légèrement les valeurs absolues de ces trois coefficients, elles n'altèreront pas sensiblement pour cela la relation de moyenne proportionnalité de $3|xyxy|$ entre $|xxxx|$ et $|yyyy|$, donnée par les termes du premier ordre en s, s', s'' (p. 379).

The calculation mentioned is made by Saint-Venant in a foot-note pp. 379—81.

The other assumption that rari-constancy holds for isotropy seems very approximately, if indeed not absolutely, true in the

[1] Saint-Venant obtains these relations among the c's by appealing to a general principle given by Cauchy in his *Nouveaux Exercices*, Prague, 1835, p. 35. It amounts to replacing 4 or 6 in (xxxiii) by the general index $2n$ and then equating general terms.

case of metals. We may then, I think, very legitimately adopt the ellipsoidal distribution indicated by the relations

$$2d + d' = \sqrt{bc},\quad 2e + e' = \sqrt{ca},\quad 2f + f' = \sqrt{ab} \quad \ldots\text{(xxxvii)}$$

together with rari-constancy $d/d' = e/e' = f/f' = 1$ for most cases of worked metal such as is used in constructions.

[145.] The fourth section of the memoir (pp. 381—414) is entitled: *Conséquence, en ce qui regarde la théorie du mouvement de la lumière dans les milieux non isotropes, en tenant compte des pressions antérieures aux vibrations excitées.*

This section more properly belongs to the history of physical optics, and I shall content myself here with referring to its chief points without reproducing the analysis.

[146.] In the first place Saint-Venant refers to Green's memoir of 1839 (see our Arts. 917—18*), and states the conditions Green thinks needful in the optical medium which doubly refracts. These conditions in our notation are:

$$\left.\begin{aligned} |xxxx| = |yyyy| = |zzzz| = 2\,|yzyz| + |yyzz| &= 2\,|zxzx| + |zzxx| \\ &= 2\,|xyxy| + |xxyy| \\ |yyyz| = |zzzy| = |zzzx| = |xxxz| = |xxxy| = |yyyx| &= 0 \\ |xxyz| + 2\,|zxxy| = |yyzx| + 2\,|xyyz| = |zzxy| + 2\,|yzzx| &= 0 \end{aligned}\right\}\text{(xxxviii)}.$$

They are obtained on the hypotheses of multi-constancy, of what Green terms extraneous pressures,—but Saint-Venant better initial stresses (*pressions antérieures*),—and finally of transverse vibrations being always accurately in the front of the wave. These conditions are practically identical with those obtained by Lamé: see our Art. 1106*.

[147.] Saint-Venant asserts that these conditions involve the isotropy of the medium in question, and therefore destroy the possibility of double refraction. If we suppose rari-constancy they are of course the conditions for isotropy,—does this however remain true in the case of multi-constancy?

Glazebrook in his *Report on Optical Theories* (*British Association Report*, 1885), p. 171, holds that Saint-Venant's criticism fails to reach Green. Let us endeavour briefly to indicate the lines of Saint-Venant's attack.

On pp. 384—393 he shews that Green's conditions flow from the hypotheses with which he has started. He then proves:

(i) From the tasinomic relation that the stretch-coefficient is the same for every direction or

$$|x'x'x'x'| = |xxxx|.$$

Thus an equal stretch always produces the same element in the traction whatever its direction.

(ii) That the second set of Green's conditions are fulfilled for all axes, i.e.

$$|y'y'y'z'| = |z'z'z'y'| = 0, \text{ etc.}$$

(iii) That the conditions whose type is

$$|z'z'z'z'| = 2\,|y'z'y'z'| + |y'y'z'z'|$$

are true for any change of rectangular axes.

(iv) That the third set of conditions of the type

$$|x'x'y'z'| + 2\,|z'x'x'y'| = 0$$

are also true for any change of rectangular axes.

(v) That the reciprocal theorems are true, i.e. if any one of the relations in (i) to (iv) hold for all rectangular axes, then Green's fourteen conditions follow.

It will thus be noted that Green's conditions are not based upon any conception of direction in the body, if fulfilled for one set of rectangular axes they are fulfilled for all. So far as these conditions are concerned the body possesses isotropy of direction, i.e. there is nothing of the nature of crystalline axes, or *the peculiarity of the medium has no relation to direction in space.* This seems to me the element of isotropy in Green's conditions which Glazebrook misses, and which Saint-Venant overstates when he identifies it with absolute elastic isotropy. Glazebrook well points out that if we give a stretch s_x only we have the following system of stresses[1]:

$$\widehat{xx} = |xxxx|\, s_x, \qquad \widehat{yz} = |yzxx|\, s_x,$$
$$\widehat{xy} = |xxyy|\, s_x, \qquad \widehat{zx} = 0,$$
$$\widehat{zz} = |xxzz|\, s_x, \qquad \widehat{xy} = 0.$$

Here we are at liberty to take the stretch in the direction of the axis

[1] By choosing as our axes the *orthotatic axes* we can reduce the stress-strain relations as given by Green to the following types:

$$\widehat{xx} = a\theta - 2fs_y - 2es_z,$$
$$\widehat{yz} = d\sigma_{yz},$$

where $a = |xxxx| =$ same for all directions

$$\left.\begin{array}{l} d = |yzyz| \\ e = |zxzx| \\ f = |xyxy| \end{array}\right\} = \text{values for orthotatic axes of direct-slide coefficients.}$$

of x, because of the directional isotropy of Green's conditions. It follows that such a stretch produces no shear on a face perpendicular to its direction. Glazebrook notes that it does produce a shear $\widehat{yz}$, and that this shear together with the tractions $\widehat{yy}$, $\widehat{zz}$ may be functions of the direction, since Green's conditions do not involve

$$|xxyy|, \quad |xxzz|, \quad \text{and} \quad |yzxx|$$

being the same for all systems of rectangular axes.

But is this the system of stresses we should expect to find in the ether in a crystallised medium? It seems to me physically very improbable, but it is best to let Saint-Venant speak himself, only remarking that the reader will do well to understand by Saint-Venant's use of the word isotropy, the independence of Green's conditions of all sense of direction, as explained above:

Il en résulte que l'exacte *transversalité* des mouvements moléculaires, ou leur parallélisme à des ondes de toutes les directions dans un milieu transparent, exige une foule de conditions qu'on ne voit remplies que dans les corps isotropes. On remarque, surtout, que non-seulement *une dilatation $s_{x'}$ ne produit qu'une pression exactement normale $\widehat{x'x'}$*, ou aucune composante tangentielle de pression sur une face qui est perpendiculaire à sa direction $(|x'y'x'x'| = |x'z'x'x'| = 0)$ et, aussi, *qu'un glissement sur une face n'y engendre jamais que des composantes tangentielles* $(|x'x'x'y'| = 0)$, mais encore qu'*en tout sens*, ou quelle que soit la direction x' dans ce milieu, une égale dilatation $s_{x'}$ y produit une pression d'égale intensité $\widehat{x'x'}$ $(|x'x'x'x'|$ constant).

Or une pareille égalité est contraire à toutes les idées qu'on peut se former, d'après les faits, des corps doués de la double réfraction. Ils sont cristallisés sous des formes polyédriques non régulières et variées; ils offrent des clivages suivant certaines directions; ils sont, en un mot, d'une contexture essentiellement inégale dans les divers sens, et qui doit, tout porte à le faire présumer, rendre inégaux les rapports $\widehat{x'x'}/s_{x'} = |x'x'x'x'|$ des pressions dans l'éther dont ils sont imprégnés, aux petites dilatations qui les engendrent, et rendre les pressions obliques aux dilatations, excepté pour certains sens principaux. Cette présomption est changée en certitude, si l'on considère la biréfringence artificiellement produite *par une compression* donnée dans un seul sens, ou inégalement dans plusieurs, à un corps amorphe primitivement isotrope et uni-réfringent, tel que le verre. On a en effet calculé, au no. xxxii (equation of our Art. 143), l'inégalité des coefficients $|xxxx|$, $|yyyy|$ due à l'inégalité des rapprochements moléculaires dans les sens x et y. Ce calcul était fondé, il est vrai, sur les expressions (equation xxx) assignées aux deux coefficients par l'analyse des actions s'exerçant entre les points matériels suivant leurs lignes de jonction deux à deux, et proportionnellement à une fonction de leur distance. Mais quelque motif qu'on puisse s'alléguer de révoquer en doute cette grande loi qui ne préjuge pourtant rien quant à la forme de la fonction, et quelque chose qu'on puisse concevoir à sa place, il est impossible de ne point convenir que l'inégal rapprochement moléculaire en divers sens doit influer sur la grandeur des *élasticités directes* $|xxxx| = \widehat{xx}/s_x$ comme elle influe bien certainement sur celle des autres élasticités, dites *latérales*, $|xxyy| = \widehat{xx}/s_y$, ou tangentielles, $|xyxy| = \widehat{xy}/\sigma_{xy}$, etc., puisque sans les inégalités au moins de celles-ci en divers sens, les formules ne donneraient pas de double réfraction. Un milieu ne peut être élastique et vibrant si ses parties n'agissent pas les unes sur les autres, et quelque soit le mode de leur action, il n'est pas possible d'imaginer qu'elles engendrent des élasticités directes parfaitement égales, lorsqu'il y a une inégalité de contexture qui rend inégales les élasticités latérales ou tangentielles. (pp. 396—8.)

This argument seems to me of great weight (see, however, a point raised in our Art. 193 (1)), and would incline me to reject Green's conditions (especially when we remember that Green himself supposed the ether-density to vary in refracting media), even were there no other grounds for questioning his hypotheses.

[148.] Saint-Venant now proceeds to deduce the *exact* wave-surface of Fresnel on the supposition that the vibrations are not accurately in the wave-front. He does this on the lines of Cauchy's memoir of 1830, but he does not assume rari-constancy and in many respects his method is an improvement on Cauchy's. This leads him to the following inter-constant conditions; the structure of the ether being supposed to have three planes of symmetry and thus its elasticity to be represented by the nine constants of our Art. 117 (a):

$$\left.\begin{array}{l}(b-d)(c-d)=(d+d')^2,\ (c-e)(a-e)=(e+e')^2\\(a-f)(b-f)=(f+f')^2\\(a-e)(b-f)(c-d)+(a-f)(b-d)(c-e)\\\qquad\qquad\qquad\qquad=2(d+d')(e+e')(f+f')\end{array}\right\}\text{...(xxxix).}$$

If the relations (xxxix) are satisfied we shall have Fresnel's wave-surface. If we make $a=b=c$ we shall reduce these conditions to Green's, which are thus only a particular case of those of Cauchy and Saint-Venant. (pp. 398—406.)

[149.] On pp. 406—411 Saint-Venant demonstrates that the relations (xxxix) give practically the same results as the ellipsoidal distribution of (xxxvii). He supposes $d/d'=i$ and then solves the first equation of both sets (xxxix) and (xxxvii) for d; let the values so obtained be respectively d_1 and d_2. Then by a numerical calculation we reach the following results:

If b/c varies from 1·1 to 1·5, then for values of i between $\frac{1}{2}$ and 2 the ratio of d_1/d_2 always lies between ·98641 and ·99962. In other words whatever the multi-constant i is between these limits, the relations (xxxix) and (xxxvii) give practically the same value of d.

Thus the Cauchy-Saint-Venant conditions correspond closely to the ellipsoidal distribution, which is the distribution we should expect in a body like the ether originally isotropic, but, owing to its presence in the doubly-refracting medium, subjected to an initial state of strain.

The fourth condition of (xxxix) is shewn to be very nearly true if the first three are satisfied (pp. 409—411).

[150.] The objections to Saint-Venant's theory are given by Glazebrook (*op. cit.* pp. 172—3). They consist in: the difficulty of reconciling the theories of double refraction and reflexion so long as we suppose the latter to depend "on difference of density and not of rigidity in the two media," and the existence of the "quasi-normal wave." The latter objection is met by Saint-Venant with the arguments of Cauchy (see his pp. 411—13), and it does not seem insuperable; the former is in some respects serious, and is not discussed by Saint-Venant. At the same time we must observe that the ellipsoid-distribution to which the Cauchy-Saint-Venant conditions approximate does suppose a change in the elastic constant $|yzyz|$ owing to the isotropic ether being rendered aeolotropic in the doubly-refracting medium: see our Art. 143, equation xxxii.

The whole subject is of peculiar interest apart from its bearing on the theory of light, as tending to introduce us by means of the elastic constants into the molecular laboratory of nature—indeed this is the transcendent merit of rari-constancy, if it were only once satisfactorily established!

[151.] Saint-Venant's fifth section (pp. 414—425) is entitled: *Distribution, en divers sens, des modules ou coefficients d'élasticité définis à la manière de Young et de Navier.* This is the determination of the stretch-modulus quartic as first given by Neumann (see our Art. 799*). It is shewn how this may be determined for multi-constancy, but it is pointed out that in the most general

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case there will be a denominator of 720 terms in the constants, and Saint-Venant wisely contents himself with the case of three planes of symmetry and a 9-constant medium.

The conclusions drawn as to the nature of the quartic and its special reduction to an ellipsoid, are all treated with somewhat fuller detail in the annotated *Clebsch,* and we have accordingly discussed them in our analysis of that work: see our Arts. 308 to 310.

[152.] We may note that Saint-Venant (pp. 424—5) attempts to apply the ellipsoidal distribution of elasticity, which leads to the ellipsoidal distribution of stretch-modulus, i.e.

$$\frac{1}{\sqrt{E_r}} = \frac{c^2_{xr}}{\sqrt{E_x}} + \frac{c^2_{yr}}{\sqrt{E_y}} + \frac{c^2_{zr}}{\sqrt{E_z}},$$

to the case of *wood.* He appeals to Hagen's results (see our Art. 1229*) and compares Hagen's empirical formula

$$\frac{1}{E_r} = \frac{c^3_{xr}}{E_x} + \frac{c^3_{yr}}{E_y} \quad\ldots\ldots\ldots\ldots(\alpha)$$

with that given by the ellipsoidal distribution

$$\frac{1}{\sqrt{E_r}} = \frac{c^2_{xr}}{\sqrt{E_x}} + \frac{c^2_{yr}}{\sqrt{E_y}} \quad\ldots\ldots\ldots\ldots(\beta).$$

He shews the theoretical impossibility of Hagen's formula, arising from the fact that if $E_x = E_y$, E_r is not equal to them, and endeavours to shew by curves that (β) and (α) coincide within the limits of experimental error. By graphical representation of the curves it is seen that only the ellipsoidal distribution gives anything like a satisfactory theoretical as well as practical figure, and Saint-Venant concludes that, although proved for a different kind of medium (see our Arts. 142 and 144), it may be practically of use in the case of fibrous material like wood. Later Saint-Venant saw occasion to alter this opinion; he treats this important material very fully in the *Leçons de Navier* (pp. 817—25) and in the annotated *Clebsch* (pp. 98—110). Under the latter heading we shall discuss his more complete treatment of the subject: see our Arts. 308—310. The memoir ends with the *résumé* to which we have before referred.

[153.] *Sur la détermination de l'état d'équilibre des tiges élastiques à double courbure. Les Mondes,* Tome 3, 1863, pp. 568—575. This note was a contribution to the *Société Philomathique,* August 8, 1863; see also *L'Institut,* 1863, pp. 324—5.

Consider a rod of double curvature; let M_t, M_n, M_ρ be the moments of the applied forces about the tangent to the central axis, the normal to the osculating plane and the principal radius of curvature. Let I, I' be the moments of inertia about the principal axes of the cross-section, and let e be the angle the radius of curvature ρ makes in the unstrained state with the axis of I'; then Saint-Venant gives the two following formulae, where ϵ is the increment in e and δs is an element of central axis:

$$\sin \epsilon = \frac{\rho}{E}\left\{M_\rho\left(\frac{\cos^2 e}{I} + \frac{\sin^2 e}{I'}\right) - M_n \sin e \cos e\left(\frac{1}{I} - \frac{1}{I'}\right)\right\},$$

$$\frac{dM_t}{ds} = \frac{M_\rho}{\rho}.$$

Hence when $e = 0$ or $\pi/2$, or $I = I'$, ϵ depends only on M_ρ the moment of the forces round the radius of curvature.

The second equation shews that the moment of torsion M_t is only constant when $M_\rho = 0$ along the whole length of the wire.

Saint-Venant refers to the work of Poisson, Wantzel and Binet: see our Arts. 1599*—1607*. He also reproduces the example of the *Comptes rendus*: see our Art. 155, and that of the horizontal semi-circular bar of rectangular cross-section built-in at both terminals and loaded at its mid-point used in the *Leçons de Navier*, p. cxxxiv, which bring out clearly the need of taking into consideration the angle ϵ.

Saint-Venant refers to Bresse: *Cours de mécanique appliquée: Résistance des matériaux*, 1859, p. 86, for a good investigation of the general formulae for elastic wires of double-curvature when the shifts are small.

[154.] *Sur la théorie de la double réfraction: Comptes rendus*, T. 57, 1863, pp. 387—391.

This is a note on a memoir by Galopin, and points out that there is no need to put the initial stresses zero in the ether in order to obtain Cauchy's conditions for double refraction: see our Art. 148. The contents of this note are practically involved in the memoir of 1863: see our Art. 127, and concern properly the historian of the undulatory theory of light.

[155.] *Sur les flexions et torsions que peuvent éprouver les tiges courbes sans qu'il y ait aucun changement dans la première ni dans la seconde courbure de leur axe ou fibre moyenne: Comptes rendus*, T. 56, 1863, pp. 1150—54. See also *L'Institut*, Vol. 31, 1863, pp. 195—6.

This memoir draws attention to the point considered by Saint-Venant in his memoirs of 1843 and 1844; see our Arts. 1598* and 1603*; namely the importance of taking into consideration the 'angle of torsion' or angle between new and old osculating planes in dealing with the elastic equilibrium of wires of double-curvature. Saint-Venant brings out the importance by a good example, namely a curved wire turned upon itself so as to have the same curvature at each point of the central axis, but so that the naturally longest and shortest 'fibres' interchange places.

He points out that the stretch in a fibre distant z from the central axis is:

$$z\sqrt{1/\rho^2 - 2/\rho\rho_0 \,.\, \cos\epsilon + 1/\rho_0^2},$$

where ρ, ρ_0 are the new and the primitive radii of curvature and ϵ the angle the new and old radii of curvature make with each other. In the example above referred to $\rho = \rho_0$ and $\epsilon = \pi$, so that the stretch becomes

$$2z/\rho_0.$$

Generally when $\rho = \rho_0$, the stretch equals

$$2z/\rho_0 \,.\, \sin\tfrac{1}{2}\epsilon.$$

In conclusion Saint-Venant refers to the contributions of Lagrange, Poisson, Binet, Wantzel and himself to the subject: see our Art. 1602* for references.

[156.] *Mémoire sur les contractions d'une tige dont une extrémité a un mouvement obligatoire; et application au frottement de roulement sur un terrain uni et élastique: Comptes rendus*, T. 58, 1864, pp. 455—8.

This memoir was written in 1845, and is an attempt to apply the theory of elasticity to the phenomena of rolling friction. The chief results were published in the *Bulletin de la Société Philomathique* of June 21, 1845. The following conclusions are given in the *résumé* in the *Comptes rendus*:

On en déduit que le frottement de roulement sur un pareil sol est: 1° proportionnel à la pression; 2° en raison inverse du rayon du cylindre; 3° indépendant de sa longueur (ou de la largeur de jante, si c'est une roue); 4° proportionnel à la vitesse; 5° d'autant moindre que le terrain élastique est plus roide ou moins compressible.

Saint-Venant remarks:

Ces résultats sont d'accord avec un certain nombre d'expériences de Coulomb et de M. Morin. (p. 457.)

There is a general indication of the method of treatment adopted in the original memoir, but it is not sufficient to replace its analysis. The memoir itself appears never to have been published.

157. *Travail ou potentiel de torsion. Manière nouvelle d'établir les équations qui régissent cette sorte de déformation des prismes élastiques. Comptes rendus,* T. 59, 1864, pp. 806—809. Translated in the *Philosophical Magazine,* January, 1865, pp. 61—64.

In his memoir on Torsion Saint-Venant used one equation which holds at every point within a body, and one which holds at every point of the convex surface: see equations (vi) of our Art. 17 on that memoir. In the present paper Saint-Venant undertakes to obtain these equations simultaneously by the aid of the principle of *Work.*

The potential of elasticity, that is to say the molecular work ϕ which a deformed element is capable of furnishing, is thus expressed for the unit of volume of the element:

$$\phi = \tfrac{1}{2}\widehat{xx}s_x + \tfrac{1}{2}\widehat{yy}s_y + \tfrac{1}{2}\widehat{zz}s_z + \tfrac{1}{2}\widehat{yz}\sigma_{yz} + \tfrac{1}{2}\widehat{zx}\sigma_{zx} + \tfrac{1}{2}\widehat{xy}\sigma_{xy}.$$

Now the values of the component stresses $\widehat{xx}$, $\widehat{yy}$,...... can, we know, be expressed as linear functions of the six strains s_x, s_y, s_z, σ_{yz}, σ_{zx}, σ_{xy}; substitute these values in ϕ, and we obtain an expression of the second degree in the strains, consisting of twenty-one terms. In the case of torsion which we are considering, the strains reduce to the two σ_{xy} and σ_{xz}, so that we have

$$\phi = \tfrac{1}{2}\mu_1 (\sigma_{xy})^2 + \tfrac{1}{2}\mu_2 (\sigma_{xz})^2,$$

where μ_1 and μ_2 are the slide-moduli in the directions of y and z: see Art. 17 of our account of the memoir on Torsion.

Now let M denote the *moment of torsion* so that

$$M = \int dy\, dz\, (\widehat{xz}\, y - \widehat{xy}\, z).$$

Thus if the moment of torsion is measured by an angle τ we have $M\dfrac{\tau}{2}$ for the molecular work; so that by equating the two expressions for this work we obtain

$$\tfrac{1}{2}\iint dy\, dz\, (\mu_1\sigma^2_{xy} + \mu_2\sigma^2_{xz}) = \tfrac{1}{2}\tau\iint dy\, dz\, (\widehat{xz}y - \widehat{xy}z) \text{.........} (1).$$

Now we assume that the body has three planes of symmetry perpendicular to the axes of x, y, z respectively; so that

$$\widehat{xy} = \mu_1\sigma_{xy}, \qquad \widehat{xz} = \mu_2\sigma_{xz};$$

also
$$\sigma_{xy} = \frac{du}{dy} - \tau z, \qquad \sigma_{xz} = \frac{du}{dz} + \tau y,$$

by equation (iii) of our Art. 17.

Substitute in the above equation (1) and we obtain

$$\iint dy\,dz\left\{\mu_1\frac{du}{dy}\left(\frac{du}{dy}-\tau z\right)+\mu_2\frac{du}{dz}\left(\frac{du}{dz}+\tau y\right)\right\}=0.$$

Integrate this equation by parts in the usual way, and it becomes

$$\int u\left[\mu_1\left(\frac{du}{dy}-\tau z\right)\cos(ny)+\mu_2\left(\frac{du}{dz}+\tau y\right)\cos(nz)\right]ds$$
$$-\tau\iint u\left[\mu_1\frac{d^2u}{dy^2}+\mu_2\frac{d^2u}{dz^2}\right]dy\,dz=0\ldots\ldots\ldots\ldots\ldots\quad(2);$$

here (ny) and (nz) denote the angles which the normal to the surface at the point (x, y, z) makes with the axes of y and z respectively; and ds is an element of the curve of intersection of the body by a plane at right angles to the axis of x.

If we equate to zero the term in brackets in the double integral we obtain the equation which must hold at every point of the interior; and if we equate to zero the term in brackets in the single integral we obtain the equation which must hold at every point of the surface.

But Saint-Venant does not explain why we must equate these terms separately to zero; that is, he does not explain why he breaks up equation (2) into *two* equations. Moreover the whole process borrows so much from the memoir on Torsion that it has not the merit of being an independent investigation.

Saint-Venant says:

Or la deuxième et la première parenthèse carrée, égalées séparément à zéro... :

by this he means the terms contained within the square brackets in (2). The English translation has very strangely "Now the squares of the second, and of the first parenthesis, each equated to zero,..."

[158.] A remark of Saint-Venant's on p. 809 may be cited:

Le calcul du potentiel de torsion a aussi, en lui-même, une valeur pratique; car les ressorts en hélice, qu'on oppose souvent à divers chocs, travaillent *presque* entièrement par la torsion de leurs fils, ainsi que je l'ai montré en 1843, et que l'ont ermarqué, au reste, Binet dès 1814, M. Giulio en 1840, et récemment des ingénieurs des chemins de fer.

See our Arts. 175*, 1220*, 1382* and 1593—5*. The 1814 and the *récemment* (1864) mark the wide interval which too often separates theory from practice!

[159.] *Théorie de l'élasticité des corps, ou cinématique de leurs déformations. Les Mondes,* Tome 6, 1864, pp. 607 and 608. If a body is deformed any small portion originally spherical becomes an ellipsoid: see our Art. 617*. In the present paper Saint-Venant undertakes to establish this proposition by simple general reasoning; the process does not seem very satisfactory.

Section III.

Researches in Technical Elasticity.

[160.] *Résumé des Leçons...sur l'application de la mécanique à l'établissement des constructions et des machines....Première section. De la Résistance des corps solides, par Navier....Troisième Édition avec des Notes et des Appendices par M. Barré de Saint-Venant.* The title-page bears the imprint, Paris, 1864. A foot-note, however, on p. 1 tells us that pp. 1—224 appeared in 1857, pp. 225—336 in 1858, pp. 337—496 in 1859, pp. 497—688 in 1860, pp. 689—849 in 1863, while the *Notices et l'Historique,* pp. i—cccxi, were finally added in 1864. Thus the whole work of more than 1100 pages occupied some seven years in the production, and thus necessarily lacks somewhat of the unity which is to be met with in other treatises. Under the form of notes to a few sections of Navier's original work (see our Art. 279*), Saint-Venant has given us a complete text-book of elasticity from the practical standpoint. At the same time, by additional notes and appendices, he has rendered his text-book of surpassing historical value and physical suggestiveness. The leading characteristics of the book are simplicity of analysis and copiousness of reference. See *Notice* I., pp. 41—2 and *Notice* II., pp. 28—9.

[161.] The cccxi. pages of introductory matter are occupied with the following subjects: Table of Contents, pp. i—xxxviii; *Notice biographique sur Navier* by de Prony extracted from the *Annales des ponts et chaussées* (1837, 1er semestre, p. 1), pp. xxxix—

li; the funeral discourses on Navier by Emmery and Girard, pp. li—liv: a bibliography of the works of Navier with copious remarks due to Saint-Venant, pp. lv—lxxxiii; the original prefaces to the editions of Navier's *Leçons* published in 1826 and 1833; pp. lxxxiv—xc; and finally Saint-Venant's *Historique abrégé des recherches sur la résistance et sur l'élasticité des corps solides,* pp. xc—cccxi.

[162.] The *Historique abrégé* is practically the only brief account of the chief stages of our science extant. Girard had written what was for his day a fair sketch of the *incunabula* (see our Art. 123*), but it remained for Saint-Venant, without entering into the analysis of the more important memoirs, to describe their purport and relationship. It fulfils a different purpose to our own history—for it makes no attempt to replace the more inaccessible memoirs—but as a model of how mathematical history should be written, we hold it to be unsurpassed, and can only regret that a recent French historian has not better profited by the example thus set[1].

We would especially recommend to the student of Saint-Venant's memoirs pp. clxxiii—cxcii, which treat of the relation of his own researches by means of the *semi-inverse* method to the work of his predecessors. The point we have referred to in our Arts. 3, 6, 8 and 9 is well brought out in relation to Lamé's problem of the right-six-face.

We will note one or two further points of the *Historique* in the following five articles.

[163.] On p. cxcviii in the footnote Saint-Venant gives the expression for the work-function in terms of the stresses when there is an *ellipsoidal* distribution of elasticity: see our Art. 144. He finds

$$W = \frac{1+i}{2(2+3i)}\left(\frac{\widehat{xx}}{a}+\frac{\widehat{yy}}{b}+\frac{\widehat{zz}}{c}\right)^2 + \frac{\widehat{yz}^2-\widehat{yy}\,\widehat{zz}}{2bc} + \frac{\widehat{zx}^2-\widehat{zz}\,\widehat{xx}}{2ca} + \frac{\widehat{xy}^2-\widehat{xx}\,\widehat{yy}}{2ab},$$

where for isotropy $i=\lambda/\mu$ and $a^2=b^2=c^2=\mu$.

[1] The essential feature of scientific history is the recognition of growth, the interdependence of successive stages of discovery. This evolution is excellently summarised in Saint-Venant's *Historique*. Our own 'history' is only a bibliographical repertorium of the mathematical processes and physical phenomena which form the science of elasticity, as a rule for the purpose of convenience chronologically grouped. M. Marie's *Histoire des sciences mathématiques* is a chronological biography, without completeness as bibliography or repertorium. Excellent fragments there are in it, but the conception of evolutionary dependence is wanting.

Generally:

$$|xxxx| = (2+i)\,a^2, \qquad |yzyz| = bc, \qquad |yyzz| = ibc,$$
$$|yyyy| = (2+i)\,b^2, \qquad |zxzx| = ca, \qquad |zzxx| = ica,$$
$$|zzzz| = (2+i)\,c^2, \qquad |xyxy| = ab, \qquad |xxyy| = iab.$$

[164.] Pages cxcix—ccix deal with the history of the problem of rupture. According to Saint-Venant, two kinds of rupture may be distinguished: *rupture prochaine* and *rupture éloignée.* The former falls outside the theory of 'perfectly elastic' bodies, the latter he thinks may be deduced from the hypothesis that when the limit of mathematical elasticity is passed,—i.e. when the stretch is greater than the limit at which stretch remains wholly elastic and proportional to traction,—then the body will ultimately be ruptured if it has to sustain the same load. The reader who has followed our analysis of the *state of ease* and the *defect in Hooke's Law* given in the appendix to Vol. I. and also our Arts 4 (γ) and 5 (a) in the present volume will recognise that this hypothesis has only a small field of application. What we have really obtained is a *limit to linear elasticity.* It is the more important to notice this because Saint-Venant argues that we must take as our limit the maximum *positive stretch,* for, as Poncelet has asserted: "*que le rapprochement moléculaire ne peut être une cause de désagrégation*" (p. cci). It is probably true that rupture can only be produced by stretch, but *squeeze* can surely produce failure of linear elasticity when the body is so loaded that no transverse stretch is possible. Hence when Saint-Venant introduces the stretch and slide-moduli into his condition for safe loading and so makes it a question of *linear elasticity,* it seems to me that he ought at the same time to alter his statement as to the *greatest positive stretch* being the only quantity we are in search of. Indeed, his condition seems partly based upon an idea associated with *rupture,* and is then applied to constants and equations deduced from the principle of linear elasticity (see his p. ccviii, § XLVIII.). The limitations to which his theory is subjected were, however, partially recognised by Saint-Venant himself (see his pp. ccv—vii). Thus he writes:

Nous ne prétendons pas, au reste, qu'une théorie subordonnant uniquement le danger de rupture d'un solide à la grandeur qu'atteint une dilatation linéaire n'importe dans quelle de ses parties, et indépendamment des autres circonstances où il se trouve en même temps, soit le dernier mot de la science et de l'art.

He refers on this point to the experiments of Easton and Amos: see our Art. 1474*.

[165.] Pages ccxiv—xxiv deal with the problems of resilience and impact.

In the footnote p. ccxvii, there is an error in the integral of the equation $\frac{d^2z}{dt^2} = g \cos \alpha - \frac{g}{f} z$ there given. It should be

$$z = f \cos \alpha + V \sqrt{\frac{f}{g}} \sin \sqrt{\frac{g}{f}}\, t - f \cos \alpha \cos \sqrt{\frac{g}{f}}\, t.$$

The error was noted by Saint-Venant himself in a letter to the Editor of this History, August, 1885.

On p. ccxxii and footnote there should have been a reference to Homersham Cox with regard to the factor $k = 17/35$. His memoir of 1849 (see our Art. 1434*) seems to have escaped Saint-Venant's attention.

A further consideration of the effect of impact on bars when the vibrations are taken into account occurs on pp. ccxxxii—viii, and then follows (pp. ccxxxix—xlix) an account of Stokes' problem of the travelling load (see our Art. 1276*). Saint-Venant refers to the researches of Phillips and Renaudot, but his account wants bringing up to date by reference to more recent researches.

[166.] On pp. ccxlix—ccliii Saint-Venant refers to the rupture conditions given by Lamé and Clapeyron and again by Lamé for cylindrical and spherical vessels. It seems to me that he has not noticed here that these conditions are, on his own hypothesis of a *stretch* and not a *traction* limit, erroneous: see the footnotes to our Arts. 1013* and 1016*.

[167.] After an excellent and succinct account of the course of the investigations of Euler, Germain, Poisson, Kirchhoff &c. with regard to the vibrations of elastic plates (pp. ccliii—cclxxi) the *Historique* closes with two sections LXI. and LXII. (pp. cclxxi—cccxi) on the experiments made by technologists and physicists previously to 1864 on the elasticity and strength of materials. Good as these pages are, they are insufficient to-day in the light of the innumerable experiments of first-class importance made during the last twenty years.

[168.] In considering Saint-Venant's edition of Navier we shall leave the original text out of consideration, and note only those points of Saint-Venant's additions (ten-fold as copious as the original text) which present novelty of treatment or result. We put aside all matters already discussed in the memoirs on Torsion and Flexure. Those memoirs are here to a great extent embodied, their processes simplified and their results extended.

[169.] (*a*) On pp. 2—3 we find Saint-Venant basing the theory of elasticity on the principle of a central inter-molecular action which is a function of the distance.

(*b*) On p. 4, § 6 we have *écrouissage* and *énervation* defined. These definitions are rather theoretical than practical. Thus Saint-Venant defines as *écrouissage* the arrangements taken by the molecules of a body when they pass by changes which are persistent from a less to a more stable condition of equilibrium, as *énervation* the arrangements when they pass to a less stable condition. It will be noted that the physical characteristics of set, yield-point and plasticity are not clearly brought out by these definitions.

(*c*) Pp. 5—14 treat of rupture by compression. Saint-Venant rejects the theory of Coulomb (see our Art. 120*) as giving a stress not a stretch limit. He adopts that of Poncelet, who in 1839 in a course given at Paris, ascribed rupture by compression to the transverse stretch which accompanies longitudinal squeeze (pp. 6 and 10, and compare with footnote p. 381). That short prisms of cast iron, cement &c. often take 8 to 10 times as great a load to rupture them by negative as by positive traction and not the 4 times of the uni-constant theory, is attributed not to bi-constant isotropy but to terminal friction which hinders the lateral expansion, or to want of isotropy (pp. 10 and 12). Such *rupture*, however, really lies at present outside theory.

(*d*) On pp. 15—19 we have the generalised Hooke's Law and the definition of the stretch-modulus (E) and the stretch-squeeze ratio (η). Saint-Venant remarks, that theoretically $\eta = \frac{1}{4}$ (i.e. on the uni-constant hypothesis), that Wertheim finds it differs little from $\frac{1}{3}$, and that it can never be $> \frac{1}{2}$ as otherwise a traction would diminish the volume of a prism of the given substance, "*ce qui n'est pas supposable.*" There is no further reason given why we cannot suppose the volume to diminish. We may, however, look at the matter thus:

Let ξ = the ratio of the slide-modulus to the dilatation coefficient $(=\mu/\lambda)$, then (Vol. I. p. 885):

$$\xi = \frac{3 - E/\mu}{E/\mu - 2}.$$

Hence, since ξ is necessarily positive, we must have $E/\mu > 2$ and < 3 (the mean of these gives the uni-constant hypothesis $E/\mu = 5/2$). But $\eta = \frac{E}{2\mu} - 1$, or η can only have values from 0 to $\frac{1}{2}$.

This proof holds only for an isotropic material. In the case of an aeolotropic material it does not seem obvious why a longitudinal stretch should not produce a negative dilatation. The ratio of dilatation to stretch

$$= \frac{s_x + s_y + s_z}{s_x} = 1 - \eta_1 - \eta_2,$$

and in the case of wood the values obtained for η_1, η_2 would seem to give this a negative value, for they are $> \frac{1}{2}$. Saint-Venant admits later this possibility: see his pp. 821—2. Hence any set of experiments which give values for $\eta > \frac{1}{2}$ may be taken to denote that the material in question is not isotropic and homogeneous.

(*e*) On pp. 20—21 it is suggested that for some substances it is advisable to consider the stretch-modulus E as varying over the cross-section of a prism. Saint-Venant refers to the experiments on this point of Collet-Meygret and Desplaces: see our Chapter XI. He also regards Hodgkinson's experiments as leading to a like conclusion notwithstanding a special experiment to the contrary: see our Arts. 952* (iii), 1484* and references there. We thus have the formula

$$P_x = s_x \int E_x d\omega$$

put forward by Bresse, where P_x is the total traction in a prism stretched s_x in the direction of its axis x, and $(\int E_x d\omega)/\omega$ is the mean value of the stretch-modulus over the cross-section ω. For metals *coulés ou laminés*, where on the lateral faces there is a surface or *skin* change of elasticity, Saint-Venant would take:

$$P_x = s_x (E_0 \omega + \epsilon\chi),$$

χ étant le périmètre de la section supposée diminuée d'un à deux millimètres tout autour, afin de représenter le développement moyen de la croûte douée généralement de plus de roideur et de nerf que le reste; et E_0 et e étant deux coefficients à déterminer par les méthodes connues de compensations d'anomalies en faisant des expériences d'extension sur des barres ayant des grosseurs ou des formes sensiblement différentes (p. 21).

(*f*) Saint-Venant returns to this same point on pp. 42—44, and pp. 115—118 when treating of the problem of flexure. In the former passage, Saint-Venant gives reasons for adopting in the case of *metal* a *skin* change only in the elastic-modulus. He proposes the formula

$$M = \frac{E_0 I + ei}{\rho},$$

for the bending-moment, $1/\rho$ being the curvature, I the moment of inertia of the section, and i that of its contour, or rather of the mean line of the skin zone (ligne qu'on peut placer à 1 ou à 2 millimètres à l'intérieur). E_0 and e are to be determined by experiments on the flexure of bars of the given material but sensibly different in size and form.

In the case of wood, Saint-Venant, referring to the experiments of Wertheim and Chevandier (see our Art. 1312*), adopts a parabolic law for the variation of the stretch-modulus. Let E_0 and E_1 be the moduli in the direction of the fibre at the centre ($r = 0$) and circumference ($r = r_1$) of the tree, then at any other point (r) we have

$$E = E_0 - (E_0 - E_1)\, r^2/r_1^2.$$

Saint-Venant determines the value of $\int E y^2 d\omega$—i.e. the 'rigidity'—for a bar of rectangular cross-section ($b \times c$) whose centre of gravity was, before it was hewn, distant r_0 from the centre of the tree (p. 44).

In the second passage to which I have referred the rupture condition (rather the failure of linear elasticity) is deduced from the like hypothesis of skin-change. Saint-Venant obtains a formula

$$M_0 = \frac{T_0}{Ey}(E_0 I + ei),$$

where M_0 is the maximum bending moment which will not cause the elasticity of a 'fibre' at distance y from the neutral axis (where the stretch-modulus $= E$) to fail by giving it a greater stretch than T_0/E. We have then to find the fibre for which T_0/Ey is smallest.

Si l'on a des raisons de penser que c'est la fibre la plus dilatée, comme quand la matière est homogène, ou que la contexture hétérogène est telle que le rapport T_0/E varie moins que y, l'équation sera, en désignant comme à l'ordinaire par y' la grandeur de l'ordonnée de cette fibre, et par E', T'_0, les valeurs correspondantes de E, T_0;

$$M_0 = \frac{E_0 T'_0 I}{E'y'} + \frac{eT'_0 i}{E'y'},$$

ou bien, C et c désignant deux constantes dépendant comme E_0 et e de la nature de la matière et de son mode de forgeage ou de fusion,

$$M_0 = C\,I/y' + c\,i/y'.$$

Saint-Venant calculates the value of M_0 for a rectangular section, and also deals with a similar expression for the case of the *wood* prism referred to above; see his pp. 117—8.

(*g*) In §§ 8—12 (pp. 22—26) the reader will find some account of the behaviour of a material under stress continued even to rupture. This account was doubtless for the time succinct and good, but there are several points which could only be accepted now-a-days with many reservations. For example the statement (§ 11): *Le calcul théorique est toujours applicable pour limiter les dilatations et établir les conditions de résistance à la rupture éloignée*—is one which requires much reservation. We have seen in Vol. I. p. 891 that a material may be in a state of ease and yet not possess linear elasticity for strains such as often occur in practice. Further that even when there is linear elasticity its limit can often be raised without *enervation* almost up to the yield-point, where one exists. Hence when Saint-Venant takes s_0 to be the stretch at which material ceases *de s'écrouir et commence à s'énerver, ce qui se manifeste par la marche des allongements persistants,* and puts $P_0 =$ or $< E\omega s_0$ as the safe tractive load—where E is the stretch-modulus and ω the sectional area—we find some difficulty in ascertaining what limit s_0 really represents. In most cases before enervation begins, *linear* elasticity will be long gone, and all the formula really can tell us is the stage at which *linear elasticity fails*; this fail-limit may be very far from the yield-point, and in some materials very far indeed from the elastic limit.

Saint-Venant refers to the 'fatigue' of a material due to repeated loading and to the question whether vibrations can change the molecular structure from fibrous to crystalline (see our Arts. 1429*, 1463* and 1464*). These are points on which we know to-day a good deal more than was accessible in 1857.

[170.] Article III. is devoted to the flexure of prisms and commences with a criticism of the Bernoulli-Eulerian hypothesis as expounded by Navier. Saint-Venant shews with the simplest analysis that the cross-sections neither retain their original contour (not even in the simple case of 'circular' flexure, § 3, p. 34) nor their original planeness (§ 4, pp. 36—9). To § 6, p. 40—2, we have already referred when dealing with the question of equipollent load-systems in Art. 8 of our account of the memoir on *Torsion.*

[171.] Pages 52—58 of this Article reproduce with some important additions the formulae of Art. 14 of our account of the memoir on *Torsion.* Saint-Venant proves the following results for the case when the load plane is not a plane of inertial symmetry:

(*a*) *The neutral line is the diameter of the ellipse of inertia conjugate to the trace of the load-plane on the cross-section.* (This theorem was given by Saint-Venant and Bresse about the same time: see our Arts. 1581* and 14.)

(*b*) *The 'deviation' or angle between the load- and flexure-planes is a maximum when the former has for trace on the cross-section a diagonal of the rectangle formed by the tangents at the extremities of the principal axes of the ellipse of inertia.*

A good illustration of a simple kind shewing the deviation is given in § 7, p. 57.

[172.] The notes on pp. 73—85 deal with the elastic line when the flexure is *not so small* that we may neglect the square of the slope which the elastic line makes with the unstrained position of the central axis. The results here given express the maximum deflection and terminal slope in series ascending according to powers of $\dfrac{\text{load} \times (\text{span})^2}{\text{rigidity}}$, further the load and maximum stretch in series of ascending powers of $\dfrac{\text{max. deflection}}{\text{span}}$, and finally the stretch-modulus in terms of max. deflection, span and load. Saint-Venant in *Notice* I. (p. 42) claims some originality for these results. This I think can only refer to the convenient form into which he has thrown them: see our Art. 908*.

[173.] Article IV. (pp. 86—186) is entitled: *Rupture par Flexion.*

This practically deals with the formula for the maximum moment

$$M_0 = \frac{T_0 \omega \kappa^2}{h},$$

where h is the distance of the 'fibre' most stretched from the neutral axis[1] and $\omega\kappa^2$ the sectional-moment of inertia about that axis. The question then arises: what is T_0? Saint-Venant holds that if T_0 be the stress at which *enervation* commences, we have in reality a condition for the safety of a permanent structure. This involves *the enervation-point being very close to the limit of linear elasticity.* In many materials this is certainly not the case, even were it possible to define exactly this enervation-point. We must treat the results of this article as applying only to the *fail-limit*, i.e. the failure of linear elasticity (p. 91). Saint-Venant indeed fully recognises that the formula does not give any condition for *immediate* rupture, and that no argument *against* the mathematical theory of 'perfect elasticity' can be drawn from experiments on absolute strength. He states clearly enough that for beams of various sections, for which $\omega\kappa^2/h$ retains the same value, T_0 varies with the form of the section and is greater than, even to the double of, the value obtained from pure traction experiments (this is the well-known 'crux' which the technicists raise against the mathematicians): see his pp. 90, 91. Yet it seems to me that even the extent to which he adopts the formula is not valid. It only gives the fail-limit, which in some cases, perhaps, may indicate *rupture éloignée.*

[174.] On pp. 95—101 our author treats of 'Emerson's paradox' or the existence of 'useless fibres'. In other words, the expression $\omega\kappa^2/h$ can be occasionally increased by cutting away projecting portions of ω.

We have the cases of beams of square, triangular and circular cross-sections fully treated, as well as that of the *croix d'équerre.*

[1] We use 'neutral *axis*' for the trace of the plane of unstrained 'fibres' on the cross-section, while we retain 'neutral *line*' for the succession of points in the plane of flexure through which pass real or imaginary elements of unstretched fibre. It will only coincide with the 'elastic line' or distorted central axis when there is no thrust.

The elastic failure of such outer fibres does not however denote that the truncated section possesses greater strength than the complete section, as Emerson argued from the formula, Rennie confirmed and Hodgkinson refuted by experiment: see our Arts. 187* and 952* (ii). Saint-Venant very aptly terms them *fibres inutiles.* We may indeed calculate the maximum elastic efficiency of such sections by supposing them truncated till $\omega\kappa^2/h$ is a maximum, but the difference is generally so small as not to repay the labour of calculation, albeit it suggests a method of economising material.

[175.] Pages 103—105 treat of the obscure point of how to determine the value of T_0 in the formula of Art. 173, so that there shall be no danger of *rupture éloignée.* Saint-Venant apparently recognises that the exact point at which *enervation* begins is difficult to discover experimentally, especially when the duration and repetition of loads have to be taken into account (p. 105).

Let T_0, T_0' be the stresses which in positive and negative traction respectively mark the limit of *rupture éloignée*; let T_1, T_1' be the corresponding easily discovered stresses which mark *cohésion instantanée.* Then Saint-Venant observes that we may learn from previous constructions and from our experience of structures submitted to long use what fraction T_0 is of T_1, and that we are justified in taking for the *same* kind of material, even in its several varieties, a constant ratio between T_0 and T_1, e.g. $T_0 = \frac{1}{8} T_1$.

On n'aura pas pour cela la dilatation limite $s_0 = T_0/E$ égale au 1/8 de la dilatation finale positive ou négative, puisque la proportionalité des efforts aux effets cesse longtemps avant. Mais on aura un certain rapport aussi à peu près constant entre ces deux dilatations (p. 106).

Saint-Venant even suggests (p. 107) that T_0 may be taken proportional to the T obtained from the formula $P = T \cdot \frac{4}{l} \cdot \frac{\omega\kappa^2}{h}$ where P is the concentrated mid-load which will rupture immediately a bar of length l terminally supported. As the T obtained from this formula when used for rupture is found to be a function of the section, this suggestion seems to me a dangerous one.

[176.] On p. 109 (§ 13) a formula is given for finding T_0' when T_0 is known. Suppose that the material is a prism with longitudinal stretch-modulus E, and that E_t is the same modulus for *all* directions transverse to the axis; let $T_{0,t}$ and $T_{1,t}$ be the limiting elastic and the rupture stresses when the material sustains a tractive load in the transverse sense, η the stretch-squeeze ratio. Then:

$$\frac{T_{0,t}}{E_t} = \text{stretch in transverse direction due to } T_{0,t},$$

$$\frac{1}{\eta}\frac{T_{0,t}}{E_t} = \text{squeeze in longitudinal direction...},$$

$$E\,\frac{1}{\eta}\frac{T_{0,t}}{E_t} = \text{safe limit to negative traction in longitudinal direction.}$$

Thus we must have:

$$T'_0 = E\,\frac{1}{\eta}\,\frac{T_{0,t}}{E_t},$$

hence
$$\frac{T'_0}{T_0} = \frac{1}{\eta}\frac{E}{E_t}\frac{T_{0,t}}{T_0}.$$

Now by what precedes, Saint-Venant holds that we can legitimately replace $T_{0,t}/T_0$ by $T_{1,t}/T_1$, a ratio easily found from rupture experiments, thus:

$$\frac{T'_0}{T_0} = \frac{1}{\eta}\cdot\frac{E}{E_t}\cdot\frac{T_{1,t}}{T_1}.$$

In the case of isotropy $T_{1,t} = T_1$, $E = E_t$, and thus on the uni-constant hypothesis we should have $T'_0/T_0 = 1/\eta = 4$.

Saint-Venant finds from experiments of Wertheim and Chevandier, that for oak $T'_0/T_0 = 1{\cdot}21$ or $1{\cdot}08$; for cast-metals he suggests 3, for stone 8 to 10, and for wrought iron 2. He holds the value 6 as obtained by Hodgkinson for cast-iron much too large to be prudently adopted, and discusses at some length Hodgkinson's experiments on the beam of strongest section: see our Art. 243*.

Finally we may note that on p. 115, he states that for different varieties of the same material it is more legitimate to take T_0 proportional to T of the formula of Art. 175, than to the stretch-modulus as some writers have done.

[177.] Pp. 122—171 are occupied with what is generally known as the comparative strength of beams of various sections—in reality it is the failure of linear elasticity and not *strength* with which we are dealing.

(*a*) On pp. 123—5 we have the fail-limit determined for cases of loading in planes of inertial asymmetry. The formula of our Art. 14 namely:

$$M_0 = \text{minimum of } \frac{T_0\omega}{\dfrac{z\cos\phi}{\kappa_y^2} + \dfrac{y\sin\phi}{\kappa_z^2}},$$

we find repeated.

When, as in the case of a rectangular section, z, y have values independent of ϕ corresponding to a maximum of the denominator, we find at once

$$M = T_0\omega \Big/ \sqrt{\frac{z^2}{\kappa_y^4} + \frac{y^2}{\kappa_z^4}}.$$

Saint-Venant applies these results to rectangular and elliptic sections.

(*b*) On pp. 143—156 we have a very full investigation of the I-section with special reference to Hodgkinson's section of greatest strength. Although Hodgkinson's experiments were made on *absolute* strength, Saint-Venant finds that his results are true for the fail-limit (*rupture éloignée*). The general conclusions given on p. 155 are: (1) When T'_0 is sensibly greater than T_0 the I-section with unequal flanges has a higher fail-limit, but a less resistance to flexure, than one with equal flanges, provided the squeeze of the smaller flange is not accompanied by lateral stretches more dangerous than the longitudinal in the larger flange, nor the smaller flange receive lateral flexure (buckle) owing to its compression. (2) When the height of the section is increased by ·4 to ·7 of itself we obtain for the same area a I-section of equal flanges with a higher fail-limit than one of unequal flanges and the lesser height; at the same time the resistance to flexure is largely increased. Such increase of height, however, increases the possibility of *déversement* being produced by a slightly oblique load and facilitates the lateral flexure of the squeezed flange.

(*c*) On pp. 156—163 we have a discussion of the fail-limit of *feathered axes.* Saint-Venant shews that their advantages are not so great as has been frequently supposed, while as we have seen (Art. 37) in the case of torsion they give no increased resistance worth mentioning.

[178.] The next point we have to notice is one of considerable interest and has recently been again attracting the attention of the technicists[1]. It is the calculation of the absolute strength from an empirical relation between stress and strain supposed to hold nearly up to rupture. That strain increases more rapidly than stress after the beginning of set even up to rupture had been long noticed by experimentalists, and various modifications of Hooke's Law had been suggested by Varignon, Parent, Bülfinger and Hodgkinson: see our Arts 13*, 29 β*, 234* and 1411*. There has been, however, considerable obscurity about the various empirical formulae suggested, and they have only been applied to the old Bernoulli-Eulerian theory of flexure with its unchanged

[1] See the discussion and references in *Stabilité des Constructions: Résistance des Matériaux* by M. Flamant, pp. 322—9, and also in the *Engineer*, Vol. LXII., 1886, pp. 351, 392, 407.

cross-sections. To begin with, they can hardly be taken as approximate for any material having a distinct yield-point; nor in the second place is it clearly stated how far they represent stress-strain relations for bodies whose elasticity is non-linear, or how far elastic-strain and set are to be treated as coexistent.

Saint-Venant after citing Hodgkinson's formulae (see our Art. 1411*) takes by preference the following for the positive and negative tractions p_1, p_2 at distances y_1, y_2 from the neutral axis of a beam under flexure:

$$p_1 = P_1\left\{1 - \left(1 - \frac{y_1}{Y_1}\right)^{m_1}\right\}, \quad p_2 = P_2\left\{1 - \left(1 - \frac{y_2}{Y_2}\right)^{m_2}\right\},$$

where P_1, P_2 are the tractions at distances Y_1, Y_2 from the axis, and m_1, m_2 are constants. On p. 177 traces of the curves for p in terms of y are given for values of m from 1 to 10, and they are compared with the curves obtained from Hodgkinson's formula.

It will be observed that the difficulty of stating exactly the physical relation between stress, elastic-strain and set is avoided by an assumption of this kind. There is, however, another assumption of Saint-Venant's which does not seem wholly satisfactory. He states it in the following words:

Observons d'abord que lorsque la dilatation d'une fibre a atteint sa limite, comme une faible augmentation qu'on lui fait subir produit la rupture ou bien fait décroître très-rapidement sa force de tension, il est naturel de regarder la courbe des tensions comme ayant à l'instant de la rupture sa tangente verticale ou parallèle à l'axe coordonné des y, d'autant plus que cet instant a été précédé d'une énervation graduelle (pp. 180—1).

This paragraph assumes that for the material dealt with the rupture stress is an *absolute maximum*, but in several automatically drawn stress-strain relations which I have examined this does not appear to be the case (see Vol. I. p. 891), and at any rate in some materials it could only refer to the maximum stress *before stricture* and not to the rupture-stress.

On pp. 178—184 the case of a rectangle is treated at some length. Saint-Venant obtains general formulae on the supposition that the curves for negative and positive traction coincide at the origin, i.e. on the supposition that the stretch- and squeeze-moduli *for very small strains* are equal ($m_1 P_1/Y_1 = m_2 P_2/Y_2$). The limiting value of the bending moment is then calculated.

In § 3 various values are assumed for m_1 and m_2; in particular if $m_1 = m_2 = 1$, it is shewn that to make the initial stretch- and squeeze-moduli unequal is to *increase the resistance to rupture by flexure.*

In the case of $m_1 = m_2$, $P_1 = P_2$, $Y_1 = Y_2$ we easily find for a rectangular cross-section $(b \times c)$:

$$M_0 = R_0 \cdot \frac{bc^2}{6} \cdot \frac{3m(m+3)}{2(m+1)(m+2)},$$

which increases from $R_0 \frac{bc^2}{6}$ to $R_0 \frac{bc^2}{4}$ as m increases from 0 to ∞.

If we take $m_2 = 1$ and m_1 any value, we obtain a more complex value for M_0, which increases with m_1 from $R_0 \frac{bc^2}{6}$ to $R_0 \frac{bc^2}{2}$. Thus in all cases the value lies between those given by Galilei's theory and by the ordinary Bernoulli-Eulerian hypothesis.

Saint-Venant does not venture into the analysis required to determine how the constant n given by $M_0 = n R_0 bc^2/6$ varies with the shape of the section, which must be the true test of any theory of this kind, i.e. the constant m must be found to have the same value for all sections.

[179.] Saint-Venant gives on pp. 186—204 an excellent elementary discussion of slide and shear; on pp. 206—214 a like discussion of the effect of slide in changing the contour and shape of the cross-sections of a beam under flexure. The method of treatment is very simple, and by the consideration of a special case the action of the slide is well brought out.

[180.] Pages 216—237 are devoted to combined strain, flexure, stretch due to pure traction and slide. The fail-limit is determined by simple geometrical considerations, and the examples, chosen from those of Chapters XII. and XIII. of the memoir on *Torsion* (see our Arts. 50 to 60), are treated with considerable numerical detail. The example on the combined flexure and slide exhibited by the strained axis of a pulley is new (p. 234).

[181.] On pp. 239—271 the general equations of torsion are deduced. The treatment is in some respects better than in the memoir of 1853. We may note a few points:

(*a*) Pages 240—242 give a fuller discussion of the resistance to torsion due to longitudinal stretch of the 'fibres': see our Art. 51.

(*b*) Pages 244—5 (§ 4). Elementary proof that the cross-sections of all prisms, except the right-circular cylinder, are distorted by torsion.

(*c*) Pages 261—2. The expressions $\int \widehat{xy}\, d\omega$ and $\int \widehat{xz}\, d\omega = 0$ for every section of a prism under torsion. This is true whether or not the axis of torsion passes through the centre of the section, supposing it to have one. Saint-Venant had only treated of this matter in the case of the elliptic section (§ 59 of the memoir on *Torsion*: see our Art. 22). A general proof is here given in a footnote.

(*d*) In § 15, pp. 264—7, we have a fuller treatment than occurs in the memoir on *Torsion* of eccentric torsion, or torsion about *any* axis parallel to the prismatic sides. Taking the equations of torsion for an isotropic material (equations vi. of Art. 17):

$$u_{yy} + u_{zz} = 0,$$

$$(u_z + \tau y)\, dy - (u_y - \tau z)\, dz = 0,$$

for which the origin lies on the axis of torsion, let us put $y' = y + \eta$, $z' = z + \zeta$ we find η and ζ being constants:

$$u_{y'y'} + u_{z'z'} = 0,$$

$$\{u_{z'} + \tau (y' - \eta)\}\, dy' - \{u_{y'} - \tau (z' - \zeta)\}\, dz' = 0.$$

These equations have for solution

$$u = u' - \tau (\zeta y' - \eta z'),$$

where u' is the value of u when $\eta = \zeta = 0$, or in other words the shift when the torsion operates round an axis through the new origin. The shifts u' and u giving the distortions in the two cases differ only by

$$\tau (\zeta y' - \eta z') = \tau (\zeta y - \eta z),$$

or the two distorted surfaces are superposable by rotating the one through small angles $\tau\eta$ and $-\tau\zeta$ round the axes of y and z respectively.

Further, $\left\{\begin{matrix} u_z + \tau y = u'_{z'} + \tau y' \\ u_y - \tau z = u'_{y'} - \tau z' \end{matrix}\right\}$ or the slides determined for either axis are equal for the same points. Thus it follows that the torsional couple will in both cases be the same.

Saint-Venant then shews how by placing two prisms of equal cross-sections with corresponding lines parallel, and fixing their terminal faces so as to remain parallel after torsion about a mid-axis, we can obtain eccentric torsion. The torsional couple will be just double of that obtained from the simple torsion of either. Their axes it is true will be bent into helices, but the bending introduced is a small quantity of the *second order* in the torsion.

(*e*) In § 17 (pp. 268—71) we have an investigation of the maximum-slide and the fail-points. We cite the following passage:

Si σ_x^2 [σ_x= le plus grand *glissement principal*] croissait toujours de l'intérieur à l'extérieur de la section pour chaque direction, ce serait constamment sur son contour qu'il faudrait chercher les points dangereux. Mais nous savons qu'il y a souvent des points du contour où le glissement est nul, et il peut y avoir, dans l'intérieur, quelque point de maximum absolu de σ_x^2 (quoique cela ne se soit présenté dans aucun des exemples ci-après traités); et il n'est pas impossible que ce maximum excède toutes les valeurs de σ_x^2 relatives aux points du contour (p. 269).

We have then an analytical investigation of the fail-points, which suggests a general method of investigation adopted in the sequel for the special cases. This method avoids the ambiguities of some of the paragraphs on this subject in the memoir of *Torsion*: see our Arts. 39 and 42.

[182.] Pages 271—372 treat very thoroughly of the torsion-problem. They reproduce to a great extent the formulae and tables of the memoir on *Torsion*, but at the same time make frequent additions and improvements. We may note the following:

(*a*) Eccentric torsion of a right-circular cylinder. The coordinates of the centre referred to the axis of torsion being η, ζ, we find with the notation of our Art 181 (*d*), a being the radius:—

$$u = \tau (\zeta y - \eta z),$$

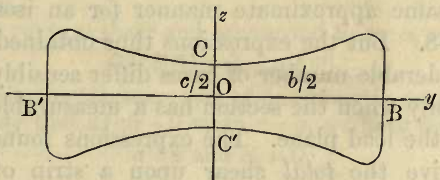
$$\sigma_{xy} = -\tau (z - \zeta), \quad \sigma_{xz} = \tau (y - \eta),$$

while $M = \mu\tau \int_0^a r^2 d\omega = \mu\tau\omega \cdot \frac{a^2}{2}$, as in the case of central torsion.

(*b*) A fuller treatment of the prisms whose cross-sections are included in the equation:

$$\frac{r^2}{2} + a_2 r^2 \cos 2\phi + a_4 r^4 \cos 4\phi = \text{const.} \quad \text{(See our Art. 49 (c).)}$$

The most interesting of the cross-sections included in this equation is entitled by Saint-Venant: *Section en double spatule analogue à celle d'un rail de chemin de fer* (p. 365). It has the shape given in the accompanying figure.



case of $c = b/5$

See pp. 305—307, 312—317, 325—335.

(*c*) The accurate investigation of the fail-points for the bi-symmetrical curves of the 4th and 8th order; see pp. 308—312, 339—341. Cf. our Arts. 37 and 39.

(*d*) In a foot-note to p. 335 Saint-Venant treats a special case of the curve of the fourth degree

$$\frac{y^2 + z^2}{2} + a_2 (y^2 - z^2) + a_4 (y^4 - 6y^2z^2 + z^4) = \text{const.}$$

By taking $a_2 = -1/\sqrt{2}$, $a_4 = 2(\sqrt{2}-1)/b^2$ and the constant $= 0$, we obtain an isosceles triangle having for base a portion of the hyperbola $y^2 = b^2/4 + (\sqrt{2}-1)^2 z^2$ and for sides lines making with the bisector of the base angles whose tangent $= \sqrt{2}-1$. The length of the bisector from vertex to hyperbolic base is then $b/2$. The torsion takes place round an axis through the *vertex*. Saint-Venant finds approximately,

$$M = .56702\, \mu\tau\omega\kappa^2.$$

This value agrees very closely with that of the equilateral triangle: see our Art. 41.

[183.] Pages 372—460 deal with the conditions for resistance to *rupture éloignée* under simultaneous torsion and flexure. Most of this matter had already been given in Chapters XII. or XIII. of the memoir on *Torsion* or in the memoir on *Flexure:* see our Arts. 50—60 and 90—8. One or two points may be noticed:

(*a*) In the memoir on *Torsion* Saint-Venant when seeking for the fail-limit neglects as a rule the flexural slides (see our Art. 56, Case (iii) etc.). Here he commences with an investigation of the values of these slides. The approximate methods of Jouravski and Bresse for obtaining the slide in a beam of small breadth are considered (see our Chapter XI.), and are applied to the rectangle, ellipse and I-cross-sections. A footnote gives the value of the slide in the same approximate manner for an isosceles triangle. See pp. 391—8. But the expressions thus obtained are not exact, and in a considerable number of cases differ sensibly from the real values, especially when the section has a measurable breadth perpendicular to the load plane. The expressions found by Jouravski and Bresse give the *total* shear upon a strip of unit-breadth taken on a section of the beam perpendicular to both the cross-section and the load plane, but they do not determine how such shear is transversely distributed, still less the magnitude of the maximum slide on the cross-section. Saint-Venant then proceeds as in the memoir on *Flexure* to deduce exact expressions for the flexural slides (pp. 399—414). The notation used differs from that in the original memoir. The reader will find the two notations placed side by side in the footnote, p. 405. The treatment in the *Leçons de Navier* is shorter and not nearly so complete as in the memoir. The diagrams reproduced in our frontispiece are given in a footnote on pp. 410—12: see our Arts. 92 and 97.

(*b*) Pages 414—60 are occupied with combined flexure and torsion in those cases where we may neglect the flexural slides. They reproduce with some modifications and extensions the results of Chapter XII. of the *Torsion* memoir. There is a good summary on pp. 453—9.

[184.] On pp. 461—9 Saint-Venant treats of rupture (*rupture immédiate*) by torsion.

(*a*) He shews that the moments capable of producing rupture are for similar sections as the *cubes* of their homologous dimensions. A footnote (p. 463) refers to Vicat's experiments which apparently contradict this result; see our Art. 731*. Saint-Venant attributes this divergence to flexure having taken place in the short prisms of *plâtre* and *brique crue* used by Vicat.

(*b*) In § 61 (p. 464) Saint-Venant endeavours to find the absolute strength of a circular prism (radius a) under torsion by the assumption of an empirical formula, similar to that of our Art. 178, for the shear q at distance r from the axis of torsion. Namely:

$$q = Q\left[1 - \left(1 - \frac{r}{b}\right)^m\right]$$

where Q is the shear at distance b, and m is a constant.

We are only told in favour of this formula, (1) that for small values of r and for very small shears q is proportional to r and thus to the slide, (2) that q increases less rapidly than r, or the slide, when the slide becomes greater.

If S_1 be the rupture shear and correspond to $r = a$, we have

$$Q = S_1/\{1 - (1 - a/b)^m\}.$$

Then, introducing the same sort of questionable condition as in our Art. 178, namely that $dq/dr = 0$ when $r = a$, we have further

$$a = b \text{ and } S_1 = Q.$$

This leads us to a rupture couple $M_1 = \int_0^\omega rq\,d\omega$,

$$= 2\pi a^3 S_1\left(\frac{1}{3} - \frac{2}{(m+1)(m+2)(m+3)}\right).$$

Or, as m changes from 1 to ∞, M_1 changes from $\frac{1}{2}$ to $\frac{2}{3}$ of $\pi a^3 S_1$ (p. 466)[1].

(*c*) Saint-Venant then attacks the problem of the prism of rectan-

[1] Saint-Venant's result seems to be $\frac{1}{4}$ of the real value, owing to the displacement of a factor 2.

gular cross section ($b \times c$) for which b is much greater than c. Here the approximate values of the slides before the linear limit is passed are:

$$\sigma_{xy} = -2\tau z, \quad \sigma_{xz} = \frac{2\,c^2\mu_1}{b^2\mu_2}\,\tau y.$$

These results may be deduced from Art. 46 by replacing the elongated rectangle by its inscribed ellipse and neglecting c^2/μ_2 as compared with b^2/μ_1. See also Table I. p. 39, and Art. 47.

He assumes that after the linear limit is passed:

$$\widehat{xy} = -Q'\left\{1-\left(1-\frac{z}{h}\right)^m\right\}, \quad \widehat{xz} = Q''\left\{1-\left(1-\frac{y}{k}\right)^n\right\}.$$

Hence, since for small slides or small values of z and y, $\widehat{xy} = \mu_1\sigma_{xy}$ and $\widehat{xz} = \mu_2\sigma_{xz}$, we must have:

$$-\mu_1 2\tau = -\frac{Q'm}{h}; \quad \mu_2\frac{2\,c^2\mu_1}{b^2\mu_2} = \frac{Q''n}{k}.$$

These give, $$Q''_2 = Q'_1\,\frac{m}{n}\,\frac{c^2}{b^2}\,\frac{k}{h}.$$

Further, since the fail-points are the mid-points of the much longer side b, the rupture points are taken there also. Thus it is necessary that:

$$d\widehat{xy}/dz = 0, \quad \widehat{xy} = S' \text{ when } z = c/2.$$

It follows that $h = c/2$ and $Q' = S'$, the *absolute* shearing strength in direction of y.

To proceed further Saint-Venant assumes that the slide σ_{xz} always remains much less than the slide σ_{xy}, so that for the former it is sufficient to retain the linear strain form, we have thus

$$\widehat{xz} = Q''ny/k = S'm\,\frac{c}{b}\cdot\frac{2y}{b},$$

together with $$\widehat{xy} = -S'\left\{1-\left(1-\frac{2z}{c}\right)^m\right\}.$$

It easily follows that

$$M_1 = bc^2 S'\left(\frac{m}{6} + \frac{2^{m-1}\,m}{(m+1)(m+2)}\right).$$

Cases (b) and (c) confirm the law of the cube stated in (a). Such formulae, although by no means satisfactory from the theoretical standpoint, are yet useful as suggesting lines for future experiment.

[185.] Pages 469—77 (§ 62) contain a useful discussion of the various methods of determining the elastic and fail-point constants, especially in the case of prisms whose material is transversely aeolotropic. Saint-Venant (p. 471) adopts the result given in Art. 5. d. of our account of the memoir on *Torsion*, $\bar{\sigma}_{yz} = 2\sqrt{\bar{s}_y\bar{s}_z}$, to obtain a plausible relation between shear fail-limit (S_0) and tractive fail-limits T_0 and T_0'. We have thus the formula $S_0/G = 2\sqrt{T_0/E \times T_0'/E}$.

[186.] Pages 477—480 deal with the problem of the torsion of circular cylinders (radius a) having a cylindrical distribution of elastic homogeneity. In this case μ is a function of the axial distance r. There will be no distortion of cross-sections. Saint-Venant supposes μ to remain constant from the axis up to a radius $a - \zeta$, and then at distances r from $a - \zeta$ to a to follow the law

$$\mu = \mu_0 + (\mu_1 - \mu_0)\left(\frac{z}{\zeta}\right)^n \text{ where } z = r - (a - \zeta).$$

He easily deduces the following formula for M,

$$M = \mu_0 \tau \frac{\pi a^4}{2} + 2\pi(\mu_1 - \mu_0)\tau \left\{\frac{(a-\zeta)^3\zeta}{n+1} + \frac{3(a-\zeta)^2\zeta^2}{n+2} + \frac{3(a-\zeta)\zeta^3}{n+3} + \frac{\zeta^4}{n+4}\right\}.$$

Special cases are:

(1) Wooden cylinder whose axis is about the same as that of the tree out of which it has been cut; here we may put $\zeta = a$, and we have:

$$M = \frac{n\mu_0 + 4\mu_1}{n+4}\,\omega\kappa^2\tau.$$

(2) Forged or cast iron cylinder with skin change of elasticity:

$$M = \tau\,(\mu_0\omega\kappa^2 + \gamma 2\pi a^3) \text{ where } \gamma = \frac{\mu_1 - \mu_0}{n+1}\zeta.$$

Supposing the fail-point to be on the surface, we have $S_0 = \mu_1 \tau a$, and eliminating τ:

$$M_0 = A\pi a^3 + B\pi a^2,$$

where A and B are two constants depending only on the elastic nature of the material. Thus the fail-couple depends partly on the cube, partly on the square of the radius of the cylinder.

[187.] The text of the work concludes with numerical examples such as are given on pp. 551—8 of the memoir on *Torsion*. The remainder of the volume is filled with five appendices and an *Appendice complémentaire* occupying pp. 510—849, which from their historical and physical aspects are perhaps the most interesting portions of the work.

[188.] Appendix I. (pp. 512—19) contains certain elementary proofs due to Poncelet as to the curvature, deflection etc. of the elastic line. A point on p. 518 on the question of built-in terminals (*encastrements*) may be noted. Poncelet remarks that for a cantilever we may suppose two forces, whose resultant is equal and opposite to that of the load, to act at the built-in end. These forces—whose points of application are very close, one on the upper and one on the lower surface of the beam—are very great and alter the surfaces of the built-in beam and the surrounding material, so that the elastic line at this end is not horizontal, but takes a certain inclination varying as the terminal moment directly and inversely as the *profondeur de l'encastrement.* Small as this inclination is, it affects sensibly the experimental accuracy of the theoretical results based on the perfect horizontality of the elastic line at the built-in end. This was noted by Vicat: see our Art. 733*. Saint-Venant holds that careful experiments ought to be made to determine its influence.

[189.] Appendix II. is entitled: *Sur les conditions de l'exactitude mathématique des formules tant anciennes que nouvelles d'extension, de torsion, de flexion avec ou sans glissement.—Démonstration synthétique de ces formules quand on suppose ces conditions remplies.* This appendix contains first an easy refutation of Lamé's ill-judged sneer at the *procédés hybrides, mi-analytiques, et mi-empiriques ne servant qu'à masquer les abords de la véritable science:* see our Arts. 1162* and 3. Saint-Venant shews that his methods have precisely the same validity as those adopted in the cases of simple traction, of the old theories of flexure, and of torsion for a circular cylinder. In the sequel he demonstrates afresh the torsion and flexure equations. He starts from an axiom and definitions involving the hypothesis of central intermolecular action as a function of the central distance only. The appendix occupies pp. 520—541.

[190.] Appendix III. contains a complete theory of elasticity for aeolotropic bodies so far as the establishment of the general equations of elasticity and the usual formulae of stress and strain are concerned. It occupies pp. 541—617. Proceeding from central intermolecular action, Saint-Venant on pp. 556—9 reduces the 36 constants of the stress-strain relations to 15. We may note one or two points of interest:

(*a*) § 23 (pp. 562—74) with its long footnote is specially worthy of the reader's attention. Saint-Venant obtains expressions for the stresses on the hypothesis that initial stress has produced *considerable* initial strain in the body. In this case the strain developed by the initial stress sensibly influences the effect of the later strain. We can no longer add initial and secondary stresses as independent factors of total stress.

Let the initial stresses be $\widehat{xx}_0$, $\widehat{xy}_0$ etc...., the secondary stresses $\widehat{xx}_1$, $\widehat{xy}_1$ etc., and the total stresses $\widehat{xx}$, $\widehat{xy}$, etc.

Let x_1, y_1, z_1 be the directions of three right-lines slightly oblique to each other which were initially the rectangular set x, y, z; let x', y', z' be three other lines rectangular or slightly oblique among themselves, taken close to the former (x, y, z) and normal to the three planes by which we determine the six stress-components. Then, if c_{rs} be the cosine of the angle between the lines r and s we have as stress types:

$$\left.\begin{aligned}\widehat{x'x'} &= \widehat{xx}_0\,(1 + s_x - s_y - s_z) + 2\,\widehat{xy}_0 c_{y_1x'} + 2\,\widehat{zx}_0 c_{z_1x'} + \widehat{xx}_1,\\ \widehat{y'z'} &= \widehat{yz}_0\,(1 - s_x) + \widehat{yy}_0 c_{y_1z'} + \widehat{zz}_0 c_{z_1y'} + \widehat{zx}_0 c_{x_1y'} + \widehat{xy}_0 c_{x_1z'} + \widehat{yz}_1.\end{aligned}\right\}\ldots \text{(i)}.$$

To these we must add the purely geometrical relations of the type:

$$c_{y_1z'} + c_{z_1y'} = \sigma_{yz} + c_{y'z'} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

which reduce if x', y', z' are taken rectangular to the type:

$$c_{y_1z'} + c_{z_1y'} = \sigma_{yz} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}.$$

When, however, the initial stress is not such that the shears are zero or can be neglected when multiplied by small strains, we may simplify equations (i) by a proper choice of x', y', z'. Thus if x', y', z' be taken perpendicular to y_1, z_1, x_1 or z_1, x_1, y_1 respectively, which is compatible with their rectangularity, then either $c_{z_1y'} = c_{x_1'z'} = c_{y_1x'} = 0$, or, $c_{y_1z'} = c_{z_1x'} = c_{x_1y'} = 0$, and we can replace the remaining cosines in (i) by the slides $\sigma_{yz}, \sigma_{zx}, \sigma_{xy}$. By taking x', y', z' bisectors of the angles between the lines x_1, y_1, z_1 and the perpendiculars to the three slightly oblique planes y_1z_1, z_1x_1, x_1y_1, i.e. the closest rectangular system to x_1, y_1, z_1, we obtain:

$$c_{y_1z'} = c_{z_1y'} = \tfrac{1}{2}\,\sigma_{yz} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

as the type of equation (iii).

In the case (*le seul qui ait été supposé par les divers auteurs de mécanique moléculaire*, ftn. p. 571) in which the shifts are very small and consequently the directions x_1, y_1, z_1 almost coincident with x, y, z, we can take the latter for the rectangular system x', y', z' and we thus find:

$$c_{y_1z} = dw/dy, \quad c_{z_1y} = dv/dz, \quad c_{x_1y} = dv/dx,$$
$$c_{y_1x} = du/dy, \quad c_{z_1x} = du/dz, \quad c_{x_1z} = dw/dx,$$

and reach the equations (i) of our Art. 129.

These again reduce to the relations of our Art. 666*, if we put $\widehat{xx}_0 = \widehat{yy}_0 = a$, $\widehat{zz}_0 = c$, and $\widehat{yz}_0 = \widehat{zx}_0 = \widehat{xy}_0 = 0$, and give the proper values to the secondary stresses.

Saint-Venant proves equations (i) by the molecular method in the footnote before referred to. He makes some remarks on the Navier-Poisson controversy, and refers to a paper of his own published in 1844 on Boscovich's system: see our Arts. 527* and 1613*.

(*b*) On p. 587 the remark is made that the stress-strain relations, the body stress-equations and the body strain-equations remain true whatever be the amount of the shifts in space provided the *relative* shifts of adjacent parts or the strain-components are small. In this case, however, the values to be given to the strains in terms of the shifts are those of our Art. 1618*. The ordinary shift-equations of elasticity hold only for small portions of an elastic body, when the total shifts are not small. Hence they cannot be directly applied to large torsional or flexural shifts. The whole treatment on pp. 587—92 is good, and better than that of the memoir of 1847: see our Art. 1618*.

(*c*) Saint-Venant points out that it is not sufficient to find values of the stress-components which satisfy the body and surface stress-equations. There are also certain *conditions of compatibility* between the strain components deduced from these stresses which also must be satisfied: see our Art. 112.

These equations hold for *all values of the shifts*, provided the strains remain small, i.e. if they take the forms given in our Art. 1618*.

(*d*) Pp. 603—17 contain a direct investigation of Saint-Venant's torsion and flexure equations from the general equations of elasticity. In both cases the method adopted assumes a given distribution of stress and deduces the corresponding shift-equations.

In dealing with torsion Saint-Venant supposes a single plane of elastic symmetry perpendicular to the axis of torsion, and starts from formulae for the shears of the form

$$\widehat{xy} = f\sigma_{xy} + h\sigma_{zx}, \qquad \widehat{zx} = e\sigma_{zx} + h'\sigma_{xy},$$

where h and h' are supposed unequal. See our Art. 4 (θ) on the memoir on *Torsion*. He deduces the general torsional equations, which now contain four constants, and solves them for the case of the ellipse. The discussion does not seem to me of much value, as all elasticians, multi- or rari-constant, would agree that $h = h'$, in which case by a change of axes we can take $h = h' = 0$: see the same Article. In the case of an elliptic contour a direct analysis gives:

$$M = \frac{4\tau\omega}{1/\mu_1\kappa_1^2 + 1/\mu_2\kappa_2^2 + (1/\kappa_1^2 - 1/\kappa_2^2)(1/\mu_2 - 1/\mu_1)\sin^2\alpha},$$

where α is the angle between the direction in which the slide-modulus is μ_1 and the axis of the ellipse about which the swing-radius is κ_1. The reader must note that μ_1 and μ_2 are not the same constants as in Art. 46 of our discussion of the memoir on *Torsion*, where we supposed 'the principal axes of elasticity' to coincide with the principal axes of the elliptic section.

[191.] The fourth Appendix occupies pp. 617—45 and contains a careful comparison of Saint-Venant's theory of Torsion with the experimental results of Wertheim, Duleau and Savart: see our Arts. 1339* and 31. It is followed by some discussion of torsional vibrations. This appendix is practically directed against Wertheim's memoir on *Torsion* of 1857: see our Art. 1343*. It will be remembered that Wertheim had asserted the theoretical accuracy of Cauchy's erroneous torsion formula (see our Art. 661*), had persisted in retaining the value for the squeeze-stretch ratio which he had deduced by a fallacious theory in 1848 (see our Art. 1319*), and finally had exhibited complete ignorance of Saint-Venant's results for the elliptic cylinder. Saint-Venant easily shews the insufficiency of Wertheim's criticism, and how the mean results of Savart and Duleau for rectangular prisms, and of Wertheim himself for elliptic prisms confirm the new theory: see our Arts. 31 and 35.

In the discussion on torsional vibrations, Saint-Venant reproduces the matter of his memoir of 1849: see our Art. 1628*. He regards of course the theory given as only *approximate* (p. 633), but sufficiently so for all practical purposes, as indeed appears from the comparison of theory and experiment (p. 643).

[192.] The fifth Appendix, devoted to the elastic-constant controversy, occupies pp. 645—762. It is an excellent piece of scientific criticism, to which some multi-constant elasticians have insufficiently replied by squeezing caoutchouc or loading pianoforte wires. The difficulty of critical experiments lies first in obtaining a purely isotropic material free from all initial stress and without any superficial elastic variation, and then in assuring the extreme nicety required to determine successfully the stretch-squeeze-ratio. In our first volume we have referred to the leading features of the controversy (see our Arts. 921*—932*) and the chief of the earlier experiments in this field (see our Arts. 470*, 1034*, 686*—90*, 1358*). We shall find other remarkable experiments as well as theoretical conclusions have sprung from the controversy in the last 40 years; these will lead us on more than one occasion to examine the validity of Saint-Venant's arguments. Meanwhile we may refer to one or two points brought forward in the present essay.

(*a*) Saint-Venant's criticism seems to me unanswerable, when he attacks the validity of the method by which Poisson, Cauchy, Green, or

Lamé have deduced the linearity of the stress-strain relation without any appeal to experiment or any statement of physical fact or any axiom of intermolecular action: see Saint-Venant's pp. 660—5 and our Arts. 553*, 614*, 928*, 1051* and 1164* footnote.

(*b*) On pp. 665—676 we have a long and careful numerical examination of the experiments of Regnault on piezometers of copper and brass. In general they accord with the uni-constant theory, or at least better with that than with Wertheim's (see our Art. 1319*). This is followed by some remarks on Wertheim's and Clapeyron's experiments on caoutchouc. The former found $\frac{\mu}{\lambda} = -\frac{1}{4}$ to $\frac{3}{4}$, and the latter $\frac{\mu}{\lambda} = \frac{1}{2202}$: see our Art. 1322* and Chapter XI. Results so discordant as these lead Saint-Venant to remark that neither uni- nor bi-constant isotropy, nor

même des formules linéaires quelconques, ne sont pas applicables au caoutchouc, liquide coagulé ou épaissi plutôt que solidifié, et d'une nature en quelque sorte intermédiaire entre les fluides et les solides (p. 678).

(*c*) Pages 679—89 are occupied with a criticism of Wertheim's hypothesis, that $2\mu = \lambda$, and with the results of his experiments. Saint-Venant points out the great probability of a want both of homogeneity and of isotropy in the cylinders used by Wertheim (see our Art. 1343*) and he examines analytically the ratio of longitudinal to transverse stretch-moduli, when such isotropy is not presupposed. We shall return to some of Saint-Venant's arguments when examining Wertheim's later memoirs.

(*d*) On pp. 689—705 we have a consideration of Cauchy's hypothesis of 1851: see our Art. 681*, namely, that it is possible if a body be crystalline that:

les coefficients des déplacements et de leurs dérivées dans les équations d'équilibre intérieur ne sont plus des quantités constantes, mais deviennent des fonctions périodiques des coordonnées (p. 689).

In other words we arrive at stress-strain relations in which the 36 constants are *not* connected by 21 relations. Saint-Venant conducts a new investigation (pp. 697—706) with fairly simple analysis. The turning point of rari- or multi-constancy for such regularly crystallised bodies is then seen to lie in the legitimacy of bringing stretches like s'_x outside certain summations of the form

$$\Sigma_x R \cos^3(rx) \,.\, s'_x, \quad \Sigma_x R \cos(rx)\cos^2(rz) \,.\, s'_z,$$

and replacing them by their mean values s_x, s_z. Here s_x is the mean value of s'_x for all the atoms under consideration, and we may replace s'_x by s_x if the body is *isotropic* or possesses *confused crystallisation.* On the other hand in regularly crystallised bodies, there may be terms in s'_x periodic in the coordinates and we cannot replace s'_x by s_x and bring the mean stretch outside the summation. Hence we have not the 21 relations between the coefficients fulfilled. Saint-Venant holds, however, that even if this periodicity be true for regularly crystallised bodies, it can only introduce small differences into the otherwise equal constants. But further, if it does exist

cette altération ne peut regarder que *certains* cristaux réguliers. Elle n'est jamais relative aux corps à cristallisation confuse, comme sont tous les matériaux de construction, et comme sont aussi tous les corps isotropes. Il n'y a donc aucune raison de changer les formules trouvées depuis un tiers de siècle pour les pressions dans ces sortes de corps (p. 705).

[193.] On pp. 706—742 we have an analysis and criticism of the various methods which English and German elasticians have adopted in order to obtain the fundamental equations of elasticity; there is also a *résumé* of their views on the elastic-constant controversy. Here the memoirs of Green, Neumann, Haughton, Clebsch, Clausius, Thomson, Kirchhoff, Maxwell and Stokes are briefly considered. Saint-Venant devotes special attention (pp. 721—32) to the value of Green's results as bearing on double refraction and the disappearance of the stretch-wave. This discussion is only of importance to us in its bearing on the elastic-constant controversy. Green's treatment of the ether demands the independence of the 21 constants, but we may question whether his results are the only possible ones, nay, even whether they are so satisfactory, as to stand *per se* as a justification of multi-constant formulæ.

In order that the vibrations may be exactly parallel to the wave-face Green finds the relations xxxviii. of our Art. 146.

If to these 14 conditions of Green we were to add the six additional conditions of rari-constancy, namely:

$$|yyzz| = |yzyz|, \quad |xxyz| = |zxxy|, \text{ etc.} \quad\text{.................(i)}$$

we should then have:

$$|xxxx| = |yyyy| = |zzzz| = 3\,|yzyz| = 3\,|zxzx| = 3\,|xyxy| = 3\,|yyzz|$$
$$= 3\,|zzxx| = 3\,|xxyy| \quad\text{.................(ii)}$$

and all the other constants zero.

Thus the condition for exact parallelism would be *isotropy*, or this parallelism would be incompatible with double refraction. Now are Green's conditions so extremely probable that we ought to reject the six molecular conditions (i) which render them nugatory? Saint-Venant argues that they are not, chiefly for the reason that they involve: $|x'x'x'x'| = |xxxx|$.

This is proved in the footnote p. 726. It denotes physically

9—2

that, in whatever direction we take x', the same stretch $s_{x'}$ will produce a traction $\widehat{x'x'}$ of the same intensity. Such an equality seems opposed to our ideas on the nature of bodies endowed with double refraction. The arguments used to support the improbability of this relation are identical with those of the memoir of 1863 and have been cited in our Art. 147.

[194.] While recognising the weight of Saint-Venant's reasoning in this Appendix and in the memoir of 1863, and admitting the difficulty of conceiving a double-refracting medium to obey such conditions as those given by Green, we have yet to notice a point with regard to the arguments Saint-Venant advances. A distinction must be drawn between an isotropic body held by external pressures in an aeolotropic state of *elastic strain*, and a body also primitively isotropic which has received *set* of different intensity in different directions. In the former case the initial stresses may enter into the elastic constants (as in our Art. 129) and so affect the elasticity in different directions. In the latter it would *appear* as if the molecules must be brought in some directions nearer together and so the direct stretch coefficients be affected and varied. But is this experimentally the fact? If a bar of metal be taken and stretched beyond the elastic limit, so that it receives set, it is found that its *stretch-modulus*, which is certainly a function of the direct stretch-coefficients remains nearly constant. Now this set may be of two kinds, first: a set occurring far below the yield-point, which is often little more than a removal of an initial state of strain due to the working: and secondly, a set which denotes a large change in the relative molecular positions and can occur after the yield-point has been reached. If it can be shewn that the stretch-modulus remains nearly constant notwithstanding one or both of these sets, it would be interesting to investigate experimentally whether such is also true for the slide-moduli and the cross-stretch coefficients before we condemn Green entirely.

Experiments on simple traction and torsion of *large bars* before and after very sensible set would throw light on this matter.

[195.] Saint-Venant further remarks that Green's conditions are not *necessary* in order that we may obtain exactly Fresnel's wave-surface. Saint-Venant in a foot-note gives a fairly easy analysis leading to Cauchy's four conditions which are compatible with rari-constancy (see the memoir of 1830, *Exercices mathématiques* 5^e année). These conditions are given in our Art. 148 as Equations xxxix.

Les *quatre* relations ou conditions (xxxix)......n'ont rien d'arbitraire ni de bizarre, bien qu'elles soient d'une forme moins simple à coup sûr que les *cinq* conditions de Green (xxxviii of our Art. 146) qui n'en sont qu'un cas particulier....En effet lorsque les trois coefficients d'élasticité directes a, b, c, entre lesquels elles permettent telle inégalité qu'on veut, ont des rapports mutuels n'excédant pas $1\frac{1}{2}$ ou 2, il est facile de s'assurer par des calculs qu'elles sont, numériquement, presque identiques aux relations $2d + d' = \sqrt{bc}$, $2e + e' = \sqrt{ca}$, $2f + f' = \sqrt{ab}$ que nous verrons être celles qui donnent la distribution la plus simple des élasticités autour de chaque point dans les corps hétérotropes, et appartenir, au moins avec une grande approximation, aux corps dont l'isotropie primitive a été altérée par de simples compressions ou dilatations inégales, c'est-à-dire généralement aux corps amorphes ou à cristallisation confuse. Or tous les physiciens admettent que c'est seulement à cet état d'inégal rapprochement moléculaire en divers sens que se trouve l'éther dans les cristaux dont la forme n'est pas un polyèdre régulier (p. 731, foot-note).

We have ventured so far from our subject into that of Light, only to shew that Saint-Venant brings forward strong reasons why, even if we dogmatically assert the elastic jelly character of the ether, it is not necessary to summarily reject the rari-constant hypothesis.

[196.] Pages 732—42 are occupied with an excellent discussion of Stokes' memoir of 1845: see our Arts. 925*—6* and 1264*. There are also a few remarks upon Maxwell's memoir of 1850: see our Art. 1536*. Saint-Venant states that Thomson and Kirchhoff while adopting multi-constancy have not added any additional reasons for its validity. This at the present time is hardly true. I may note Kirchhoff's memoir of 1859: see *Poggendorff's Annalen*, Bd. 108, p. 369, and Thomson's of May, 1865: see *Proceedings of Royal Society* for that date. Saint-Venant's objections to those arguments of Stokes which are drawn from the 'doctrine of continuity,'—practically from the equivalence of the plasticity of metals and the viscosity of fluids—seem to me very forcible and should be read by all scientists interested in the ultimate molecular constitution of bodies. In the question of rari- or multi-constancy are involved, not merely points of technical expediency, but principles going to the base of our knowledge of matter,—such as our *proofs* of the equation of energy and the application of the laws of motion to inter-molecular action.

[197.] Pages 742—46 are occupied with a review of Clausius' memoir of 1849: see our Art. 1398*. It is only necessary to remark here that recent experiments would, we think, have removed Saint-Venant's doubt as to the existence of elastic after-strain in metals (p. 745). The appendix concludes with a *résumé* of all the arguments brought forward in favour of rari-constancy (pp. 746—62).

[198.] The *Appendice complémentaire* is chiefly occupied with an examination of the elastical researches of Rankine, Clebsch and Kirchhoff, which Saint-Venant tells us had not then been properly studied in France. We note one or two points:

(*a*) In § 78 (pp. 764—7) Saint-Venant cites experiments of Morin to prove the linearity of the stress-strain relation. These experiments are really not conclusive, and I especially distrust the results cited for cast-iron. For elastic strains of such magnitude as occur in structures, the stress-strain relation for this material is certainly *not* linear. Nor again can arguments drawn from wires reduced to a state of ease serve the purpose Saint-Venant has in view of demonstrating the linearity and perfect elasticity of all materials for small strains.

(*b*) § 80 (pp. 771—4) treats of what Saint-Venant terms *l'état dit naturel ou primitif.* This is the state of no internal stress. It is used as a means of deducing the uniqueness of the solution of the elastic equations. If there be no body force or surface load the internal stresses are all zero, and *vice-versâ.* I have already had occasion to remark on the caution with which this principle must be accepted: see our Arts. 6 and 10. The arguments of this section do not seem to me very convincing.

(*c*) On pp. 783—86 the reader will find some interesting notes and valuable historical references on the origin of the terms *potential* and *potential function.*

(*d*) § 84 (pp. 789—96) reproduces the erroneous method of the memoir of 1863, for finding the stresses when there is an *initial state of stress.* C. Neumann (see our Chapter XI.) had previously obtained similar results for the case when the initial stress is given by an *uniform traction:* see our Arts. 129—31.

(*e*) Pages 801—25 are occupied with an important discussion of the distribution of elasticity in aeolotropic bodies. Saint-Venant using the symbolic method of Rankine arrives at some of the results of his memoir of 1863 : see our Arts. 135—7.

The investigation of the *tasinomic* equation for particular cases, of the distribution of the stretch-moduli, and of the ellipsoidal distribution of elasticity in amorphic solids or cases of *confused crystallisation* follow the lines of the memoir of 1863: see our Arts. 136 and 151. They are accompanied by a discussion of the experimental results of Hagen, Chevandier and Wertheim, as bearing upon this theoretical distribution of elasticity. We shall return to this point when treating of the annotated *Clebsch*: see our Arts. 306—13.

(*f*) The remaining pages of the volume (825—49) are occupied with a sketch of Clebsch's treatment of the problem of torsion and flexure (see his *Theorie der Elasticität* § 23) and Kirchhoff's memoir on rods (see *Crelles Journal,* T. 56, p. 285, *Ueber das Gleichgewicht und die Bewegung eines unendlich-dünnen elastischen Stabes*). Saint-Venant shews how they are in agreement with his treatment of the problem, but does not contribute any additional matter.

[199.] Our analysis of Saint-Venant's edition of the *Leçons de Navier* will, we hope, have gone some way to convince the reader of the thorough study which this work deserves. Taken in conjunction with the annotated *Clebsch* (see our Art. 297) it forms the best introduction to the wide subjects of elasticity and the strength of materials yet published.

Section IV.

Memoirs of 1864—1882.

Impulse, Plasticity, etc.

[200.] *Compléments au Mémoire lu le* 10 *août* 1857 *sur l'impulsion transversale et la résistance vive des barres, verges ou poutres élastiques.*

Comptes rendus, T. LX. 1865, pp. 42—47 and pp. 732—35, T. LXI. 1865, pp. 33—37 and T. LXII. 1866, pp. 130—134. These extracts of additions to the memoir of 1857 (see our Arts. 104—8) are all more fully developed in the annotated *Clebsch*: see our Arts. 342 *et seq.*

[201.] *Note sur les pertes apparentes de force vive dans le choc des pièces extensibles et flexibles, et sur un moyen de calculer élémentairement l'extension ou la flexion dynamique de celles-ci*: *Comptes rendus*, T. LXII. 1866, pp. 1195—99.

This note suggests the application of the principle of virtual displacements and of the hypothesis that dynamical strain is of the same form as statical strain to the problem of impact. Saint-Venant apparently considers that in his papers of 1865—66 he had been the first to adopt this method, but as we have seen it is really due to Cox: see our Art. 1434*. The discussion in this *Note* appears in a more consistent form in the annotated *Clebsch*: see our Art. 368. It is Saint-Venant's great service to have shewn that the accurate and approximate methods agree fairly closely, and *why* they agree. Cox's method gives a result which is almost the same as that given by taking the term involving the principal vibration only. This point is well brought out in the concluding paragraphs of the *Note*, pp. 1198—9.

[202.] *Démonstration élémentaire:* (1°) *de l'expression de la vitesse de propagation du son dans une barre élastique;* (2°) *des formules nouvelles données, dans une communication précédente, pour le choc longitudinal de deux barres: Comptes rendus*, Tome

LXIV. 1867, pp. 1192—5. This is an extract from a memoir afterwards published in the *Journal de Liouville*: see our Arts. 203—20. Other parts of the same memoir are extracted in *Comptes rendus,* T. LXIII. 1866, pp. 1108—1111, and T. LXIV. 1867, pp. 1009—1013.

[203.] *Sur le choc longitudinal de deux barres élastiques de grosseurs et de matières semblables ou différentes, et sur la proportion de leur force vive qui est perdue pour la translation ultérieure; ...Et généralement sur le mouvement longitudinal d'un système de deux ou plusieurs prismes élastiques: Journal de Liouville,* T. XII. 1867, pp. 237—376, (the last two pages containing errata).

This is a long and theoretically very interesting memoir on the longitudinal impact of rods. It is the first complete treatment of the subject published. German writers have made some claim in this respect for Franz Neumann, who in his Königsberg lectures of 1857—8 dealt with the problem in somewhat the same fashion. But Neumann's investigations as first published in the *Vorlesungen über die Theorie der Elasticität,* 1885, pp. 340—346, are very insufficient and incomplete as compared with Saint-Venant's. Experimental investigations have been made by Boltzmann, W. Voigt, Hausmaninger and Hamburger with a view to testing the theory. Their results are not in full accordance with Saint-Venant's formulae. I shall refer to certain points of difference in discussing the present memoir, but the articles devoted to their memoirs must be consulted for fuller details.

[204.] The memoir is divided into two parts, the first treats of the impact of two rods of the same material and of equal cross-section. It is divided into seven articles. The first of these (pp. 237—244) deals with the history of the problem. At the invitation of Coriolis in 1827 Cauchy had investigated the influence of the vibrations produced by impact in altering the translational energy of two rods; Coriolis having recognised that these vibrations must be a source of loss in visible energy. Cauchy accordingly presented on February 19, 1827, a short note to the Academy, which was printed in the *Bulletin...de la Société Philomathique,* December 1826, pp. 180—182, and afterwards in the *Mémoires de l'Institut.* Cauchy treated only of the longitudinal impact of two rods of the same material and section. He concluded that the impulse terminated whenever the two bars had not the same speed at their impellent terminals. This, as we shall see, is not true, and the conclusion vitiated some of Cauchy's results, the analysis of which does not appear to have been published.

Poisson in the second edition of the *Traité de Mécanique* (1833, Vol. II. pp. 331—47) also attacked the problem supposing his rods of the same material and cross-section. He used a double condition for separation, namely, not only that the bar which precedes shall have a greater speed at the impelled terminal than that which follows, but that the squeeze in both at the impellent terminals shall be simultaneously zero. This condition led Poisson to the singular conclusion that two unequal bars would never separate. He had forgotten that physically they can never sustain a stretch at the impellent terminals. In fact Cauchy's condition of excess of speed in the preceding bar is insufficient, and Poisson's additional one of no squeeze is superabundant. The true condition is clearly excess of speed at a time when there is zero squeeze at the impellent terminals, which can never sustain a stretch. It will also be necessary to shew that the bars thus separated are separated for good, and do not, owing to their vibrations, come again into contact.

[205.] Saint-Venant's method of treatment is to investigate the vibrations of a bar, of which the initial condition is given by zero stretch throughout, and by speeds constant for each of the several parts into which the rod may be supposed divided. The first instant at which a zero stretch at the section between any two of these parts is accompanied by an excess speed in the terminal of the preceding section marks a disunion if the parts are not those of a continuous rod. In this manner Saint-Venant shews that if two bars of the same section and material are in impact the shorter takes ultimately and uniformly, while losing all strain, the initial speed of the longer.

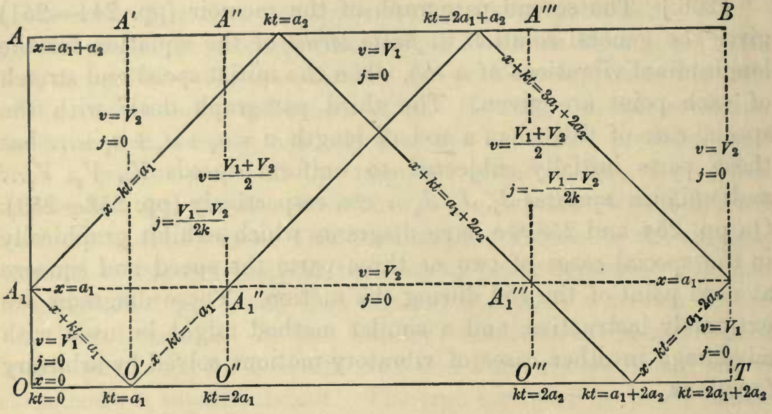
This result was stated by Cauchy in 1826. Saint-Venant refers to the elementary proof of it given by Thomson and Tait in §§ 302—304 of their *Treatise on Natural Philosophy* which in 1867 was in the press. His notice had been drawn to this proof by an article in *The Engineer* (February 15, 1867) due to Rankine who, reviewing the extract in the *Comptes rendus* of Saint-Venant's memoir, had also given an elementary proof of one of his results for rods of different materials and cross-sections.

[206.] The second paragraph of the memoir (pp. 244—251) gives the general solution in *finite terms* of the equation for the longitudinal vibrations of a rod, when the initial speed and stretch of each point are given. The third paragraph deals with the special case of this when a rod of length $a = a_1 + a_2 + a_3 + \ldots$ has these parts initially subjected to uniform speeds V_1, V_2, $V_3 \ldots$ and uniform *squeezes* J_1, J_2, $J_3 \ldots$ etc. respectively (pp. 252—259). On pp. 254 and 258 we have diagrams which exhibit graphically in the special cases of two or three parts the speed and squeeze at each point of the rod during the motion. These diagrams are extremely instructive, and a similar method might be used with advantage in other cases of vibratory motions solved by arbitrary functions.

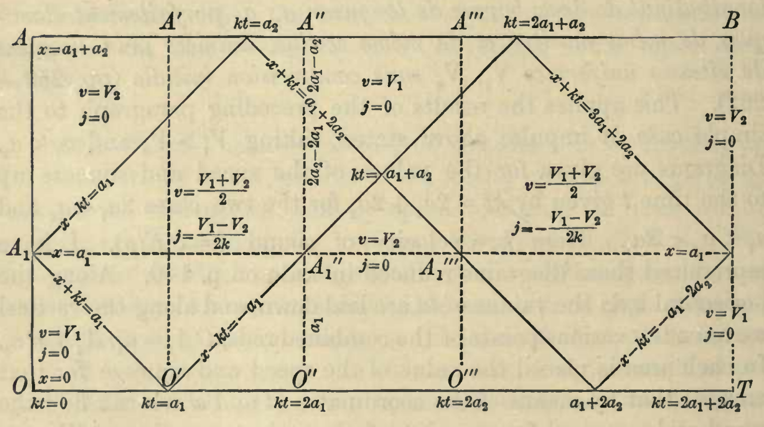
[207.] The fourth paragraph is entitled: *Problème du choc longitudinal de deux barres de longueur a_1, a_2 parfaitement élastiques, de même matière et de même section, animées primitivement de vitesses uniformes* V_1, V_2 *sans compression initiale* (pp. 259—262). This applies the results of the preceding paragraph to the simple case of impulse above stated, taking $V_1 > V_2$ and $a_1 < a_2$. Diagrams are given for the values of the speed and squeeze up to the time t given by $kt = 2a_1 + 2a_2$ for the two cases $2a_1 < a_2$ and $a_1 < a_2 < 2a_1$. Here k = velocity of sound $(= \sqrt{E/\rho})$. I have reproduced these diagrams reduced in scale on p. 140. Along the horizontal axis the values of kt are laid down, and along the vertical we have the various points of the combined rods, $OA_1 = a_1$, $A_1A = a_2$. In each area is placed the value of the speed and squeeze for that area, so that by means of the coordinates kt and x we can find the speed and squeeze of any point of the rod at any time. We see from this that at time $t = 2a_1/k$ the contiguous terminals will be moving with unequal velocities V_2 and $\frac{1}{2}(V_1 + V_2)$ but that this is *only for the instant*, and as there is no stretch at those terminals, the bars will not separate. They afterwards move till $t = 2a_2/k$ with the same velocity at the contiguous terminals and no squeeze. *The impulse is terminated, but the bars do not yet separate.*

Unequal speeds occur again when $t = 2a_2/k$, and now the upper bar has a negative squeeze, $j = -(V_1 - V_2)/2k$ at the

CASE $2a_1 < a_2$.



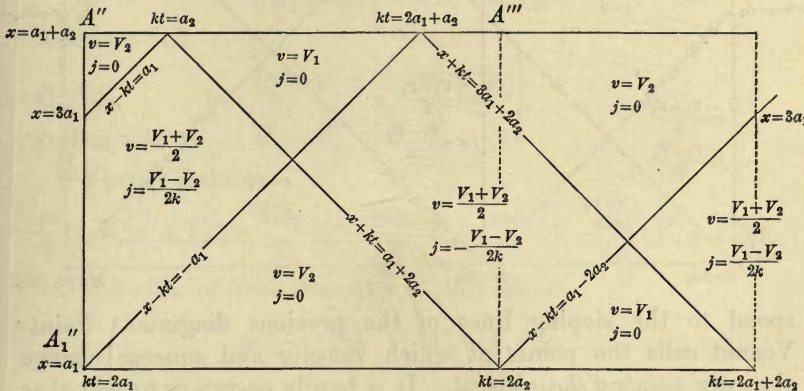
CASE $a_1 < a_2 < 2a_1$.



impelled terminal. Hence the solution no longer holds, and we have to treat each bar from this epoch as a distinct one. The bar a_1 moves obviously without strain and with the speed V_2 which the bar a_2 initially had. To deal with the bar a_2, we have to distinguish two cases. Let us suppose:

(1) $2a_1 < a_2$. We have to enquire how a bar of which a

portion $2a_1$ has initially a speed $v=\frac{1}{2}(V_1+V_2)$ and a negative squeeze $j=-\frac{1}{2}(V_1-V_2)/k$, and a portion a_2-2a_1 a speed $v=V_2$ and a squeeze $j=0$ subsequently moves. This has been ascertained in the second paragraph of the memoir and is represented by Saint-Venant in the accompanying diagram.

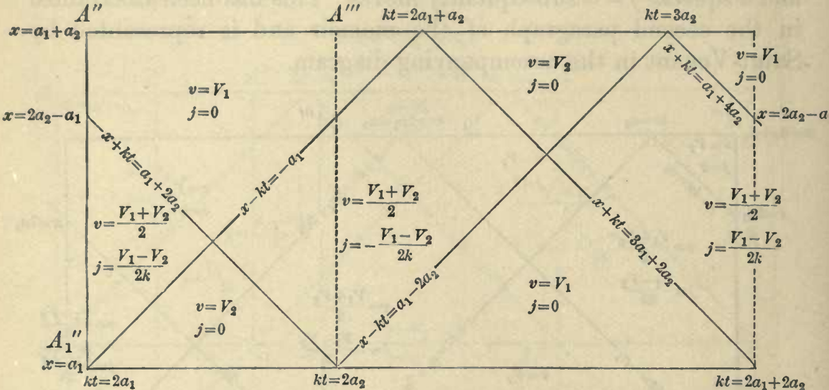


We see at once that after $t=2a_2/k$ the terminal moves with speed V_1 and therefore separates from the terminal of a_1 with speed V_1-V_2. This lasts till $t=2(a_1+a_2)/k$, when what happened at time $t=2a_1/k$ repeats itself and the terminal moves with speed V_2, i.e. with the same speed as the terminal of a_1. Thus it alternately moves with greater and equal speed, or the two terminals never again come into contact.

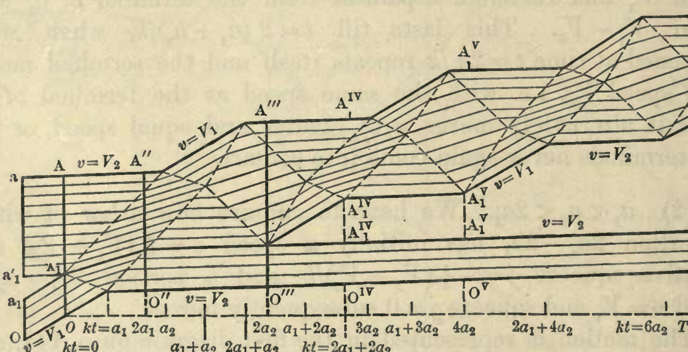
(2) $a_1<a_2<2a_1$. We have to enquire how a bar of which a portion $2a_2-2a_1$ has initially a speed $v=\frac{1}{2}(V_1+V_2)$ and negative squeeze $j=-\frac{1}{2}(V_1-V_2)/k$, and a portion $2a_1-a_2$, a speed $v=V_1$ and squeeze $j=0$ subsequently moves.

The motion is represented in the first diagram on p. 142, and we see that after the time $t=2a_2/k$ these bars never again come into contact.

[208.] The second diagram on p. 142 represents the whole motion of the two bars supposing them to be endowed with a uniform velocity perpendicular to their lengths during and subsequent to the impact. The full lines give the paths of various points of the rods, the dotted lines give the points at which the speed or squeeze of the rods changes abruptly. They corre-



spond to the sloping lines of the previous diagrams. Saint-Venant calls the points at which velocity and squeeze change abruptly *points d'ébranlement.* It is hardly necessary to add that the stretch and squeeze of the rods are for diagrammatic purposes enormously exaggerated.



The separation of the two rods is discussed in Saint-Venant's sixth paragraph, the fifth having been devoted to a verification by means of the solution in trigonometrical series of the general results of the fourth paragraph: see pp. 262—269 of the memoir.

[209.] The seventh paragraph (pp. 278—86) is entitled; *Conséquences.—Force vive translatoire perdue dans le choc des deux* *barres élastiques de même grosseur et de même matière.—Vitesses de translation après le choc.* Let U_1, U_2 be the *centroidal* speeds after the impulse, i.e. at time $t = 2a_1/k$; then, as we have seen on p. 140, $U_1 = V_2$. To obtain U_2 we have only to make use of the principle of conservation of momentum, or

$$a_1 U_1 + a_2 U_2 = a_1 V_1 + a_2 V_2,$$

whence we find

$$\left.\begin{aligned} U_2 &= V_2 + \frac{a_1}{a_2}(V_1 - V_2) \\ \text{together with}\quad U_1 &= V_2. \end{aligned}\right\} \text{..................... (i).}$$

We easily deduce

$$\frac{m}{2}(a_1 V_1^2 + a_2 V_2^2) - \frac{ma_1}{2} U_2^2 - \frac{ma_2}{2} U_1^2 = \frac{ma_1}{2}\left(1 - \frac{a_1}{a_2}\right)(V_1 - V_2)^2.$$

Or, the loss of kinetic energy of translation

$$= \frac{ma_1}{2}\left(1 - \frac{a_1}{a_2}\right)(V_1 - V_2)^2 \text{ .... (ii).}$$

Writing $M_1 = ma_1$, $M_2 = ma_2$, we see the following differences between Saint-Venant's theory and the ordinary theory of the impact of *perfectly elastic* bodies:

	Saint-Venant's theory	Ordinary theory
$U_1 =$	V_2	$V_1 - \frac{2M_2}{M_1 + M_2}(V_1 - V_2)$
$U_2 =$	$V_2 + \frac{M_1}{M_2}(V_1 - V_2)$	$V_2 + \frac{2M_1}{M_1 + M_2}(V_1 - V_2)$
Loss of Energy	$\frac{1}{2} M_1 \left(1 - \frac{M_1}{M_2}\right)(V_1 - V_2)^2$	0

Comparison with the so-called *inelastic* bodies of the ordinary theory gives no better agreement.

[210.] It may be noted here that Voigt's results for rods of equal cross-sections do not agree with Saint-Venant's theory when the shorter is the *impelling* rod. (*Annalen der Physik*, Bd. XIX. 1883, p. 51.) Further Saint-Venant makes the duration of the impulse $= 2a_1/k$, or $= 2a_2/k$ if we take it till the instant when the rods actually separate. In either case the duration of the impulse is proportional to the length of one of the rods and independent of the area of the cross-section. These results do not agree with Hamburger's experiments (*Untersuchungen über die Zeitdauer des Stosses elastischer cylindrischer Stäbe: Inaugural-Dissertation*, Breslau 1885, pp. 23—27). Hamburger finds that the duration is a function of the velocity of impact, which contradicts Saint-Venant's results.

[211.] The second part of Saint-Venant's memoir is entitled: *Choc de deux barres dont les sections et les matières sont différentes.*

The first paragraph (§ 8, pp. 286—98) gives in a double form the solution of the problem of the motion of two contiguous rods:

1° in trigonometrical series. This result Saint-Venant had obtained in an earlier memoir: see our Arts. 107 and 200. He adds the solution for beams in the form of truncated cones as given in the *Comptes rendus*, LXVI.; see our Art. 223. He remarks of these solutions:

> Au mémoire cité, complément de ceux que j'ai présentés depuis 1857 et qui vont être imprimés au *Journal de l'École Polytechnique*, on trouvera le développement de cette solution, à laquelle il convient de recourir quelquefois même pour les barres prismatiques, comme nous verrons plus loin, notamment quand une des deux parties a une section relativement fort grande, une longueur fort petite ou une résistance élastique considérable; suppositions qui poussées plus loin encore, permettent de réduire l'une des deux parties ou barres à une masse étrangère parfaitement dure, pouvant être venue heurter l'autre barre supposée libre aussi, ce qui constitue un problème dont la solution directe, a été présentée en 1865 (*Comptes rendus*, T. LXI., p. 33: see our Arts. 200 and 221).

2° in finite terms. This solution is somewhat lengthy, but is accompanied by diagrammatic representations of speed and squeeze of the same character as in the simpler case when the bars have equal cross-sections and sound-velocities. It is of a more complex nature, however, in particular the sloping lines become more numerous and change their slope abruptly at the horizontal line which marks the contiguous terminals: see p. 297 of the memoir.

[212.] The general solution is applied to the special case of the impact of two rods where initially the squeeze is zero throughout and the velocities are respectively V_1, V_2: see the ninth and tenth paragraphs. The results are again of a somewhat complex nature, but are rendered more intelligible by the aid of diagrams. They occupy pp. 299—326 of the memoir.

[213.] The eleventh paragraph is entitled: *Conséquences, en ce qui regarde le mouvement des deux barres après l'instant de leur choc, leur séparation, et les vitesses à l'instant où elle s'opère* (pp. 327—336).

Let $M_1 (= m_1 a_1)$, a_1, k_1, E_1, V_1, v_1, j_1 be the mass, length, velocity of sound, stretch-modulus, initial velocity, and velocity and squeeze of any point at any time of the first bar; similar quantities with the subscript 2 will refer to the second bar. Let $r = m_2k_2/(m_1k_1)$, and $\tau_1 = a_1/k_1$, $\tau_2 = a_2/k_2$. We shall suppose $\tau_1 < \tau_2$ or that sound traverses the following in less time than it does the preceding bar; this supposition is allowable as we can choose arbitrarily which sense of the velocity shall be considered positive. In discussing the results of the investigation we have to consider three possible cases:

$$r = 1, \quad r > 1, \quad \text{and} \quad r < 1.$$

Case (i). $\qquad r = 1$, or $m_2k_2 = m_1k_1$.

The impulse ends when $t = 2\tau_1$, but the bars do not separate until $t = 2\tau_2$. We have for the centroid-velocities after impact,

$$U_1 = V_2, \quad U_2 = V_2 + \frac{M_1}{M_2}(V_1 - V_2).$$

Thus the two rods behave in this case exactly like bars of the same material and of equal cross-section.

Case (ii). $\qquad r > 1$, or $m_2k_2 > m_1k_1$.

The impulse ends and the bars separate when $t = 2\tau_1$.
In this case:

$$U_1 = V_2 - 2\frac{m_2k_2}{m_1k_1 + m_2k_2}(V_1 - V_2),$$

$$U_2 = V_2 + 2\frac{M_1}{M_2}\frac{m_2k_2}{m_1k_1 + m_2k_2}(V_1 - V_2).$$

Case (iii). $\qquad r < 1$ or $m_2k_2 < m_1k_1$.

The bars no longer separate when $t = 2\tau_1$, but at the instant given by $t = 2\tau_2$.

If n be a whole number such that

$$n\tau_1 < \tau_2 < (n+1)\tau_1,$$

then:

$$U_1 = V_2 + \left(\frac{m_1k_1 - m_2k_2}{m_1k_1 + m_2k_2}\right)^n \left\{1 - \frac{2m_2k_2}{m_1k_1 + m_2k_2}\left(\frac{\tau_2}{\tau_1} - n\right)\right\}(V_1 - V_2),$$

$$U_2 = V_2 + \frac{M_1}{M_2}(V_1 - U_1).$$

where the value for U_1 on the right of the value for U_2 must be substituted from the first expression.

[214.] It will be observed that these formulae are again widely removed from those of the ordinary theory. They have been tested by Voigt for the velocities U_1 and U_2 of rebound, and by Hamburger for the duration of the impact. Neither find a really sufficient experimental accordance. Voigt attributes the discrepancy to the hypothesis adopted for the contiguous terminals, and considers that the rods cannot, while the contiguous terminals are in contact, be replaced by a single rod. He proposes a new theory, which introduces an elastic couch of some indefinite material (*Zwischenschicht*) between the terminals. This in a limiting case reduces the expressions for U_1 and U_2 to those of the ordinary theory, which in the same case agrees fairly with the results of experiment. In the general case, however, he has neither sufficiently specialised his hypotheses nor worked out his analytical results, so that we are unable to form any but the vaguest comparison of theory and experiment. His constants are unknown functions of material and of cross-section, and there seems no means of determining their form: see our discussion of his memoir later. A good test of Saint-Venant's theory might be made by experimenting in a vacuum and so removing a portion of Voigt's couch. I am inclined to think the discrepancy has more to do with thermal effect than with the couch of air, and that we ought to seek for results corresponding to those of the ordinary theory not when the coefficient of elastic impact is taken as unity, or the 'elasticity perfect,' but when it has a value differing from unity and so allowing for a loss of energy by heat. The problem ought not to be impossible with the aid of Duhamel's thermo-elastic equations.

[215.] In the twelfth paragraph (pp. 336—342) it is shewn that the bars after separating at time $t = 2\tau_1$, or $= 2\tau_2$ as the case may be, do not again come into contact. The thirteenth paragraph represents by diagrams similar to the figure on our p. 142 the motion of the two bars before, during and after the impact. These diagrams bring out very clearly the time of separation, and in *Case* (iii.), $r < 1$, shew how both bars retain a portion of the energy in the vibrational form, while in the previous case one bar only has any vibrational energy: see pp. 342—7 of the memoir, especially the diagram p. 345.

[216.] The following or fourteenth paragraph (pp. 347—50) is entitled: *Condition générale de séparation des barres à un instant donné quelconque, exprimée en fonction des vitesses et des compressions de leurs extrémités jointives à cet instant.*

Let V_2', J_2' be the velocity and squeeze of the bar a_2, supposed to be the impelled or preceding bar, at the point of contact.

Let V_1', J_1' be the like quantities for the impelling or following bar.

Saint-Venant deduces the necessary and sufficient condition for separation as follows:

Supposons en premier lieu, ce qui est permis, qu'elles se séparent pendant un temps *infiniment petit*. Le diagramme (23) du no. 3 relatif aux barres se mouvant isolément, ou le théorème qu'on en déduit, énoncé à la fin de ce même numéro, montre que leurs vitesses, au point de leur jonction, deviendront immédiatement après:

$$V_2' - k_2 J_2' \text{ pour } a_2,$$

$$V_1' + k_1 J_1' \text{ pour } a_1.$$

Cette soustraction $-k_2 J_2'$ et cette addition $k_1 J_1'$, faites à leurs vitesses positives, viennent, comme on a dit alors, de la *détente* de compressions J_1', J_2'. Si la nouvelle vitesse de a_2 excède la nouvelle vitesse de a_1, elles s'éloignent alors l'une de l'autre.

La condition de séparation ou d'éloignement est donc

$$V_2' - k_2 J_2' - V_1 - k_1 J_1' > 0.$$

This arises from the fact that a wave of squeeze j is propagated along the rod with the velocity k of sound; $\pm kj$ is then the velocity at which a cross-section is shifted (*vitesse de détente*), and if the whole of the rod were moving with velocity v, the rate of transfer of the section through space would be $v \pm kj$. But in the case of a free terminal section this must denote its absolute velocity, where v now becomes the velocity through space of the element at the end of the rod: cf. pp. 357—8 of the memoir with p. 347.

[217.] The fifteenth paragraph treats of the loss of kinetic energy, or the energy of translation transformed into energy of vibration. All the formulae of our Art. 213 may be included in the forms

$$U_1 = V_1 - \alpha (V_1 - V_2), \quad U_2 = V_2 + \alpha \frac{M_1}{M_2} (V_1 - V_2).$$

The energy lost is then represented by

$$M_1 \{2\alpha - (1 + M_1/M_2)\,\alpha^2\} \tfrac{1}{2} (V_1 - V_2)^2.$$

Since there must always be a loss of energy, it is necessary that

$$\alpha < \frac{2M_2}{M_1 + M_2}.$$

Saint-Venant shews from the values of α in the various cases referred to in Art. 213 that this is always true (pp. 361—355).

The coefficient of dynamic elasticity e as investigated by Newton (*Principia, Ed. Princeps*, p. 22) has probably relation to the energy lost not only in vibrations, but also in the form of heat. To make Newton's formula agree with the above, it is necessary to take $\alpha = \frac{M_2}{M_1 + M_2}(1 + e)$, supposing for a moment Newton's laws to hold for rods and that the energy lost is *principally vibrational, not thermal.* This gives us, for example in *Case* (ii) of Art. 213,

$$1 + e = 2\,\frac{m_1 a_1 + m_2 a_2}{m_1 k_1 + m_2 k_2} \cdot \frac{k_2}{a_2}.$$

Thus if the rods were of different materials, it is difficult to see how e could be independent of *their masses,* which Newton proved for the impact of spherical bodies. Further in the case of equal rods of the same material e would always equal unity. This again is not true for most bodies. Hence we are driven to conclude either that the amount of thermal energy generated is generally of importance or that the conditions at the surface of impact adopted by Saint-Venant are not satisfactory. It would be interesting to make experiments for a material for which e is nearly unity, the rods being of equal cross-section and the same material, and then endeavour to ascertain by varying their masses whether there was any change in e. Haughton's experiments seem to indicate that e is not constant but a function of the velocity of impact; this does not suggest Saint-Venant's form, but it is interesting as pointing out a want of constancy in this coefficient: see our Arts. 1523* and also 941*, 1183*.

[218.] In his sixteenth paragraph (pp. 355—373) Saint-Venant proceeds to give an elementary proof of the formulae of Art. 213. This proof does not involve differential or integral processes, but it seems to me that, while luminous and suggestive to the reader of the previous analysis, it would not in the more complex cases be of equal value to the student who approached in this manner for the first time the problem of the impact of bars. Similar proofs for the *simpler* cases have been given by Thomson and Tait (§§ 302—305 of their *Natural Philosophy*), and by Rankine (*The Engineer*, February, 1867, p. 133).

[219.] The elementary discussion opens with a deduction of the value of the velocity of longitudinal sound vibrations in a rod ($=\sqrt{E/\rho}$). At that time Saint-Venant thought it novel, believing that no elementary proof had been offered since Newton's rather obscure demonstration of the velocity of sound. In a Note in the *Comptes rendus*, LXXI. 1867, p. 186, Saint-Venant acknowledges the priority of Babinet, who had given the proof in oral lectures 40 years previously and published it in his *Exercices sur la Physique*, Second Edn, 1862. In the same Note Saint-Venant gives in a footnote an elementary demonstration of the velocity of slide waves ($=\sqrt{\mu/\rho}$).

[220.] We shall not reproduce any of Saint-Venant's elementary treatment, but merely refer the curious reader to the sixteenth paragraph of his memoir. We conclude with a short extract on this point from the *résumé* of his memoir which he gives in the seventeenth paragraph:

J'aurais pu borner mon travail à ces sortes de démonstrations. Mais les solutions analytiques, telles que celles qui m'ont conduit aux résultats présentés, portent leur genre de conviction comme les solutions synthétiques, et ce n'est pas trop du concours de deux genres de recherches et de raisonnements pour établir complètement des résultats tout nouveaux et controversés. Et puis, il eut manqué quelque chose, savoir la preuve que les deux barres, après s'être séparées pendant un temps fini, ne se rejoindront pas en vibrant (p. 374).

[221.] *Choc longitudinal de deux barres élastiques, dont l'une est extrêmement courte ou extrêmement roide par rapport à l'autre: Comptes rendus*, LXVI. 1868, pp. 650—3.

This may be looked upon as a supplement to the memoir in the *Journal de Liouville*: see our Art. 203. Saint-Venant had treated this case in that memoir by expressions involving trigonometrical series; he now proposes to give its solution in finite terms.

If a_1, a_2 be the lengths of the two bars, k_1, k_2 the corresponding velocities of sound, M_1, M_2 the masses, V_1, V_2 the initial velocities, U_1, U_2 the final *mean* velocities of the impelling and impelled bars, then Saint-Venant had obtained in that memoir the following results for the case in which a_2/k_2, or the time sound takes to traverse the second bar, is an exact multiple n of the time a_1/k_1 it takes to traverse the first bar:

$$\left.\begin{aligned} U_1 &= V_2 + \left(\frac{1-r}{1+r}\right)^n (V_1 - V_2), \\ U_2 &= V_2 + \frac{M_1}{M_2}\left\{1 - \left(\frac{1-r}{1+r}\right)^n\right\}(V_1 - V_2), \\ \text{where,}\quad r &= \frac{M_2}{M_1}\frac{a_1k_2}{a_2k_1} < 1. \end{aligned}\right\} \text{............ (i)},$$

Now if the impelling bar is infinitely short or infinitely hard (if $a_1 = 0$ or $k_1 = \infty$), the number $n\left(=\frac{M_2}{M_1}\frac{1}{r}\right)$ will be infinitely great, hence it follows that:

$$\left(\frac{1-r}{1+r}\right)^{1/r} = e^{-2},$$

and the formulae (i) become:

$$U_1 = V_2 + (V_1 - V_2)\, e^{-2M_2/M_1},$$

$$U_2 = V_2 + M_1/M_2 \,.\, (1 - e^{-2M_2/M_1})\,(V_1 - V_2).$$

[222.] Saint-Venant also shews in this memoir how to obtain from the results of his previous memoir the velocity and squeeze of each bar at each instant of the impact. Thus:

(1°) For the impelling bar. From $t = 0$ to $2a_2/k_2$,

$$\text{velocity} = V_2 + (V_1 - V_2)\, e^{-M_2/M_1 \,.\, k_2t/a_2},$$

$$\text{squeeze} = 0.$$

(2°) For the impelled bar. First from $t = 0$ to a_2/k_2:

From $x = 0$ to k_2t, $\begin{cases}\text{velocity} = V_2 + (V_1 - V_2)\, e^{-M_2/M_1 \,.\, (k_2t - x)/a_2}, \\ \text{squeeze} = (V_1 - V_2)/k_2 \,.\, e^{-M_2/M_1 \,.\, (k_2t - x)/a_2}.\end{cases}$

From $x = k_2t$ to a_2, $\begin{cases}\text{velocity} = V_2, \\ \text{squeeze} = 0.\end{cases}$

Secondly from $t = a_2/k_2$ to $2a_2/k_2$:

From $x = 0$ to $2a_2 - k_2t$ the velocity and squeeze have the same values as previously from $x = 0$ to k_2t.

From $x = 2a_2 - k_2t$ to a_2,

$$\begin{cases}\text{velocity} = V_2 + (V_1 - V_2)\left\{e^{-M_2/M_1 \,.\, (k_2t - x)/a_2} + e^{-M_2/M_1 \,.\, (k_2t + x - 2a_2)/a_2}\right\}, \\ \text{squeeze} = (V_1 - V_2)/k_2 \,.\left\{e^{-M_2/M_1 \,.\, (k_2t - x)/a_2} - e^{-M_2/M_1 \,.\, (k_2t + x - 2a_2)/a_2}\right\}.\end{cases}$$

This gives the whole state of the bars up to the end of the impact or until $t = 2a_2/k_2$.

Saint-Venant tests these results: 1° by the principle of conservation of momentum, 2° by that of conservation of energy, 3° by comparing the above finite forms with the solutions in trigonometrical series. He finds them verified in all cases. In the concluding paragraph he promises in a future communication to deal with the case of a bar with one terminal fixed and the other terminal struck by a load represented by an infinitely short second bar. This is a fundamental problem in suspension bridge bars, and solutions in trigonometrical series had been given by Navier and Poncelet: see our Arts. 272* and 991*. Saint-Venant promises one in *finite* terms.

[223.] *Solution, en termes finis, du problème du choc longitudinal de deux barres élastiques en forme de tronc de cône ou de pyramide: Comptes rendus*, LXVI. 1868, pp. 877—81.

This is again a complement to the memoir in the *Journal de Liouville* (see our Art. 213). It gives in finite terms a solution for a case in that memoir, which Saint-Venant had only solved in trigonometrical series. Namely the case when the bars instead of being prismatic are truncated cones or pyramids.

The equations for the vibrations are in this case of the form:

$$\frac{d\,(E_1\Omega_1\,du_1/dx_1)}{dx_1}=\Omega_1\rho_1\,\frac{d^2u_1}{dt^2},$$

where ρ_1 is the density and Ω_1 the cross-section $=\omega_1(1+x_1/h_1)^2$, ω_1 and h_1 being constants. If we put $E_1/\rho_1=k_1^2$, we have an integral of the form:

$$u_1=\frac{f_1(x+h_1+k_1t)+F_1(x_1+h_1-k_1t)}{x_1+h_1}.$$

Similarly there will be two arbitrary functions f_2, F_2 for the second bar. The problem is to determine these four functions by the initial conditions $du_1/dt=V_1$ from 0 to a_1, $du_2/dt=-V_2$ from 0 to a_2, while the initial squeeze is zero throughout the bars. The terminal conditions have also to be satisfied throughout the motion. The forms of the functions are given on pp. 879—80 of the memoir, and the general treatment of the problem indicated, without, however, any numerical details for special cases.

La solution s'étendrait même à plusieurs barres juxtaposées bout à bout, et par conséquent au choc de deux solides allongés quelconques à axe rectiligne, car ces solides peuvent toujours être approximativement décomposés en troncs de pyramide à base quelconque (p. 881).

[224.] *Leçons de mécanique analytique*, par M. l'Abbé Moigno. *Statique.* Paris, 1868. The last two *Leçons* of this work, the twenty-first and twenty-second, pp. 616—723, contain a general theory of elasticity by Saint-Venant. This is the fourth such general theory that we have from his pen, the former three being respectively in the memoir on *Torsion*, in that on *Flexure*, and in the *Leçons de Navier:* see our Arts. 4, 72, and 190. Saint-Venant's treatment is in the main a modified and improved form of that of the second, third and fourth years of Cauchy's *Exercices de mathématiques;* that is to say it starts from the molecular definition of stress (p. 617). After a very full analysis of stress and strain we reach the general elastic equations. The hundred odd pages form one of the best introductions to the subject of elasticity, though they naturally contain no new results. We may refer to one or two points.

[225.] Saint-Venant rejects like Lamé that definition of stress across a plane, which considers stress as the force necessary to retain the plane in equilibrium if it were to become rigid (footnote p. 619). This apparently simple definition conveys, he holds, no exact notion and its simplicity is a pure delusion. In other words he insists upon the importance of the molecular-definition of stress: see Lamé's *Leçons sur l'élasticité*, § 5, and our Arts. 1051* and 1164*.

[226.] The well-known theorems of Cauchy and the equations to his ellipsoids are reproduced with short proofs: see our Arts. 603*—12*. We may note also on p. 630 a demonstration of Hopkins' theorem: see our Art. 1368*. Relations for change of direction of stretch and slide, such as those of our Art. 133, are given on pp. 644—5. Saint-Venant remarks that these relations were first given by Lamé in 1851, but that he assumes that the *shifts are small;* the proof given by Saint-Venant holds for any shifts, provided the relative shifts, i.e. the local strains, are small.

[227.] On pp. 652—3 Saint-Venant states as a Lemma and proves the principle of linearity of the stress-strain relations, i.e. the generalised Hooke's Law. The proof appeals to the rariconstant hypothesis. The reader will remember that there is an unjustifiable assumption often made in the proof of the generalised Hooke's law by Green's method: see our Art. 928*. We may note here how Saint-Venant as a rari-constant elastician proves his Lemma. After stating that the stresses must be functions of the strains he continues:

Et elles en sont fonctions linéaires ou du premier degré; car, comme les actions réciproques entre molécules sont fonctions continues de leurs distances mutuelles r, celles que développent de *très-petites* augmentations rs_r des distances leur sont proportionnelles; et les changements très-petits des inclinaisons mutuelles de ces actions à composer ensemble pour avoir les pressions sont proportionnelles aussi à des augmentations rs_r de distances. Or ces petites augmentations positives ou négatives:

$$rs_r = rc^2_{rx}s_x + rc^2_{ry}s_y + rc^2_{rz}s_z + rc_{ry}c_{rz}\sigma_{yz} + rc_{rz}c_{rx}\sigma_{zx} + rc_{rx}c_{ry}\sigma_{xy},$$

[see our Art. 547*]

sont sommes de produits des premières puissances des dilatations et glissements s, σ par des quantités $rc^2_{rx}, \ldots rc_{rx}c_{ry}$ qui ne dépendent que de l'état antérieur aux déformations......Les composantes $\widehat{xx} \ldots \widehat{xy}$ des pressions sont donc fonctions du premier degré des mêmes six quantités très-petites s et σ, ce qui est le lemme énoncé.

It will be noted that a clear reason is here given for the legitimacy of Taylor's theorem and the retention of the first powers. *It depends on the rari-constant hypothesis.* A slight discussion of this point with a reference to the *Appendice V.* of the *Leçons de Navier* will be found on pp. 654—6: see our Arts. 192 and 298. There is a footnote on the arbitrary assumption of the stress-strain relations for isotropic bodies by Cauchy and Maxwell: see our Arts. 614* and 1537*.

[228.] On p. 670 there is a footnote citing the values of the stretches and slides for large shifts. This requires modifying in the sense of my remarks in Art. 1619*—22*.

There is an excellent proof on rari-constant lines following Cauchy of the most general elastic equations with initial stresses on pp. 673—689. It is followed on pp. 694—7 by some useful remarks on the difficulties which occur in the treatment of stresses as the sums of intermolecular actions: see our Arts. 443* and 1400*. The pitfalls into which Poisson, Navier and others have fallen are well brought out.

[229.] This discussion on elasticity concludes with a deduction of the expression for the strain-energy (Green's function) by means of Lagrange's process and the rari-constant hypothesis (p. 717). The method is similar to that used by C. Neumann in his memoir of 1859: see our Chap. XI. It is pointed out that if Navier's error of taking $(r_1 - r)f'(r)$ instead of $f(r) + (r_1 - r)f'(r)$ for $f(r_1)$ be avoided, and if the summations be not replaced by integrals, then Poisson's objection to the application of the Calculus of Variations to molecular problems falls to the ground (p. 719): see our Arts. 266* and 446*. Finally there is an account of Green's process and an unfavourable criticism of his theory of double refraction (pp. 719—23): see our Arts. 147 and 193.

[230.] *Formules de l'élasticité des corps amorphes que des compressions permanentes et inégales ont rendus hétérotropes. Journal des mathématiques*, Tome XIII. 1868, pp. 242—254. In his memoir of 1863 Saint-Venant has shewn on the rari-constant hypothesis that the ellipsoidal distribution of elasticity holds for aeolotropic, but amorphic bodies, i.e. bodies such as the metals, whose primitive isotropy has been altered by a permanent strain, which has not converted their elements into crystals; such a permanent strain for instance as would be produced by the processes of rolling, forging, etc. This ellipsoidal distribution he has applied to explain the phenomena of double refraction, without adopting exact transversality of vibration, but obtaining without approximation Fresnel's wave-surface. The ellipsoidal conditions are of two kinds:

$$\left.\begin{array}{ll}\text{(i) a group of the type} & 2d + d' = \sqrt{bc} \\ \text{or, (ii)} \quad ,, \quad ,, \quad ,, & 2d + d' = \tfrac{1}{2}(b + c)\end{array}\right\}\dots\dots\text{(i)}.$$

If the differences of the direct-stretch coefficients ($b - c$, $c - a$, $a - b$) are so small that their squares may be neglected, these two groups of conditions are identical; this is probably the case in the metals used for construction, and in doubly-refracting media: see our Arts. 142—7. The conditions by which Saint-Venant would replace Green's relations—the Cauchy-Saint-Venant conditions as we have termed them—amount to an ellipsoidal distribution of elasticity (see our Art. 149), but this distribution Saint-Venant has only discussed on the basis of rari-constant equations. Boussinesq in a memoir entitled: *Mémoire sur les ondes dans les milieux isotropes déformés*, which immediately precedes the present memoir (pp. 209—241 of the same volume) has deduced the Cauchy-Saint-Venant conditions for double refraction on the basis of the ellipsoidal distribution without any appeal to rari-constancy. The ellipsoidal distribution is proved by Boussinesq for amorphic bodies on the multi-constant hypothesis,—provided we assume the elastic coefficients to be themselves *linear* functions of three small quantities corresponding respectively to the three principal rectangular directions of the permanent strain given to the initially isotropic material. Saint-Venant proposes to give a new proof of Boussinesq's result, so that the ellipsoidal distribution may be accepted for the amorphic bodies in question even by multi-constant elasticians.

[231.] Suppose the body initially isotropic to be permanently strained in such manner that at each point there are three planes of elastic symmetry, then the stress-strain relations are of the form:

$$\left.\begin{array}{ll} \widehat{xx} = as_x + f's_y + e's_z, & \widehat{yz} = d\sigma_{yz}, \\ \widehat{yy} = f's_x + bs_y + d's_z, & \widehat{zx} = e\sigma_{zx}, \\ \widehat{zz} = e's_x + d's_y + cs_z, & \widehat{xy} = f\sigma_{xy}. \end{array}\right\} \text{.................. (ii).}$$

Let ϵ, ϵ', ϵ'' be the three small quantities corresponding to the three rectangular directions x, y, z of which the elastic constants are, according to hypothesis, to be functions, or let the types be

$$a = \mathfrak{a} + l_1\epsilon + m_1\epsilon' + n_1\epsilon'',$$
$$d' = \delta' + p_1\epsilon + q_1\epsilon' + k_1\epsilon'',$$
$$d = \delta + r_1\epsilon + s_1\epsilon' + h_1\epsilon''.$$

Then since the original condition is isotropy, a must be related to ϵ' and ϵ'' in the same way, and further in the same way to ϵ as b to ϵ' and c to ϵ''. Thus $l_1 = m_2 = n_3$, and $m_1 = n_1 = l_2 = n_2 = l_3 = m_3$. Similar relations hold for the constants of d and d'. Thus we may write as types:

$$a = \mathfrak{a} + l\epsilon + m(\epsilon' + \epsilon''),$$
$$b = \mathfrak{a} + l\epsilon' + m(\epsilon + \epsilon''),$$
$$d' = \delta' + p\epsilon + q(\epsilon' + \epsilon''),$$
$$e' = \delta' + p\epsilon' + q(\epsilon + \epsilon''),$$
$$d = \delta + r\epsilon + s(\epsilon' + \epsilon''),$$
$$e = \delta + r\epsilon' + s(\epsilon + \epsilon'')\text{...etc.}$$

Now if we take $\epsilon' = \epsilon''$, or the stretch the same all round the direction x, we ought to have not only $b = c$, $e = f$, $e' = f'$, which easily follows, but in addition the values of the constants ought not to be affected by a rotation of the axes round that of x. This however is easily shewn to involve

$$b = 2d + d',$$

or what is the same thing

$$\mathfrak{a} + m\epsilon + (l + m)\epsilon' = 2\delta + \delta' + (2r + p)\epsilon + (4s + 2q)\epsilon'.$$

This involves, as an identity true for all values of ϵ and ϵ', the further results

$$a = 2\delta + \delta', \quad m = 2r + p, \quad l + m = 4s + 2q.$$

Whence we easily find generally:

$$\begin{aligned} b + c &= 2a + (\epsilon' + \epsilon'')(l + m) + 2m\epsilon, \\ &= 2(2\delta + \delta') + 2(\epsilon' + \epsilon'')(2s + q) + 2(2r + p)\epsilon, \\ &= 2(2d + d'), \end{aligned}$$

or $$2d + d' = \tfrac{1}{2}(b + c),$$

the type of ellipsoidal condition for the second group. It will be identical with the group of type $(2d + d') = \sqrt{bc}$, when we may neglect the squares of the differences of a, b, c, or quantities like $(l - m)^2 (\epsilon - \epsilon')^2$. Hence the ellipsoidal conditions have been deduced on a hypothesis very probable in character and not opposed to multi-constancy.

[232.] The memoir concludes by noting that to the stress-strain relations (ii) subject to the inter-constant relations (i), we must add terms of the type:

$$\widehat{xx}_0 (1 + u_x - v_y - w_z) \text{ to } \widehat{xx},$$

$$\widehat{zz}_0 v_z + \widehat{yy}_0 w_y \qquad \text{to } \widehat{yz},$$

if there be an initial stress $\widehat{xx}_0$, $\widehat{yy}_0$, $\widehat{zz}_0$ symmetrical with regard to the planes of symmetry of the primitive strain. Saint-Venant appeals for these to his memoir of 1863, but as we have seen he has really only proved them there for rari-constancy (see our Art. 129).

[233.] *Calcul du mouvement des divers points d'un bloc ductile, de forme cylindrique, pendant qu'il s'écoule sous une forte pression par un orifice circulaire; vues sur les moyens d'en rapprocher les résultats de ceux de l'expérience: Comptes rendus*, LXVI. 1868, pp. 1311—24. This memoir deals only with the motion of the parts of a ductile mass, and does not take into consideration the stresses which produce those motions. Its methods thus approach those of hydrodynamics rather than of elasticity; it belongs as Tresca's own theory, to which it refers, to the pure kinematics of deformation. A report drawn up by Saint-Venant on Tresca's communications to the Academy immediately precedes the above memoir (pp. 1305–11). It deals with and criticises Tresca's pure kinematic theory.

Memoirs by Saint-Venant treating of the flow of a ductile solid or of a liquid out of a vessel will be found in the *Comptes rendus*, LXVII. 1868, pp. 131—7, 203—211, 278—282 and LXVIII. 1869, pp. 221—237, 290—301. They cannot be considered to fall in any way under the title of the elasticity or even the *strength* of materials.

[234.] *Note sur les valeurs que prennent les pressions dans un solide élastique isotrope lorsque l'on tient compte des dérivées d'ordre supérieur des déplacements très-petits que leurs points ont éprouvés: Comptes rendus,* LXVIII. 1869, pp. 569—571. This note gives without proof expressions for the traction and shear at any point of an elastic solid. when we do not neglect the squares of the shift-fluxions. Saint-Venant says that his results have been obtained from rari-constant considerations. He finds:

$$\widehat{xx} = \epsilon_0 \left(\theta + 2\frac{du}{dx}\right) + \epsilon_1 \left\{2\frac{d^2\theta}{dx^2} + \nabla^2\left(\theta + 2\frac{du}{dx}\right)\right\}$$
$$+ \epsilon_2 \left\{4\nabla^2\frac{d^2\theta}{dx^2} + \nabla^2\nabla^2\left(\theta + 2\frac{du}{dx}\right)\right\}$$
$$+ \epsilon_3 \left\{6\nabla^2\nabla^2\frac{d^2\theta}{dx^2} + \nabla^2\nabla^2\nabla^2\left(\theta + 2\frac{du}{dx}\right)\right\} + \ldots$$

$$\widehat{yz} = \epsilon_0 \left(\frac{dv}{dz} + \frac{dw}{dy}\right) + \epsilon_1 \left\{2\frac{d^2\theta}{dydz} + \nabla^2\left(\frac{dv}{dz} + \frac{dw}{dy}\right)\right\}$$
$$+ \epsilon_2 \left\{4\nabla^2\frac{d^2\theta}{dydz} + \nabla^2\nabla^2\left(\frac{dv}{dz} + \frac{dw}{dy}\right)\right\}$$
$$+ \epsilon_3 \left\{6\nabla^2\nabla^2\frac{d^2\theta}{dydz} + \nabla^2\nabla^2\nabla^2\left(\frac{dv}{dz} + \frac{dw}{dy}\right)\right\} + \ldots$$

Here θ is as usual the dilatation, ∇^2 is the Laplacian $\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$, and $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3 \ldots$ are constants depending on the elastic nature of the body.

Saint-Venant concludes his note with the remark:

Ces formules serviront peut-être à expliquer des faits relatifs à certaines substances élastiques pour lesquelles le rapport entre les efforts et les effets varie plus rapidement lorsqu'on les comprime que lorsqu'on les étend, en sorte que les vibrations qui y seraient excitées augmenteraient leurs dimensions comme fait la chaleur, dont les effets de dilatation peuvent être attribués, comme j'ai eu l'occasion de le faire remarquer (*Société Philomathique*, October 20, 1855: see our Art. 68), à ce que les actions entre les derniers atomes suivraient une loi analogue, (p. 571).

[235.] *Sur un potentiel de deuxième espèce, qui résout l'équation aux différences partielles du quatrième ordre exprimant l'équilibre intérieur des solides élastiques amorphes non isotropes: Comptes rendus,* LXIX. 1869, pp. 1107—1110.

This note merely refers to E. Mathieu's discussion of the potential of the second kind

$$\phi = \iiint f(\alpha, \beta, \gamma) \sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}\, d\alpha d\beta d\gamma,$$

by means of which the equation $\nabla^2\nabla^2\phi = 0$ can be solved. This equation occurs in the treatment of an isotropic solid. Saint-Venant notices the form

$$\phi = \iiint f(\alpha, \beta, \gamma) \sqrt{\frac{(x-\alpha)^2}{A} + \frac{(y-\beta)^2}{B} + \frac{(z-\gamma)^2}{C}}\, d\alpha d\beta d\gamma,$$

which solves the equations of elasticity when there is an ellipsoidal distribution of elasticity: see our Arts. 140–1.

Saint-Venant speaks highly of Cornu's memoir of 1869 and its bearing on the constant-controversy: see our Articles below on that physicist's work.

[236.] *Preuve théorique de l'égalité des deux coefficients de résistance au cisaillement et à l'extension ou à la compression dans le mouvement continu de déformation des solides ductiles au delà des limites de leur élasticité: Comptes rendus*, LXX. 1870, pp. 309—11.

The object of this note is to prove the equality between the coefficient of resistance to slide and the coefficient of resistance to stretch or squeeze, when both slide and stretch are plastic.

Saint-Venant takes a right six-face of edges $a, b, c,$ and supposes the two faces $a \times b$ to be subjected to shearing forces in direction of a which produce a plastic slide-set $\sigma \times c$, so that the limit of elasticity is passed. If K' be the force necessary per unit of area, the work expended in producing this set is

$$K'ab \times \sigma \times c,$$

or, it equals $K'\sigma$ *per* unit of volume.

Now this same slide-set could have been produced by diagonal stretch and squeeze of magnitude $\sigma/2$: see our Art. 1570*. Let us take the right six-face abc and divide it up into others of the same breadth b, but of length a' and height c' making angles of $45°$ with a and c and having their end-faces $a' \times c'$ in the faces $a \times c$. In order to produce set-stretch it is necessary to apply to the faces bc' a traction given by Kbc' and to the faces ba' a negative traction given by Kba', where K is the coefficient of resistance to *both* stretch and squeeze. Hence to produce a stretch of $\sigma/2$ and a squeeze of $\sigma/2$ parallel to a' and c' respectively, we require work equal to

$$Kbc' \cdot \frac{\sigma}{2} a' \text{ and } Kba' \cdot \frac{\sigma}{2} c',$$

or, *per* unit volume of the little prism $a'bc'$, we require work equa to

$$K\sigma.$$

But this quantity must equal the previous $K'\sigma$ or

$$K' = K,$$

the result experimentally ascertained by Tresca.

Saint-Venant concludes the note as follows:

Ce raisonnement me paraît, aussi, justifier l'hypothèse, hardie au premier aperçu, mais, en y réfléchissant, très-rationnelle, de l'égalité des résistances à l'extension et à la compression permanente, par unité superficielle des bases des prismes qu'on y soumet; bien entendu, sous la condition générale, que tout ceci suppose remplie, de mouvements excessivement lents, ou tels que leur vitesse n'entre pour rien dans les résistances aux déformations qu'ils produisent.

In a footnote he refers to a method by which the flow-lines of a plastic material might be obtained experimentally.

It must be noted that the proof assumes the coefficients K_1, K_2 of resistance to squeeze- and stretch-set to be equal, otherwise we should have

$$K_1 + K_2 = 2K'.$$

The reader may compare Coulomb's results on shearing and tractive *strength* referred to on p. 877 of our first volume.

[237.] *Formules des augmentations que de petites déformations d'un solide apportent aux pressions ou forces élastiques, supposées considérables, qui déjà étaient en jeu dans son intérieur.—Complément et modification du préambule du mémoire: Distribution des élasticités autour de chaque point, etc. qui a été inséré en* 1863 *au Journal de Mathématiques*, (see our Arts. 127—152). This memoir is published in the *Journal de Mathématiques*, Tome XVI. 1871, pp. 275—307, and is divided into two parts; the *Première Partie* (pp. 275—291) is occupied with correcting an error which Brill and Boussinesq had pointed out in the memoir of 1863 (see our Art. 130); the *Deuxième Partie* deals with the relations between the elastic constants $|xxxx|$, etc. and the six components of initial strain. It occupies pp. 291—307 and forms the subject of a note on pp. 355 and 391 of the *Comptes rendus*, T. LXXII. 1871.

[238.] The error in question was really indicated in our first volume (see Art. 1619*), namely that the true relations between the strains, s_x, σ'_{yz} and the shift-fluxions are in their most general form of the types[1]:

$$\left.\begin{aligned} s_x + \tfrac{1}{2} s_x^2 &= u_x + \tfrac{1}{2}(u_x^2 + v_x^2 + w_x^2) \\ \sigma'_{yz}(1+s_y)(1+s_z) &= v_z + w_y + u_y u_z + v_y v_z + w_y w_z \end{aligned}\right\} \ldots\text{(i)},$$

but that these are not the values taken by Saint-Venant in his memoirs of 1847 and 1863: see our Arts. 1622* and 130. Accordingly Saint-Venant's attempt to deduce Cauchy's equations from a multi-constant hypothesis is erroneous.

The full value of the potential energy is

$$\left.\begin{aligned} \phi = \phi_0 + \widehat{xx}_0 &(s_x + \tfrac{1}{2}s_x^2) + \ldots\ldots + \ldots\ldots \\ &+ \widehat{yz}_0\, \sigma'_{yz}(1+s_y)(1+s_z) + \ldots\ldots + \ldots\ldots \\ &+ \phi_1 \end{aligned}\right\} \ldots\ldots\ldots\ldots\text{(ii)},$$

as Boussinesq had pointed out, and not

$$\phi = \phi_0 + \widehat{xx}_0\, s_x + \ldots\ldots + \ldots\ldots + \widehat{yz}_0\, \sigma'_{yz} + \ldots\ldots + \ldots\ldots + \phi_1,$$

as assumed in the memoir of 1863 (see our Art. 130). But the expression (ii) *has been deduced only from molecular considerations on the rari-constant hypothesis.* The fact is that we can on the multi-constant hypothesis expand ϕ in linear and quadratic terms of the strain-components ϵ_x, ϵ_y, ϵ_z, η_{yz}, η_{zx}, η_{xy} of our Art. 1619*, as Green in fact did (*Collected Papers*, pp. 298–9), but we cannot determine to what extent the resulting coefficients are functions of the initial stress-components. This apparently requires us also to make some molecular assumption.

[239.] Starting with expression (ii) for the potential energy, we should arrive at the equations of Cauchy (as Saint-Venant had done in his memoir of 1863 by a double self-correcting error), but we must renounce the hope of arriving at (ii) on the simple assumption of a generalised Hooke's Law. We may note one or two further points in the first part of the memoir:

(*a*) To the second order of small quantities,

$$\left.\begin{aligned} s_x &= u_x + \tfrac{1}{2}(v_x^2 + w_x^2) \\ \sigma_{yz} &= v_z + w_y + u_y u_z - v_y w_y - v_z w_z \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

This was first noticed by Brill: see p. 279 of Saint-Venant's memoir.

[1] σ'_{yz} differs from the σ_{yz} of our Art. 1621*, it being the *cosine* and not the cotangent of the slide-angle. See Saint-Venant's definition of slide in Art. 1564*.

(*b*) If we assume that the work-function may be expanded in powers of s_x, s_y, s_z, σ_{zy}, σ_{xz}, σ_{yx}, and write

$$\left.\begin{aligned}\phi = \phi_0 + \widehat{xx}_0\, s_x + \widehat{yy}_0\, s_y + \widehat{zz}_0\, s_z \\ + \widehat{zy}_0\, \sigma_{zy} + \widehat{xz}_0\, \sigma_{xz} + \widehat{yx}_0\, \sigma_{yx} \\ + \phi_2\end{aligned}\right\}\text{..................(iv)},$$

then we are throwing a portion of ϕ involving initial stresses into ϕ_2, which thus differs from the ϕ_1 of (ii). We thus obtain for the stresses the types:

$$\left.\begin{aligned}\widehat{xx} &= \widehat{xx}_0\,(1 - v_y - w_z) - \widehat{xy}_0\,(v_x - u_y) + \widehat{zx}_0\,(u_z - w_x) + \widehat{xx}_2 \\ \widehat{yz} &= \widehat{yz}_0\,(1 - u_x - v_y - w_z) + \widehat{yy}_0\, w_y + \widehat{zz}_0\, v_z + \widehat{zx}_0\, v_x + \widehat{xy}_0\, w_x + \widehat{yz}_2\end{aligned}\right\}\text{...(v)}.$$

But $\widehat{xx}_2$ and $\widehat{yz}_2$ while being of the same form as Cauchy's $\widehat{xx}_1$, $\widehat{yz}_1$ [see our Art. 129, (ii)], will in reality have constants increased by the corresponding initial stresses, as is shewn by the rari-constant investigation. Thus:

$$\left.\begin{aligned}|xxxx|_2 &= |xxxx| + \widehat{xx}_0 \\ |yyyz|_2 &= |yyyz| + \widehat{yz}_0 \\ |zzyz|_2 &= |zzyz| + \widehat{yz}_0\end{aligned}\right\}\text{........................(vi)}.$$

It is the impossibility of determining on the multi-constant theory how these initial stresses occur in the changed values of the constants, which throws us back on rari-constancy for a proof of (ii). Results (vi) combined with (v) convert the latter into Cauchy's formulae: see our Art. 129, (i).

[240.] The second part of the memoir deals with the following problem: *If* $|xxxx|$, $|xxxy|$, $|xxyz|$, *etc. are the elastic constants when there is an initial state of stress* $\widehat{xx}_0$, $\widehat{xy}_0$, *etc. it is required to determine these constants in terms of* $|xxxx|_0$, $|xxxy|_0$, $|xxyz|_0$, *etc. the elastic constants before this initial state of stress.*

Saint-Venant deals with the problem on rari-constant lines. We have, with abbreviated symbols (see our Art. 143):

$$|x^4| \text{ or } |y^2z^2| \text{ or } |y^3z| \text{ or } |x^2yz| = \frac{\rho}{2}\,\Sigma m \frac{d}{rdr}\frac{f(r)}{r}\,\{x^4 \text{ or } y^2z^2 \text{ or } y^3z \text{ or } x^2yz\}\text{ ...(vii)}.$$

Further we have, if x_0, y_0, z_0 be the position of the molecule m relative to a second before the initial strain, u_0, v_0, w_0 its shift due to that strain, and x, y, z the relative position after the strain,

$$x = x_0 + x_0\frac{du_0}{dx_0} + y_0\frac{du_0}{dy_0} + z_0\frac{du_0}{dz_0} + \frac{1}{1\,.\,2}\left(\frac{d^2u_0}{dx_0^2}\,x_0^2 + \ldots\right),$$

$$r - r_0 = \frac{1}{r_0}\left[x_0^2\frac{du_0}{dx_0} + \ldots\ldots + \ldots\ldots \right.$$
$$\left. + y_0z_0\left(\frac{dv_0}{dz_0} + \frac{dw_0}{dy_0}\right) + \ldots\ldots + \ldots\ldots\right]$$

$$
\begin{aligned}
&+\frac{1}{2r_0^2}\left[\left(\frac{du_0}{dx_0}x_0+\frac{du_0}{dy_0}y_0+\frac{du_0}{dz_0}z_0\right)^2\right.\\
&\qquad+\left(\frac{dv_0}{dx_0}x_0+\ldots\ldots+\ldots\ldots\right)^2\\
&\qquad\left.+\left(\frac{dw_0}{dx_0}x_0+\ldots\ldots+\ldots\ldots\right)^2\right]\\
&-\frac{1}{2r_0^2}\left[x_0^2\frac{du_0}{dx_0}+y_0^2\frac{dv_0}{dy_0}+\ldots\ldots+x_0y_0\left(\frac{du_0}{dy_0}+\frac{dv_0}{dx_0}\right)\right]^2\\
&\qquad+\frac{1}{2r_0}\left(x_0^3\frac{d^2u_0}{dx_0^2}+\ldots\ldots\right),
\end{aligned}
$$

$$\frac{d}{rdr}\left(\frac{f(r)}{r}\right)=\frac{d}{r_0dr_0}\left(\frac{f(r_0)}{r_0}\right)+(r-r_0)\frac{d}{dr_0}\frac{1}{r_0}\frac{d}{dr_0}\left(\frac{f(r_0)}{r_0}\right)+\ldots\ldots,$$

$$\rho=\frac{\rho_0}{\left(1+\frac{du_0}{dx_0}\right)\left(1+\frac{dv_0}{dy_0}\right)\left(1+\frac{dw_0}{dz_0}\right)-\frac{dv_0}{dz_0}\frac{dw_0}{dy_0}-\frac{dw_0}{dx_0}\frac{du_0}{dz_0}-\frac{du_0}{dy_0}\frac{dv_0}{dx_0}+\ldots}$$

$$=\rho_0\left(1-s_{x_0}-s_{y_0}-s_{z_0}\right),$$

if we suppress squares and products.

Substituting in Equation (vii) and remembering that

$$|x^4|_0 \text{ or } |y^2z^2|_0 \text{ or } |y^3z|_0 \text{ or } |x^2yz|_0=\frac{\rho_0}{2}\Sigma m\frac{d}{r_0dr_0}\left\{\frac{f(r_0)}{r_0}\right\}\{x^4_0 \text{ or } y^2_0z^2_0 \text{ or } y^3_0z_0 \text{ or } x^2_0y_0z_0,\}$$

we obtain the typical results :

$$
\begin{aligned}
|x^4| &= |x^4|_0\left(1-3u_{x_0}-v_{y_0}-w_{z_0}\right)+4\left(|x^3y|_0\,u_{y_0}+|x^3z|_0\,u_{z_0}\right),\\
|y^2z^2| &= |y^2z^2|_0\left(1-u_{x_0}+v_{y_0}+w_{z_0}\right)+2\left(|yz^3|_0\,v_{z_0}+|xyz^2|_0\,v_{x_0}+|xy^2z|_0\,w_{x_0}+|y^3z|_0\,w_{y_0}\right),\\
|y^3z| &= |y^3z|_0\left(1-u_{x_0}+2v_{y_0}\right)+3\left(|y^2z^2|_0\,v_{z_0}+|xy^2z|_0\,v_{x_0}\right)+\left(|xy^3|_0\,w_{x_0}+|y^4|_0\,w_{y_0}\right),\\
|x^2yz| &= |x^2yz|_0\left(1+u_{x_0}\right)+2\left(|xy^2z|_0\,u_{y_0}+|xyz^2|_0\,u_{z_0}\right)+|x^3z|_0\,v_{x_0}+|x^2z^2|_0\,v_{z_0}+|x^3y|_0\,w_{x_0}\\
&\qquad+|x^2y^2|_0\,w_{y_0}.
\end{aligned}
$$

Here $u_{x_0},\ldots$ denote $du_0/dx_0\ldots$, and since the stresses $\widehat{xx}_0$, $\widehat{yz}_0$ are given functions of $u_{x_0}\ v_{y_0}\ldots u_{z_0}\ldots$etc., we can express the new coefficients $|x^4|\ldots$ in terms of the old $|x^4|_0\ldots$ and the initial stresses. These results are obviously only a more general case of the formulae of our Art. 616*. The following pages 297—304 are concerned with other modes of looking at these results or expressing the stresses in terms of them.

[241.] Let us take as a special case that of a bar of primitively isotropic material subjected to a traction $\widehat{xx}$, there being an initial traction $\widehat{xx}_0$. We have

$$s_{x_0}=\widehat{xx}_0/E_0,\quad s_{y_0}=s_{z_0}=-s_{x_0}/4.$$

Further, if $|x^2y^2|_0=\lambda=\mu$, then $|x^4|_0=3\lambda$ and $E_0=5\lambda/2$.

Thus, $|x^4| = 3\lambda\,(1 - \frac{5}{2}s_{x_0}), \quad |y^4| = |y^4|_0 = 3\lambda,$

$$|x^2y^2| = |x^2z^2| = \lambda\,(1 + s_{x_0}),$$

$$|y^2z^2| = \lambda\,(1 - \tfrac{3}{2}\,s_{x_0}),$$

$$|y^3z| = |x^2yz| = \text{etc.} = 0.$$

Substituting in the traction-type as given by Cauchy's formula, Eqn. (i) Art. 129, we have

$$\widehat{xx} = \widehat{xx}_0\,\{1 + s_x - 2s_y\} + 3\lambda\,(1 - \tfrac{5}{2}\,s_{x_0})\,s_x + 2\lambda\,(1 + s_{x_0})\,s_y,$$

$$\widehat{yy} = 0 = 3\lambda s_y + \lambda\,(1 + s_{x_0})\,s_x + \lambda\,(1 - \tfrac{3}{2}\,s_{x_0})\,s_y.$$

Whence we find from the second equation:

$$s_y\,(4 - \tfrac{3}{2}\,s_{x_0}) = -\,(1 + s_{x_0})\,s_x,$$

or $$s_y = -\tfrac{1}{4}\,s_x\,\{1 + \tfrac{11}{8}\,s_{x_0}\}, \text{ neglecting } s^2_{x_0}.$$

Substituting in the first we easily deduce

$$\widehat{xx} = \widehat{xx}_0 + \frac{5\lambda}{2}\,s_{x_0} \times \tfrac{3}{2}s_x + \lambda s_x\,\{\tfrac{5}{2} - \tfrac{139}{16}\,s_{x_0}\},$$

$$= \widehat{xx}_0 + \frac{5\lambda}{2}\,s_x\,(1 - \tfrac{79}{40}\,s_{x_0}),$$

or $$\frac{\widehat{xx} - \widehat{xx}_0}{s_x} = E_0 - \frac{79}{40}\,\widehat{xx}_0.$$

Thus if E be the new stretch-modulus, we have $E = E_0 - \frac{79}{40}\,\widehat{xx}_0$.

This shows that a large initial traction can alter to some extent the value of the stretch-modulus. It slightly *decreases it.* Saint-Venant obtains in our notation

$$E = E_0 + \tfrac{11}{2}\,\widehat{xx}_0\,,$$

but I do not think this result is correct. It would denote an *increase* of the stretch-modulus. Saint-Venant in fact puts the stretch-squeeze ratio after the initial stress $= \frac{1}{4}$, (thus on p. 305 he writes $s_z = s_y = -\,\frac{1}{4}\,s_x$), but it seems to me that this ratio

$$= -\,(1 + s_{x_0})/(4 - \tfrac{3}{2}\,s_{x_0}) = -\,\tfrac{1}{4}\,(1 + \tfrac{11}{8}\,s_{x_0}),$$

and is only $= -\,1/4$ when $s_{x_0} = \widehat{xx}_0/E_0 = 0$, or, when there is no initial stress.

The matter is one of theoretical rather than practical interest, for supposing E were 30,000,000 lbs. per sq. inch, it is unlikely that $\widehat{xx}_0$ could be at most more than 40,000 to 60,000 lbs. per sq. inch; hence the change in E would not amount to more than 140,000 to 200,000 lbs., or at most to 1/150 of E, which with the want of uniformity in any material is in practice almost within the limits of experimental error.

[242.] In Tome XV. of the *Journal de Liouville*, 1870, there are two articles by Saint-Venant, but they refer to a matter which I have thought it well to treat as lying outside our field, namely the stability of masses of loose earth. The history of the memoirs in question may be briefly referred to. Maurice Lévy in 1867 had presented to the Academy a memoir entitled: *Essai sur une théorie rationnelle de l'équilibre des terres fraîchement remuées, et ses applications au calcul de la stabilité des murs de soutènement* (published in the *Journal de Liouville* T. XVIII. 1873, pp. 241—300). This memoir had been referred to a committee including Saint-Venant for report. The report appeared in the *Comptes rendus*, T. LXX. 1870, pp. 217—28, and was reprinted in Vol. XV. of the *Journal*, pp. 237—49. Lévy as well as the committee appear to have been ignorant of Rankine's memoir: *On the Stability of Loose Earth* (*Phil. Trans.* 1857, pp. 9—27) which had contained most of Lévy's results. Lévy had started from Cauchy's stress-theorems (see our Arts. 606* and 610*), and arrived at certain general equations. Saint-Venant in his first note solves to a first approximation Lévy's equation (pp. 250—63 of Tome XVIII.) and hopes some mathematician will proceed further. This was done by Boussinesq, who proceeded to a second approximation in a memoir occupying pp. 267—70 of Tome XV. of the *Journal*. Saint-Venant then reconsidered the whole matter in a second memoir, which occupies the following pp. 271—80. In a footnote he recognises Rankine's priority of research. The memoirs of Saint-Venant and Boussinesq appear also in the *Comptes rendus*. T. LXX. 1870, pp. 217—28, 717—24, 751—4 and 894—7.

[243.] *Rapport sur un mémoire de Maurice Lévy: Comptes rendus*, T. LXXIII. 1871, pp. 86—91. This is a report by Saint-Venant and others on Lévy's memoir establishing the general body-stress equations of plasticity in three dimensions: see our Art. 250. The *Rapport* speaks well of Lévy's memoir as advancing the new branch of mechanics, "pour laquelle l'un de nous a hasardé, sans le préconiser comme le meilleur, le terme *d'hydrostéréo-dynamique*." This branch of research has been called later *plastico-dynamics*, a better word, and we shall refer to it simply as *plasticity*.

[244.] *Sur la mécanique des corps ductiles: Comptes rendus*,

T. LXXIII. 1871, pp. 1181—1184. Saint-Venant here replaces his first name—hydrostereo-dynamics—by *plastico-dynamics.* He refers to the *Complément* to his memoirs on this subject in the *Journal de Liouville:* see our Art. 245, (iii), and to the two examples of the plasticity of a cylinder under torsion and of a prism under circular flexure dealt with there. The object of this note is to show that a formula obtained by Tresca for the torsion of a semi-plastic cylinder contributes no more than Saint-Venant's formula of the above-mentioned *Complément,* while it is at the same time obtained in a semi-empirical fashion. While Tresca's formula involves a new constant K', Saint-Venant depends only on the elastic slide-modulus μ and the plastic-modulus K. Saint-Venant distinguishes in his cylinder only two zones, an elastic and a plastic one, Tresca supposes a mid-zone in which elasticity alters to plasticity or, as Tresca terms it, fluidity. Saint-Venant's discussion has the theoretical advantage, but it seems not improbable that physically something corresponding to Tresca's mid-zone has an existence.

[245.] We have next to turn to a series of interesting and important memoirs by Saint-Venant in which he deals with the *plastic* equations. These are:

(i) *Mémoire sur l'établissement des équations différentielles des mouvements intérieurs opérés dans les corps solides ductiles au delà des limites où l'élasticité pourrait les ramener à leur premier état. Journal de Mathématiques.* Tome XVI. 1871, pp. 308—316. [See also *Comptes rendus,* T. LXX. 1870, p. 473.]

(ii) *Extrait du mémoire sur les équations générales des mouvements intérieurs des corps solides ductiles au delà des limites où l'élasticité pourrait les ramener à leur premier état. Par M. Maurice Lévy. Ibid.* pp. 369—372. [See also *Comptes rendus,* T. LXX. p. 1323, and Saint-Venant's correction referred to in our Art. 263. Some account of the memoir itself will be given under the year 1870.]

(iii) *Complément aux mémoires du* 7 *mars* 1870 *de M. de Saint-Venant et du* 19 *juin* 1870 *de M. Lévy sur les équations différentielles 'indéfinies' du mouvement intérieur des solides ductiles etc.;...Equations 'définies' ou relatives aux limites de ces corps;*—Applications, *Ibid.* pp. 373—382.

[246.] The first paper begins with an interesting account of the history of the theory of plasticity. It refers to Tresca's memoirs and to the attempts of Tresca and Saint-Venant himself to obtain solutions by means of pure kinematics. It is pointed out that the problem is essentially mechanical as well as kinematical and involves a consideration of stress as well as of mere continuity.

In the first place the ordinary equations of fluid-motion must be replaced by others involving inequality of pressure in different directions. Thus the well-known type of hydrodynamic equation:

$$\frac{dp}{dx} = \rho \left(X - \frac{du}{dt} - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz} \right),$$

becomes the plastico-dynamic type:

$$\frac{d\widehat{xx}}{dx} + \frac{d\widehat{xy}}{dy} + \frac{d\widehat{xz}}{dz} = -\rho \left(X - \frac{du}{dt} - u\frac{du}{dx} - v\frac{du}{dy} - w\frac{du}{dz} \right) \text{.........(i)}.$$

The change of sign is due to change from pressure to *traction*.
To this we must add the equation of continuity:

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \text{.........................(ii)}.$$

The four equations given by (i) and (ii) represent the relation between the flow (velocity-components u, v, w) of the material and the stress-components. The material in the plastic state is treated as incompressible.

[247.] Now Tresca has demonstrated that, if a material is in the plastic stage, the maximum shear across any face must have a constant value K, which he has ascertained experimentally for a variety of materials. This constant resistance to maximum slide we shall term in future the *plastic modulus*. Hence to obtain the plastico-dynamic equations we must express the fact that

the maximum shear across any face $= K$..........(iii).

Again, Tresca has demonstrated that the direction of the maximum shear is also that of the maximum velocity of slide. This forms then our last condition:

$$\left.\begin{array}{c}\text{maximum shear and maximum slide-}\\ \text{velocity are co-directional}\end{array}\right\} \text{.........(iv)}.$$

Equations (i) and (ii), with conditions (iii) and (iv) should give the complete plastico-dynamic equations.

[248.] Saint-Venant only treats the case of what we may term uniplanar plasticity, or the motion the same in all planes parallel to that of x, z. Thus the co-ordinate y disappears from his results.

Let x', z' be two rectangular axes making an angle α with those of x, z, then it easily follows from the first formula in our Art. 1368* that,

$$\widehat{x'z'} = \frac{\widehat{zz} - \widehat{xx}}{2} \sin 2\alpha + \widehat{zx} \cos 2\alpha.$$

This takes its maximum value for

$$\tan 2\alpha = \frac{\widehat{zz} - \widehat{xx}}{2\widehat{xz}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)},$$

and is then of the intensity

$$\tfrac{1}{2}\sqrt{4\widehat{zx}^2 + (\widehat{zz} - \widehat{xx})^2}.$$

Thus condition (iii) becomes

$$\widehat{zx}^2 + \left(\frac{\widehat{zz} - \widehat{xx}}{2}\right)^2 = K^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(vi)}.$$

Further the slide-velocity is easily found to be given by

$$\frac{dw'}{dx'} + \frac{du'}{dz'} = \left(\frac{dw}{dz} - \frac{du}{dx}\right) \sin 2\alpha + \left(\frac{dw}{dx} + \frac{du}{dz}\right) \cos 2\alpha,$$

and therefore takes its maximum when

$$\tan 2\alpha = \frac{\dfrac{dw}{dz} - \dfrac{du}{dx}}{\dfrac{dw}{dx} + \dfrac{du}{dz}}.$$

Hence condition (iv) becomes

$$\frac{\widehat{zz} - \widehat{xx}}{2\widehat{xz}} = \left(\frac{dw}{dz} - \frac{du}{dx}\right) \Big/ \left(\frac{dw}{dx} + \frac{du}{dz}\right) \ldots\ldots\ldots\ldots\text{(vii)}.$$

Finally equations (i) and (ii) take in this case the simpler forms:

$$\left.\begin{aligned} \frac{d\widehat{xx}}{dx} + \frac{d\widehat{xz}}{dz} &= -\rho\left(X - \frac{du}{dt} - u\frac{du}{dx} - w\frac{du}{dz}\right) \\ \frac{d\widehat{xz}}{dx} + \frac{d\widehat{zz}}{dz} &= -\rho\left(Z - \frac{dw}{dt} - u\frac{dw}{dx} - w\frac{dw}{dz}\right) \\ \frac{du}{dx} + \frac{dw}{dz} &= 0 \end{aligned}\right\} \ldots\ldots\ldots\text{(viii)}.$$

Equations (vi), (vii), and (viii) are those for uniplanar plasticity.

[249.] Saint-Venant remarks that even these equations will be difficult to solve for any except the simplest cases. He suggests, however, that those for a cylindrical plastic flow would not be difficult to obtain.

In a final paragraph (p. 316) to the first paper Saint-Venant remarks:

Je ferai seulement une dernière remarque: c'est que si, aux six composantes de pressions ci-dessus, $\widehat{xx}, \ldots\ldots \widehat{xy}$, l'on ajoute respectivement les termes: $2\epsilon u_x$, $2\epsilon v_y$, $2\epsilon w_z$, $\epsilon\,(v_z + w_y)$, $\epsilon\,(w_x + u_z)$, $\epsilon\,(u_y + v_x)$, représentant, comme on sait, ce qui vient du frottement dynamique dû aux vitesses de glissement relatif dans les fluides non visqueux se mouvant avec régularité, les équations des solides plastiques, ainsi complétées, s'étendront au cas où les vitesses avec lesquelles leur déformation s'opère, sans être considérables, ne seraient plus excessivement petites, et pourraient engendrer ces résistances particulières, ordinairement négligeables, dont on a parlé au No. 3. Les mêmes équations, avec tous ces termes, seraient propres, aussi, à exprimer les mouvements réguliers (c'est-à-dire pas assez prompts pour devenir tournoyants et tumultueux) des *fluides visqueux*, où il doit y avoir des composantes tangentielles de deux sortes, les unes variables avec les vitesses u, v, w, et mesurées par les produits de ϵ et de leurs dérivées, les autres indépendantes de ces grandeurs des vitesses, ou les mêmes quelle que soit la lenteur du mouvement, et attribuables *à la viscosité*, dont K *représenterait alors le coefficient spécifique.*

[250.] In the second paper to which we have referred in our Art. 245, Maurice Lévy establishes two sets of results. In the first place he obtains the general equations of plasticity; in the next he considers the special case of a cylindrical plastic flow.

We cite the general equations here, but refer to our later discussion of Lévy's memoir for remarks on his method of obtaining them.

The general equations (i) and (ii) hold for this case. The condition (iii) becomes:

$$4\,(K^2 + q)\,(4K^2 + q) + 27r^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ix)},$$

where
$$q = \Delta_y\Delta_z + \Delta_z\Delta_x + \Delta_x\Delta_y - \widehat{yz}^2 - \widehat{zx}^2 - \widehat{xy}^2,$$
$$r = \Delta_x\widehat{yz}^2 + \Delta_y\widehat{zx}^2 + \Delta_z\widehat{xy}^2 - \Delta_x\Delta_y\Delta_z - 2\widehat{yz}\,\widehat{zx}\,\widehat{xy},$$

and
$$\widehat{xx} - \Delta_x = \widehat{yy} - \Delta_y = \widehat{zz} - \Delta_z = \tfrac{1}{3}\,(\widehat{xx} + \widehat{yy} + \widehat{zz}).$$

The condition (iv) becomes

$$\frac{\widehat{yz}}{v_z + w_y} = \frac{\widehat{zx}}{w_x + u_z} = \frac{\widehat{xy}}{u_y + v_x} = \frac{\widehat{yy} - \widehat{zz}}{2\,(v_y - w_z)} = \frac{\widehat{zz} - \widehat{xx}}{2\,(w_z - u_x)} \quad \ldots\ldots \text{(x)}.$$

Thus (i), (ii), (ix) and (x) are the requisite equations.

[251.] On p. 371 Lévy remarks that Saint-Venant in the case of uniplanar plasticity has not considered the stress $\widehat{yy}$. From equation (x) since $v_y = 0$, and therefore $w_z + u_x = 0$ from (ii), we have

$$\frac{\widehat{yy} - \widehat{zz}}{2u_x} = \frac{\widehat{zz} - \widehat{xx}}{-4u_x},$$

or,

$$\widehat{yy} = \tfrac{1}{2}(\widehat{zz} + \widehat{xx}) \quad \text{(xi)}.$$

[252.] On p. 372 we have the equations for a cylindrical plastic flow. If z be the axial, r the radial directions, ϕ the meridian angle, u, w radial and axial velocities, they take the form:

$$\left.\begin{aligned} \frac{d\widehat{rr}}{dr} + \frac{d\widehat{rz}}{dz} + \frac{\widehat{rr} - \widehat{\phi\phi}}{r} &= -\rho\left(R_0 - \frac{du}{dt} - u\frac{du}{dr} - w\frac{du}{dz}\right) \\ \frac{d\widehat{rz}}{dr} + \frac{d\widehat{zz}}{dz} + \frac{\widehat{rz}}{r} &= -\rho\left(Z_0 - \frac{dw}{dt} - u\frac{dw}{dr} - w\frac{dw}{dz}\right) \end{aligned}\right\} \quad \text{(xii)},$$

$$\frac{du}{dr} + \frac{u}{r} + \frac{dw}{dz} = 0 \quad \text{(xiii)},$$

$$4\widehat{rz}^2 + (\widehat{rr} - \widehat{zz})^2 = 4K^2 \quad \text{(xiv)},$$

$$\frac{\widehat{rz}}{w_r + u_z} = \frac{\widehat{rr} - \widehat{zz}}{2(u_r - w_z)} = \frac{\widehat{rr} - \widehat{\phi\phi}}{2(u_r - u/r)} \quad \text{(xv)}.$$

We shall see later that the condition (xiv) is not sufficient nor always correct: see our Art. 263.

As a rule when the plastic movements are very small and the effects of gravity can be neglected, we may put the right-hand sides of equations (i) and (xii) equal to zero.

[253.] In the third paper whose title is given in our Art. 245 Saint-Venant first makes the remark that if the velocities be neglected the equations of uniplanar plasticity reduce to the discovery of an unknown auxiliary ψ, where:

$$\widehat{xx} = \frac{d^2\psi}{dz^2}, \quad \widehat{zz} = \frac{d^2\psi}{dx^2},$$

$$\widehat{xz} = -\frac{d^2\psi}{dzdx},$$

and

$$4\left(\frac{d^2\psi}{dxdz}\right)^2 + \left(\frac{d^2\psi}{dx^2} - \frac{d^2\psi}{dz^2}\right)^2 = 4K^2 \quad \text{(xvi)}.$$

He suggests that this equation might be solved by approximation.

[254.] Saint-Venant next passes to the treatment of the limiting or surface conditions of plasticity, i.e. the conditions which hold at the boundary of the portion of the material in a plastic condition. He terms them the *équations définies ou déterminées;* the previous equations being called the *équations indéfinies.*

These conditions are of various kinds. A certain portion of the block of matter alone is plastic (called by Tresca the *zône d'activité*), other portions may remain elastic, or after passing through a plastic condition return to elasticity (e.g. a jet of metal after passing an orifice).

The conditions break up into three classes:

1st. Those which relate to the surface of the material at points which have retained or resumed their elasticity. Let such a surface be exposed to a traction T_e and let the elastic stresses be $\widehat{xx}_e \ldots\ldots \widehat{xy}_e$, the suffix e merely referring to their elastic character. The type of surface condition will be

$$\widehat{xx}_e \cos(nx) + \widehat{xy}_e \cos(ny) + \widehat{xz}_e \cos(nz) = T_e \cos(lx) \ldots\ldots \text{(xvii)},$$

where n is the direction of the surface-normal and l that of the applied traction T_e.

2nd. The material is in a plastic stage at the bounding surface, T_p being the traction: the type of equation, if $\widehat{xx}_p \ldots\ldots \widehat{xy}_p$ denote the plastic stresses, is:

$$\widehat{xx}_p \cos(nx) + \widehat{xy}_p \cos(ny) + \widehat{xz}_p \cos(nz) = T_p \cos(lx) \ldots\ldots \text{(xviii)}.$$

3rd. Equations which must hold at the surface at which the material changes from plasticity to elasticity. These are of the type:

$$(\widehat{xx}_e - \widehat{xx}_p) \cos(nx) + (\widehat{xy}_e - \widehat{xy}_p) \cos(ny) + (\widehat{xz}_e - \widehat{xz}_p) \cos(nz) = 0 \ldots \text{(xix)}.$$

In the equations (xvii)—(xix) the elastic stresses and plastic stresses must be obtained from the general equations of elasticity and of plasticity respectively.

[255.] On pp. 378—380, Saint-Venant treats the special case of a right circular cylinder of radius r subjected to torsion till plasticity commences in the outer zone from r_0 to r. He easily finds if M be the torsional couple, μ the slide-modulus and τ the torsional angle:

$$M = 2\pi \left[\mu\tau \frac{r_0^4}{4} + K\left(\frac{r^3}{3} - \frac{r_0^3}{3}\right)\right],$$

while at the surface of elasticity and plasticity we must have

$$\mu\tau r_0 = K.$$

There will be no plasticity then so long as

$$\tau < \frac{K}{\mu r}, \text{ or } M < \frac{\pi r^3}{2} K.$$

If τ be greater than this we have:

$$M = \pi K\left(\frac{2}{3} r^3 - \frac{1}{6} \frac{K^3}{\mu^3\tau^3}\right).$$

[256.] On pp. 380—381 we have the case of plasticity produced by the equal or 'circular' flexion of a prism of rectangular section.

Let $2c$ be the height in the plane of flexure, $2b$ the breadth of the section, $2c_0$ the height of the middle portion which remains elastic, and $1/\rho$ the uniform curvature. Then it is easy to see that the bending moment M is given by:

$$M = \frac{4}{3}\frac{E}{\rho} bc_0^3 + 4Kb(c^2 - c_0^2).$$

At the surface of separation of the plastic and elastic parts:

$$\frac{Ec_0}{\rho} = 2K.$$

Whence we find:

$$M = 4Kb\left(c^2 - \frac{4}{3}\frac{K^2\rho^2}{E^2}\right),$$

where we must have $\rho < \frac{Ec}{2K}$ or the prism will remain elastic.

[257.] Saint-Venant in conclusion indicates that only after first ascertaining *experimentally* the general form taken by the flow in special cases will it be possible to attempt approximate solutions of the equations of plasticity.

I may remark that Saint-Venant assumes that elasticity and plasticity are continuous. This does not seem to me at all borne out by experiment, the stresses have long ceased to be proportional to the strains before plasticity commences: see the diagram on p. 890 of our Vol. I. and my remark in Art. 244.

[258.] Two memoirs by Saint-Venant on *plastico-dynamics* or plasticity occur in Vol. LXXIV. 1872, of the *Comptes rendus.* They are entitled:

(1) *Sur l'intensité des forces capables de déformer avec continuité des blocs ductiles, cylindriques, pleins ou évidés, et placés dans diverses circonstances* (pp. 1009—1015 with footnotes to p. 1017).

(2) *Sur un complément à donner à une des équations présentées par M. Lévy pour les mouvements plastiques qui sont symétriques autour d'un même axe* (pp. 1083—1087).

These memoirs may be looked upon as supplements to those of Saint-Venant and Lévy in the *Journal de Liouville:* see our Arts. 245—57.

[259.] The general principle, Saint-Venant tells us in his first memoir, of plastic deformation is that the greatest shear at each point shall be equal to a specific constant (denoted by K in Tresca's memoir of 1869). It follows by Hopkins' theorem that at each point the greatest difference between the tractions across different faces ought to equal $2K$: see our Art. 1368*.

Saint-Venant treats two special cases, and a third by approximation. We will devote the following three articles to their discussion.

[260.] The first is that of a right six-face of ductile metal.

If the axes of coordinates be taken parallel to its edges, and its faces be subjected to uniform tractions $\widehat{xx}$, $\widehat{yy}$, $\widehat{zz}$, then these tractions will be the principal tractions at any point of the material, and it will be necessary if $\widehat{xx} \sim \widehat{zz}$ be the greatest difference that:

$$\widehat{xx} \sim \widehat{zz} = 2K \text{.............................(i).}$$

This condition is fulfilled if

$$\widehat{xx} = -\widehat{yy} = -\widehat{zz} = K,$$

or if

$$\widehat{xx} = -\widehat{yy} = -\widehat{zz} = -K.$$

Of this Saint-Venant remarks:

C'est dans ce sens qu'il faut entendre, avec M. Tresca, que la résistance, soit à l'allongement, soit à l'accourcissement du solide plastique, est constante, et égale à sa résistance au cisaillement (p. 1010).

An extension of this case is that of a cylinder on any base, for which $\widehat{yy} = \widehat{zz}$ without being equal to K, that is to say the transverse or radial tractions which we will denote by $\widehat{rr}$ are all equal and the longitudinal tractions $\widehat{xx}$ are greater than them. We have then for the condition of plasticity:

$$\widehat{xx} = 2K + \widehat{rr} \text{.............................(ii).}$$

If the radial tractions are greater than the longitudinal we have:

$$\widehat{rr} = 2K + \widehat{xx} \text{.............................(iii).}$$

Either equation (ii) or (iii) gives us by variation

$$\delta\widehat{xx} = \delta\widehat{rr},$$

or, any increment of longitudinal, is accompanied by an equal increment of transverse traction. This is Tresca's principle that *in plastic solids pressure transmits itself as in fluids,* although he proves it by the principle of work.

[261.] The second case dealt with by Saint-Venant is that of a hollow right circular cylinder placed between two rigid fixed planes perpendicular to its axis. The external face being submitted to a pressure p, we require the internal pressure p_1, necessary to reduce the material to a plastic condition.

This problem can be solved by introducing the velocities, here solely radial, of the points of the material. The principles which determine these velocities for a plastic material are: 1st, that there is no change in the volume of the element; and 2nd, that on each elementary area in the material the direction for which the shear is zero, must be that for which the slide-velocity is zero. The latter principle involves the ratios of the half-differences of the tractions to the corresponding stretch-velocities being equal two and two.

Let r be the radial distance from the axis of any element of the material, R and R_1 the external and internal radii of the cylinder; V the radial velocity of the element at r, and $\widehat{rr}$, $\widehat{\phi\phi}$, $\widehat{zz}$ the tractions along the radius, in the meridian plane and parallel to the axis at the same element. Then for the equilibrium and conservation of volume of an elementary annulus $2\pi r dr dz$, it is necessary that:

$$\frac{d\widehat{rr}}{dr} + \frac{\widehat{rr} - \widehat{\phi\phi}}{r} = 0, \quad \frac{dV}{dr} + \frac{V}{r} = 0 \ldots\ldots\ldots\ldots\ldots \text{(iv)}.$$

Further from the second principle it follows that:

$$\frac{\widehat{rr} - \widehat{zz}}{dV/dr} = \frac{\widehat{rr} - \widehat{\phi\phi}}{dV/dr - V/r} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}.$$

Eliminating dV/dr between the second equation of (iv) and (v) we have

$$\widehat{rr} - \widehat{\phi\phi} = 2\,(\widehat{rr} - \widehat{zz}) = 2\,(\widehat{zz} - \widehat{\phi\phi}),$$

whence it results that $\widehat{rr} - \widehat{\phi\phi}$ is the greatest difference, and therefore by Eqn. (i)

$$\widehat{rr} - \widehat{\phi\phi} = -2K \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(vi)}.$$

It follows from the first equation of (iv) that

$$d\widehat{rr}/dr = 2K/r.$$

Or, integrating $\qquad \widehat{rr} = -p_1 + 2K \log (r/R_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(vii)}.$

Hence from (vi) we deduce:

$$\left.\begin{aligned} \widehat{\phi\phi} &= -p_1 + 2K + 2K \log (r/R_1), \\ \widehat{zz} &= -p_1 + K + 2K \log (r/R_1). \end{aligned}\right\} \ldots\ldots\ldots\ldots \text{(viii)}.$$

We see from these equations that: (a) the pressure on the rigid faces is not uniformly distributed over the surface of the material in contact with them, (b) the meridian traction will increase and generally change from a negative to a positive value as we pass outwards.

If we make $r = R$, we have $\widehat{rr} = -p$,

or, $$p_1 = p + 2K \log (R/R_1) \quad \text{(ix).}$$

If the pressure applied p_1 has a less value than this, the 'annular fibres' near to the inside face can very well acquire stretches exceeding the limit of elasticity and even that of cohesion for isolated straight fibres; but as the fibres in the neighbourhood of the external face remain elastic, there will not be rupture, nor sensible deformation. Saint-Venant refers to the well-known experiment of Easton and Amos: see our Art. 1474*.

In the last Section 5 of the *Note* Saint-Venant refers to Tresca's somewhat unsatisfactory proof of the formula (ix).

[262.] In a foot-note pp. 1015—1017, Saint-Venant deals approximately with the following case: the outer surface of a right circular hollow cylinder (radii R, R_1) is supposed to rest on a rigid envelope, the internal surface is then subjected to great pressure which diminishes the thickness $(R - R_1)$, but increases the height (h), to determine the pressure which will produce this plastic effect. Tresca had obtained a solution of this problem on two hypotheses, which cannot be considered as entirely satisfactory. The general equations of plasticity are indeed too complex to offer much hope of an exact solution for this case. Saint-Venant gives a solution involving only the acceptance of Tresca's second hypothesis namely: that the upper base of the cylinder and all the plane-sections parallel to it remain plane and perpendicular to the axis of the block, and that lines parallel to the axis preserve their parallelism. It is obvious that this hypothesis is only approximately true; but Saint-Venant's investigation is an interesting one, as it deals with one of those cases, in which the maximum difference of the principal tractions is not given by the same pair for all values of the radial distance. This breaks up the solution into two parts corresponding to $3r^2 <$ or $> R^2$, and the case itself into two sub-cases corresponding to $3R_1^2 <$ or $> R^2$. Saint-Venant's results are not in accordance with Tresca's.

[263.] *Sur un complément à donner à une des équations présentées par M. Lévy pour les mouvements plastiques qui sont symétriques autour d'un même axe: Comptes rendus,* T. LXXIV. 1872, pp. 1083—7.

Saint-Venant refers to Lévy's third equation for plasticity with axial symmetry. This equation is (see our Art. 252, Eqn. xiv.):

$$4\widehat{rz}^2 + (\widehat{rr} - \widehat{zz})^2 = 4K^2.$$

He remarks that this equation is only the true condition for plastic motion, when the greatest and least of the *negative* tractions (*pressures*) are in the meridian plane of the point considered. This is not always true and Lévy's third condition requires to be replaced by the following one:

$2K =$ the greatest in absolute value of the three quantities:

$$\left.\begin{array}{c} 2\sqrt{\widehat{rz}^2 + \frac{1}{4}(\widehat{rr} - \widehat{zz})^2} \\ \widehat{\phi\phi} - \dfrac{\widehat{rr} + \widehat{zz}}{2} - \sqrt{\widehat{rz}^2 + \frac{1}{4}(\widehat{rr} - \widehat{zz})^2} \\ \widehat{\phi\phi} - \dfrac{\widehat{rr} + \widehat{zz}}{2} + \sqrt{\widehat{rz} + \frac{1}{4}(\widehat{rr} - \widehat{zz})^2} \end{array}\right\}.$$

This follows at once from the consideration that the discriminating cubic for the principal tractions is:

$$(T - \widehat{xx})(T - \widehat{yy})(T - \widehat{zz}) - \widehat{yz}^2(T - \widehat{xx}) - \widehat{zx}^2(T - \widehat{yy}) \\ - \widehat{xy}^2(T - \widehat{zz}) - 2\,\widehat{yz}\,\widehat{zx}\,\widehat{xy} = 0,$$

and this becomes when we put:

$$\widehat{rr},\ \widehat{\phi\phi},\ \widehat{zz},\ 0,\ \widehat{rz},\ 0 \text{ for } \widehat{xx},\ \widehat{yy},\ \widehat{zz},\ \widehat{yz},\ \widehat{zx},\ \widehat{xy},$$

respectively

$$(T - \widehat{\phi\phi})\left(T - \frac{\widehat{rr} + \widehat{zz}}{2} - \sqrt{rz^2 + \tfrac{1}{4}(\widehat{rr} - \widehat{zz})^2}\right) \\ \left(T - \frac{\widehat{rr} + \widehat{zz}}{2} + \sqrt{\widehat{rz}^2 + \tfrac{1}{4}(\widehat{rr} - \widehat{zz})^2}\right) = 0.$$

Lévy appears to have divided out by $T - \widehat{\phi\phi}$ and neglected this root.

[264.] Saint-Venant remarks that $\widehat{\phi\phi}$ is, however, sometimes the greatest or least of the three principal tractions, as for example in the problem of our Art. 261, for in that case

$$\widehat{zz} = \frac{\widehat{rr} + \widehat{\phi\phi}}{2}.$$

In the approximate solution of our Art. 262, the traction $\widehat{\phi\phi}$ is involved also in the maximum difference when $3r^2 < R^2$. Thus Lévy's memoir requires to be corrected so far as this equation is concerned.

In a foot-note Saint-Venant points out that his solutions (see our Arts. 261—2), are really obtained by the *semi-inverse* method and he suggests that the same method might be used to solve other plastic problems.

[265.] *Sur les diverses manières de présenter la théorie des ondes lumineuses. Annales de Chimie et de Physique*, 4e série, T. XXV. 1872, pp. 335—381. This memoir was also separately published by Gauthier-Villars in the same year.

The contents belong essentially to the history of the undulatory theory of light. Saint-Venant considers at considerable length the researches of Cauchy, Briot, and Sarrau in this field and points out the defects in the various theories which they have propounded. Finally he deals with Boussinesq's method of obtaining from a general type of equation the special differential equations which fulfil the conditions necessary for explaining the various phenomena of light. Saint-Venant praises highly Boussinesq's hypothesis, and considers that his theory:

> qui offre à la fois plus de simplicité, d'unité, de probabilité, et je crois aussi, de rigueur que les autres (quel que soit le remarquable talent avec lequel ont été présentés ces autres essais, qui ont toujours avancé les questions), mérite d'être enseignée de préférence (pp. 380—1).

I must remark, however, that convenient as Boussinesq's hypotheses may be as a grouping together of analytical results under one primitive formula, it cannot be held as sufficient till we understand the reasons why and how the molecular shifts are functions of the ether-shifts and their space and time fluxions, and are able to deduce the *form* of these functions from some more definite physical hypothesis.

§§ 1—2 treat of the early history of elasticity. As in the memoir of 1863 (see our Art. 146—7) Saint-Venant holds that the conditions presented by Green for exact parallelism and those suggested by Lamé for double refraction are only consistent with isotropy.

> Aussi Lamé et Green ne sont pas compris dans l'analyse que je fais des recherches de divers auteurs sur la lumière. Il importe que des hommes de talent ne s'égarent plus, en pareille matière, sur les errements des deux illustres auteurs de tant d'autres travaux plus dignes d'eux. (Footnote, p. 341.) See our Arts. 920*, 1108*, 146 and 193.

[266.] *Rapport sur un Mémoire de M. Lefort présenté le* 2 *août* 1875. This report is by Tresca, Resal and Saint-Venant (*rapporteur*) and will be found in the *Comptes rendus*, T. LXXXI. 1875, pp. 459—464. It speaks favourably of the memoir, which deals with the problem of finding the bending moment at the several sections of simple and continuous beams traversed by moving loads. We shall refer to the memoir under Lefort.

[267.] *De la suite qu'il serait nécessaire de donner aux recherches expérimentales de Plastico-dynamique: Comptes rendus,* T. LXXXI. 1875, pp. 115—122.

This note refers to the need of new plastico-dynamic experiments with a view of extending the number of solutions hitherto obtained and also the basis of the existing plastico-dynamic theory. Saint-Venant points out the insufficiency of Tresca's method of dividing the plastic solid into separate portions and applying to these the laws of fluid-continuity; he refers to his own researches in this purely *kinematic* direction: see our Art. 233, and then to his later theory and equations, as supplemented by Lévy, and based on Tresca's law of the equality of the stretch and slide coefficients of resistance: see our Arts. 236, 245 and 258. He points out that to develop this theory, what we want is not the form taken by jets of plastic material, but the *absolute paths of the elements in the material.* He suggests how this might be ascertained by allowing the same load to act in the same manner but for different periods on a number of like plastic blocks, in which a series of points had been previously marked by a three-dimensional wire netting placed in the molten metal. He notes also other methods likely to give the same result. In the course of the note he refers to the simple cases of plasticity solved by Lévy, Boussinesq and himself: see our Arts. 255—61. At the end are a few lines from Tresca, who recognises the importance of the experiments proposed by Saint-Venant, which, I believe, he did not live to undertake.

[268.] *Sur la manière dont les vibrations calorifiques peuvent dilater les corps, et sur le coefficient des dilatations; Comptes rendus,* 1876, T. LXXXII., pp. 33—38.

This is an attempt to represent thermal effects by the change produced by thermal vibrations directly in intermolecular distance rather than indirectly by their influence in altering the constants of molecular attraction. Saint-Venant deals with two molecules only and supposes one fixed.

Let r_0 be the intermolecular distance in equilibrium, $r = r_0 + v$ the displaced distance and $f(r)$ the law of intermolecular action, then we easily find for our equation of vibration:

$$m\,\frac{d^2v}{dt^2} = f(r) = f(r_0 + v),$$

$$= v\,f'(r_0) + \frac{v^2}{2} f''(r_0) + \frac{v^3}{6} f'''(r_0) + \dots$$

If $dv/dt = \dot{v}_0$, for v and $t = 0$, we have as a first approximation

$$v = \frac{\dot{v}_0}{a} \sin at, \text{ where } f'(r_0)/m = -a^2.$$

For a second approximation:

$$v = \frac{\dot{v}_0}{a} \sin at + \frac{b^2 \dot{v}_0^{\,2}}{6a^4 r_0} (1 - \cos at)^2, \text{ where } \frac{r_0 f''(r_0)}{m} = b^2.$$

Let us find the mean value v_m of v from $t = 0$ to $2\pi/a$; we have:

$$v_m = \frac{b^2 \dot{v}_0^{\,2}}{4a^4 r_0}.$$

Hence the stretch due to the thermal vibration

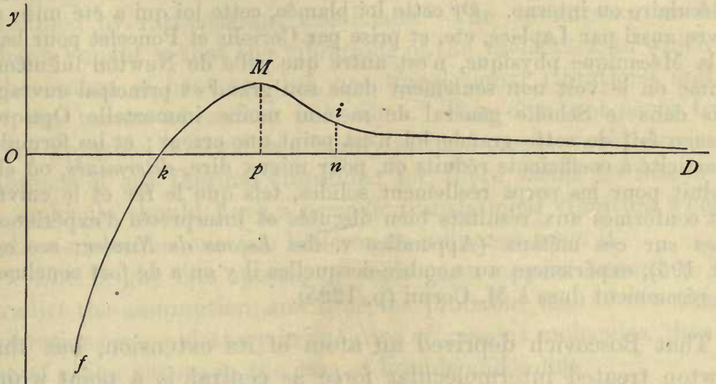
$$= \frac{v_m}{r_0} = \frac{m\dot{v}_0^{\,2}}{2} \, \frac{1}{2r_0} \, \frac{f''(r_0)}{\{f'(r_0)\}^2}.$$

Thus we see that the stretch is proportional to the kinetic energy $m\dot{v}_0^{\,2}/2$, which is generally regarded as a measure of the absolute temperature, and will be positive if $f''(r_0)$ is positive.

Saint-Venant states that these conclusions will still hold, if the two molecules be replaced by a system. The thermal effect would thus depend on *the derivatives of the second order of the function* $f(r)$.

If there should be a point of inflexion in the curve which represents the law of intermolecular action plotted out to distance, we should have a case in which increase of temperature reduced the volume, as occurs in certain exceptional substances. Saint-Venant suggests the form of the figure below for the curve $y = f(r)$; OD being the distance and Oy the force axis.

Here $Ok = r_0$ marks the point at which the action changes from repulsion to attraction; if the axes Oy, OD are asymptotic in character, we have the infinitely great force and infinitely small force at infinitely small and infinitely great distances respectively well marked. pM marks the maximum attractive force between the molecules, and any force greater than this, if maintained, will produce rupture. It corresponds to a distance Op, which defines that of rupture. Great thermal vibrations which impose such a velocity on the molecule that the intermolecular distance exceeds Op may perhaps, indicate liquefaction by heat. The point i corresponds to a point of inflexion, and to a contraction due to heating the substance in the liquid state.



The discussion, if not very conclusive, is interesting especially in its bearings on rari-constancy. See our Arts. 439* and 977*.

[269.] *Sur la constitution atomique des corps: Comptes rendus*, T. LXXXII. 1876, pp. 1223—26.

Saint-Venant in this note refers to a remark of Berthelot on the paradox involved in the indivisibility of an atom supposed to be endowed with matter and therefore of necessity extended. He refers to his memoir of 1844 (see our Art. 1613*), and declares that he considers partly for metaphysical, partly for physico-mathematical reasons, *continuous* extension to belong neither to bodies nor to their component atoms. The point which alone concerns us here is his reference to the rari-constant hypothesis:

A cette occasion je ferai une remarque. Plusieurs auteurs, soit anglais, soit allemands, dans des œuvres qui sont du reste d'une haute portée, voulant étendre à des substances élastiques celluleuses, ou spongieuses, ou demi-fluides, telles que le liége, les gelées, les moelles végétales, le caoutchouc, les formules d'élasticité des solides, découvertes et établies en France de 1821 à 1828 par Navier, Cauchy, Poisson, Lamé et Clapeyron, et ayant besoin, pour une pareille extension, d'augmenter en nombre ou de rendre indépendants les uns des autres des coefficients de ces formules, se sont pris à condamner vivement, sous le nom de *théorie* *de Boscovich*, non pas son idée capitale de réduction des atomes à des centres d'action de forces, mais la loi même, la loi physique générale des actions fonctions des distances mutuelles des particules qui les exercent réciproquement les unes sur les autres. Et ils attribuent ainsi au célèbre religieux *l'erreur grave* où sont tombés, suivant eux, Navier, Poisson et nos autres savants, créateurs, il y a un demi-siècle, de la Mécanique moléculaire ou interne. Or cette loi blamée, cette loi qui a été mise en œuvre aussi par Laplace, etc. et prise par Coriolis et Poncelet pour base de la Mécanique physique, n'est autre que celle de Newton lui-même, comme on le voit non seulement dans son grand et principal ouvrage, mais dans le Scholie général de sa non moins immortelle Optique. L'usage fait de cette grande loi n'est point une erreur; et les formules d'élasticité à coefficients réduits ou, pour mieux dire, *déterminés*, où elle conduit pour les corps réellement solides, tels que le fer et le cuivre, sont conformes aux résultats bien discutés et interprétés d'expériences faites sur ces métaux (Appendice V. des *Leçons de Navier*: see our Art. 195), expériences au nombre desquelles il y en a de fort concluantes, récemment dues à M. Cornu (p. 1225).

That Boscovich deprived an atom of its extension, but that Newton treated intermolecular force as central, is a point which deserves to be recalled to mind: see our Art. 26*.

[270.] *Sur la plus grande des composantes tangentielles de tension intérieure en chaque point d'un solide, et sur la direction des faces de ses ruptures. Comptes rendus*, 1878, T. LXXXVII., pp. 89—92.

Potier had given the following formulae for the shear $\widehat{rs}$ across a face whose normal r makes angles α, β, γ with the directions of the principle tractions T_1, T_2, T_3:

$$\widehat{rs} = (T_1 - T_2)\cos^2\alpha\cos^2\beta + (T_2 - T_3)\cos^2\beta\cos^2\gamma + (T_3 - T_1)\cos^2\gamma\cos^2\alpha,$$

maximum value of $\widehat{rs} = \frac{1}{2}$ (difference of greatest and least principal tractions).

He had then proceeded to apply these formulae to the conditions of rupture. Saint-Venant notices that these results had been given by Kleitz in 1866, by Lévy in 1870, and by himself in 1864. He might also have added by Hopkins in 1847. The note then points out that rupture in the direction of maximum shear is hardly confirmed by experiments, which point rather to rupture in the direction of maximum stretch. Saint-Venant finally considers the results of some then recent experiments, but remarks on the need for further research in this direction.

[271.] *Sur la dilatation des corps échauffés et sur les pressions qu'ils exercent. Comptes rendus*, 1878, T. LXXXVII., pp. 713—18.

This memoir should be read in conjunction with that of 1876: see our Art. 268. It shews us how the phenomena of heat may possibly be accounted for by the law of intermolecular force as assumed by rari-constant elasticians. The assumption made by Saint-Venant is that the vibrations of the molecules, to which the phenomena of heat are due, are translational vibrations, and not of the nature of surface pulsations. This does not seem to me very probable, because in a highly rarified gas, it would denote the absence of any thermal vibration; for, there seems no reason why a molecule should have a *periodic* translational vibration when its fellow-molecules exercise little or no influence upon it.

The bright line spectra of such gases appear indeed to contradict the assumption, and it seems probable that if the thermal vibrations are pulsatory in the case of gaseous molecules, they will be of a like nature in the case of liquids and solids.

[272.] Saint-Venant commences his article with some account of his earlier memoirs, namely the communication made to the *Société Philomathique* in 1855 (see our Art. 68), and the first memoir of 1876 (see our Art. 268). He deduces by similar analysis to that of the latter memoir the same result

$$v = \frac{\dot{v}_0}{a} \sin at \qquad \text{(i)},$$

shewing that it is necessary to take into account the terms of the second order, if we are to deal on these lines with thermal phenomena.

[273.] In addition, however, he here proceeds to consider the effect that translational vibrations would have on the pressure exerted by a system of molecules on a surrounding envelope. To obtain some idea of this he supposes a free molecule placed between two fixed ones at a distance $2r_0$ from each other. He easily obtains for the vibrations of the free molecule the equation:

$$m\frac{d^2v}{dt^2} = 2vf'(r_0) + \frac{v^3}{3}f'''(r_0) + \dots \qquad \text{(ii)}.$$

If we put $2f'(r_0) = -ma'^2$, and neglect *only the cubes*, we find

$$v = \frac{\dot{v}_0}{a'} \sin a't.$$

As the other two molecules are fixed, there is no question here of dilatation. To find the reaction on either molecule we have to substitute this value of v in $f(r_0 + v)$ and we obtain

$$f(r_0 + v) = f(r_0) + \frac{f'(r_0)}{a'} \dot{v}_0 \sin a't + \frac{f''(r_0)}{2a'^2} \frac{\dot{v}_0^2}{2} 2 \sin^2 a't + \ldots\ldots\ldots \text{(iii)}.$$

Thus the mean value of p, the pressure upon the envelope of the vibrating elementary mass, would be

$$= f(r_0) - \frac{f''(r_0)}{4f'(r_0)} \frac{\dot{v}_0^2}{2} m \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}.$$

Saint-Venant remarks that as $f'(r_0)$ is obviously negative $(= -ma'^2/2)$, we have only to suppose $f''(r_0)$ negative in order that this may connote an increase of pressure due to the vibration.

Referring to the value of the pressure as given by Eqn. (iv) he suggests in a footnote:

Cette sorte de considération, avec mise en compte, comme il est fait ici, des *dérivées* du second ordre $f''(r)$ des actions, n'est-elle pas propre à remplacer, avec avantage, ces chocs brusques des molécules des gaz contre les parois de leurs récipients, avec réflexions multiples et répétées, que des savants distingués de nos jours ont inventés ou revivifiés, dans la vue de rendre compte mathématiquement des pressions exercées sur ces parois, etc.? (p. 717.)

[274.] Saint-Venant in his fourth paragraph (p. 717) asks whether we can extend the results here found for two or three molecules to a multitude of molecules. He replies, yes, because it is easy to see that the new terms of the second degree due to the first derivatives $f'(r)$ will *add* to the second derivatives in $f''(r)$. On this point he refers to a footnote on p. 281 of his memoir in the *Journal de Liouville*, 1863 (see our Art. 127), and to one by Boussinesq in the same *Journal*, 1873, pp. 305—61.

Saint-Venant concludes therefore that when on the *rari-constant* hypothesis, we calculate the stresses by means of the linear terms only for the shifts, we destroy all dilatation and all stress due to increase of temperature; *we annul in fact all thermodynamics.* According to his theory then thermal effect is entirely due to the second derivatives of the intermolecular action expressed as a function of intermolecular distance. The point is obviously important in its bearing on the rari-constant hypothesis. Do the constants of $f(r)$, the law of intermolecular reaction, vary with the temperature—as would be the case if the "strength of the intermolecular reaction" were to vary with the energy of pulsational vibrations,—or, does heat only affect the mean distance of the molecules by producing molecular translational vibrations, so that $f(r)$ is no direct function of the thermal state of the body?

[275.] *De la Constitution des Atomes.* This paper was contributed to the *Annales de la Société scientifique de Bruxelles*, 2e année, 1878. No copy of this Journal is to be found in the British Museum, the Royal Society Library, or the Cambridge Libraries, and my references will therefore be to the pages of an off-print (*Hayez, Bruxelles*) for which I am indebted to the kindness of M. Raoul de Saint-Venant. The off-print contains 78 pages, and deals—with considerable historical, philosophical and scientific detail—with the continuity of matter and Boscovich's theory of atoms. It may be considered as Saint-Venant's final *résumé* of the arguments brought forward in the memoirs of 1844 and 1876: see our Arts. 1613*, 268 and 269[1].

[276.] The theoretical basis of the theory of elasticity and the strength of materials must be ultimately sought for in the law of molecular cohesion; the discovery of that law will revolutionize our subject as the discovery of gravitation revolutionized physical astronomy. Hence it is that the elastician looks for aid to the atomic physicist, who in his turn will find much that is suggestive for the theory of molecular structure in experiments on the constants of elastic and plastic materials. Bearing this in mind, a great deal that is profitable may be obtained by a perusal of the above memoir, although many scientists would disapprove of much of the method and of several of the conclusions of the author.

In order to place clearly before the reader the scope of the memoir, I preface my discussion of it with one or two remarks. We may legitimately question whether the laws of motion as based upon our experience of sensible bodies really apply to those elementary entities which form the basis of the kinetic properties of sensible bodies[2]. It is, however, most advisable to investigate

[1] In a footnote (p. 1) Saint-Venant remarks from hearsay that the memoir of 1844 (of which I have only seen the extracts in *l'Institut*), appeared in full in a Belgian Journal *Le Catholique* in 1852.

[2] For example the *Second Law of Motion* depends on the masses of the reacting bodies A and B not being influenced by the presence of a third body C, but it is conceivable that the 'apparent mass' of an atom is a function of its internal vibratory velocities, and that these are themselves dependent upon the configuration of surrounding atoms (see Arts. 51 and 52 of a paper in the *Camb. Phil. Trans.* Vol. XIV., p. 110).

what results must flow from applying the principles of dynamics to atoms and throwing back the origin of those principles on some still more simple entity. There is much that would induce us to believe (e.g. bright line spectrum of elementary gas at small pressure and not too high temperature) that an atom has an independent motion of its parts, and this suggests that we should try the effect of applying the principles of dynamics not only to the action of one atom upon another, but also to the mutual action of an atom's parts. If multi-constancy be experimentally demonstrated, then we must suppose either (i) the law of intermolecular action is a function of *aspect*, or (ii) the action of the element A upon another B is not independent of the configuration of surrounding elements (*Hypothesis of Modified Action:* see our Vol. I., p. 814). There may be other possibilities, but these, as the most probable, deserve at least early investigation. If the law of intermolecular action is a function of *aspect*, then we should expect to find that intermolecular distance is commensurable with molecular dimensions. According to Ampère and Becquerel the former is immensely greater than the latter; according to Babinet, they are in the ratio of at least 1800 : 1 (see § 13 of Saint-Venant's memoir). It is difficult to understand under these circumstances how aspect could be of influence, it would be sufficient to treat each molecule as a mere point or centre of action, which is practically Boscovich's hypothesis. According to the more recent researches of Sir William Thomson, who deals with a molecule as an extended, material body, the mean distance between two contiguous molecules of a solid is less than the $\frac{1}{1000000000}$ of a centimetre while the diameter of a *gaseous* molecule is greater than $\frac{1}{5000000000}$ of a centimetre (*Natural Philosophy*, Part II., p. 502). Thus intermolecular distance would be less than five times molecular dimensions. In this case it would seem probable that the law of action between parts of two molecules must be the same as the law of action between parts of the same molecule, for it is difficult, although, perhaps, not impossible to understand how one could begin and the other cease to be of importance at such small relative distances as 5 to 1. Resistance alike to positive and negative traction shews that the mean intermolecular distance cannot differ much from that at which intermolecular action changes its sign; further the capacity of the molecule itself to vibrate or suffer relative motion of its parts must point to a further change of sign in the action between parts of the molecule, or if this action be really intermolecular action, we are compelled, on the hypothesis of the elementary parts of a substance having extension, to presuppose a law of mutual action capable of *thrice* changing its sign within very narrow limits indeed. An analytical expression for such a law may not be hard to discover, but it would probably be difficult to conceive any mechanical system which could give rise to such an expression. On the other hand, it is perhaps impossible to conceive "aspect" as a factor of a centre of action according to Boscovich. Nor is it easy to picture the latter centre as the source of a vibration such as seems required by the bright line gas spectrum, such a vibration, on the other hand, being easily explained as the free vibration of an extended material molecule. If we turn, however, to the hypothesis of action modified by surrounding elements, there seems no reason why we should not apply it to the Boscovichian centre just as well as to the materially extended molecule of Thomson. The essential characteristic of the theory of Boscovich is the non-extension of the ultimate source of action, not the hypothesis that intermolecular action is a function of the individual molecular distance only. Rari-constancy is not then a necessity of the fundamental portion of Boscovich's doctrine, the two do not stand or fall together as some writers have assumed. Thus Saint-Venant's supposition as to the constitution of atoms in the present memoir is essentially Boscovichian, but he writes:

> La supposition dont nous parlons entraîne celle que l'intensité de chaque action entre deux particules très-proches soit généralement fonction non-seulement de leur distance mutuelle propre, mais encore, à un certain degré, de leurs distances aux particules environnantes, et même des distances de celles-ci entre elles (p. 17, § 7).

This hypothesis of modified action leading to multi-constancy[1]

[1] The *Hypothesis of Modified Action* leads to results akin to those I have referred to in the second footnote to p. 183, and which are expressed by Saint-Venant in the following sentences on p. 17:

> On remarquera qu'elle entraîne aussi que la force *totale* sollicitant une particule n'est pas exactement la résultante géométrique, composée par la règle statique du parallélogramme ou du polygone que l'on connaît, de toutes les forces avec lesquelles la solliciteraient séparément les autres particules si chacune existait seule avec elle, comme on l'a cru jusqu'à nos jours; cette règle ne serait plus vraie que pour les actions à des distances perceptibles, dont l'intensité, réciproque aux carrés des

is presupposed by Saint-Venant throughout the memoir, although, as he remarks, he does not agree with it. It forms indeed the essential difference between this memoir and that of 1844: see his § 7, p. 18.

[277.] Saint-Venant's arguments in favour of the ultimate atom being without extension are of a threefold character:

(i) arguments from the known physical properties of atoms;

(ii) metaphysical arguments;

(iii) theological arguments.

We will briefly refer to some points with regard to these in the following three articles.

[278.] §§ 3—21 deal with more purely scientific arguments based on known properties of atoms. In § 3 we have arguments from the theory of elasticity with special reference to the controversy between Navier and Poisson: see our Arts. 527*—534*. Saint-Venant points out how the continuity of matter is related to the possibility of replacing atomic summations by definite integrals. He proves with great clearness in a footnote pp. 12—15, on the hypothesis of continuity, the following propositions, which are really involved in the result of Poisson's memoirs of 1828 and 1829 and Cauchy's memoir of 1827 (see especially *Journal de l'École polytechnique*, 1831, p. 52, and the *Exercices de mathématiques*, 1828, p. 321, comparing our Arts. 443*, 548* and 616*):

1°. The stress across an elementary plane in a solid body will like that of a liquid at rest have no shearing component.

2°. The traction at any point varies as the square of the density.

3°. If there were no *initial stresses*, no state of strain would produce stress.

Thus *on the rari-constant hypothesis*, we reach impossible physical results or it follows that matter cannot be continuous. This applies also to the ether which could not propagate slide

distances, est celle de la pesanteur universelle, toujours négligeable vis-à-vis des actions à des distances imperceptibles qui produisent l'élasticité, la capillarité, les chocs, les pressions et les vibrations; et ces dernières et énergiques actions se soustrairaient à la règle statique dont nous parlons.

vibrations if continuous. Elementary proofs of the same propositions apparently not involving the hypothesis of rari-constancy are given in §§ 9 and 10.

The following §§ 11—17 contain various arguments against the continuity as well as against the extension of the ultimate elements of matter; they are certainly not conclusive, but they are extremely suggestive, especially with regard to the difficulty I have indicated on pp. 184—5 of the rapid changes in sign which must be attributed on the hypothesis of extension to the law of action. §§ 18—21 consider the explanation of various phenomena —e.g. crystallisation and inertia—on the Boscovichian hypothesis, while a footnote pp. 36—7 deals with a possible form for the law of action and some results of it: compare our Arts. 268 and 273.

[279.] §§ 22—39 deal with what Saint-Venant terms the metaphysical objections, which he says have been the only ones raised by those to whom he has communicated his theory. Some light on Saint-Venant's method of treatment may be gained from his remark on p. 9 :

Je soumets d'avance, du reste, ce que j'énoncerai, dans les cas surtout où je serai forcé de me placer plus avant sur le terrain métaphysique, à toute autorité ayant pouvoir pour prononcer sur ce qui serait faux, et condamner ou *signaler* ce qui pourrait être dangereux.

It is beyond our province to enter into a discussion of the metaphysical arguments propounded, or the very wide range of philosophical reading evidenced by these sections[1]. It must suffice to say that Saint-Venant shews a decided preference for the scholastic writers, and an occasional tendency to imitate late-scholastic quibbles, as for example the arithmetical paradox on p. 55 by which

Sans être donc dans les secrets du Créateur nous pouvons prononcer ...qu'il n'a composé ni les corps perceptibles ni leurs dernières parties, d'un nombre infini de points de matière.

[280.] A consideration of the theological arguments up to which the metaphysical lead would be out of place here, as they are, I venture to think, out of place in the pages of a scientific

[1] There is a good criticism of the antinomy of Kant (*du terrible penseur*) with regard to the divisibility of matter on pp. 37—9.

journal. Those who are anxious to determine the real source of cohesion will not be hindered from adopting the principle of extended material atoms, if it agrees best with the facts of observation, by the assertion that if they accept and comprehend thoroughly the system of Boscovich it will preserve them from the

deux principales et plus funestes aberrations philosophiques de notre temps et des temps anciens, le panthéisme et le matérialisme (p. 74).

Notwithstanding that many readers will find themselves unable to approve either the method or conclusions of the latter portion of the memoir, the whole should certainly be read for the interesting questions it raises with regard to the physics of elasticity.

[281.] *Des paramètres d'élasticité des solides et de leur détermination expérimentale. Comptes rendus*, T. LXXXVI, 1878, pp. 781—5.

This is a good *résumé* of the relations holding between the various elastic coefficients and moduli in the case of a body possessing three planes of elastic symmetry, and of the experimental methods of finding their values.

[282.] (1) The stress-strain relations will be those of our Art. 117 (a). The coefficients are now nine in number; namely, the three direct stretch-coefficients, a, b, c the three direct slide-coefficients d, e, f and the three cross-stretch-coefficients d', e', f'. We have the following special cases:

(2) Elastic isotropy in planes perpendicular to the axis of x:

$$e=f,\ e'=f',\ b=c=2d+d'.$$

Saint-Venant states the conditions erroneously and says they reduce the *nine* constants to *six*, a, b, d, e, d', e', but d' is known in terms of b and d, or we reduce them to *five*.

(3) Complete elastic isotropy, or as Saint-Venant puts it, isotropy in two of the axial planes:

$$a=b=c=2d+d'=2e+e'=2f+f', \text{ and } d=e=f.$$

This reduces the nine coefficients to *two*, namely $d'=\lambda$ the dilatation coefficient, and $d=\mu$ the slide modulus. Saint-Venant has forgotten to state the relations $d=e=f$.

(4) Que si, sans vouloir (ce qui n'a aucune utilité) étendre l'applicabilité de ces formules aux déformations perceptibles de corps spongieux stratifiés, comme est le liége, ou de mélanges celluleux de solides et de

liquides, tels que sont les *gelées*, et même le caoutchouc, on se borne aux *vrais* solides, et si l'on admet que chacune des actions mutuelles entre deux molécules, dont les $\widehat{xx}\ldots\widehat{xy}$ sont les sommes de composantes, est *fonction d'une seule distance*, savoir celle des deux molécules qui l'exercent l'une sur l'autre, on peut prouver très-facilement (sans user de ces *intégrations autour d'un point* que Lamé a désapprouvées en 1852) que l'on a

$$d' = d, \quad e = e', \quad f = f'.$$

This reduces the coefficients in cases (1), (2) and (3) to six, three and one, respectively (p. 782).

In the second case Saint-Venant says *four*, but this is an error.

With regard to these rari-constant conditions the memoir continues:

Et ces égalités peuvent être admises; car, outre la presque évidence de leur principe, l'unité de paramètre ($\lambda = \mu$ ou $d' = d$) dans tout corps réellement isotrope se trouve prouvée par des faits nombreux, dont les derniers sont fournis par les ingénieuses expériences de 1869 de M. Cornu (p. 782).

In a footnote Saint-Venant refers to the experiments cited by Sir W. Thomson in the *Philosophical Magazine*, Jan. 1878, p. 18: see our Chapter devoted to that scientist. He holds that the discordant results there given for copper, prove either a fault in the experimental method adopted, or aeolotropy in each specimen of a diverse kind...*probablement écroui de manière à rendre, dans plusieurs d'entre elles, E_x, beaucoup plus grand que E_y ou E_z.*

The results for flint-glass and iron are he considers sufficiently near the rari-constant values, while those for cork and caoutchouc may be dismissed as proving nothing either way.

Turning to the stretch-modulus we easily find:

(5) in case (2),

$$E_x = a - 2\eta_{xy} e', \text{ and } \eta_{xy} = \eta_{xz} = \tfrac{1}{2}\, \frac{e'}{d + d'};$$

(6) in case (3),

$$E_x = 2d\,(1 + \eta_{xy}), \quad \eta_{xy} = \eta_{xz} = \tfrac{1}{2}\, \frac{d'}{d + d'};$$

(7) in case (4),

$$E_x = \tfrac{5}{2} d\, (= \tfrac{5}{2}\mu), \quad \eta_{xy} = \eta_{xz}\, (= \eta) = \tfrac{1}{4}.$$

(8) For *amorphic* materials, or bodies without regular crystallisation, such as drawn or rolled metals, stratified stone, wood etc., the aeolotropy of which can be regarded as due to unequal initial

stresses in three directions, or to a fibrous formation, three relations of the type:

$$a = \frac{(2e+e')(2f+f')}{2d+d'} \quad \text{......................(i)},$$

will sensibly hold, provided E_x, E_y, E_z have not ratios exceeding $\frac{3}{2}$ or at most 2 among themselves. This is the ellipsoidal distribution of elasticity: see our Arts. 138 and 142.

For the case of rari-constant isotropy we have:

$$a = \frac{3ef}{d}, \quad b = \frac{3fd}{e}, \quad c = \frac{3de}{f} \quad \text{......................(ii)},$$

relations admissible in general for the metals.

(9) For wood, where the ratio of E_x to E_y (the axis of x having the sense of the fibres) can amount to 10, 20, 40 and more, we can only take two of the above relations, namely:

$$b = \frac{3fd}{e}, \quad c = \frac{3de}{f} \quad \text{..........................(i)},$$

which give:

$$E_x = a - \frac{ef}{2d}, \quad E_y = \frac{fd}{e}\,\frac{8ad-4ef}{3ad-ef}, \quad E_z = \frac{e^2}{f^2}E_y \text{.........(ii)},$$

$$\eta_{xy} = \frac{1}{4}\frac{e}{d}, \quad \eta_{xz} = \frac{1}{4}\frac{f}{d}.$$

For a modification of the statements in (8) and (9) with regard to *wood*: see our Arts. 308, 312 and 313.

[283.] Saint-Venant now proceeds to indicate experimental methods of arriving at the values of the following moduli and coefficients.

(1) To find the three direct slide-coefficients, or the slide-moduli d, e, f.

Case (a). If there be isotropy in the plane perpendicular to axis of x ($e=f$). We experiment on the torsion of a right circular cylinder.

Case (b). If e be not equal to f, we use the formula of Art. 29 (modified by Art. 47 and Table I) for the torsion of a prism on rectangular base. Let the base be $2b' \times 2c'$ and let b' be much $> c'$,

$$M_x = \frac{16\alpha}{l} f \frac{b'c'^3}{3}, \text{ sensibly.}$$

If c' be much $> b'$:

$$M_x = \frac{16\alpha}{l} e \frac{b'^3 c'}{3}, \text{ sensibly,}$$

where α is the *total angle* of torsion ($= l\tau$). These give the values of e, f, and similar experiments with prisms whose axes are parallel to y and z give d, f, and d, e, so controlling the former results.

(2) To find the three direct stretch-coefficients a, b, c,

(i) They are given in cases (4), (7) and (8) Eqn. (ii) of the previous article, so soon as we know, d, e, f.

(ii) In case (9) we know b and c, while a will be given from equation (ii), or $a = E_x + \dfrac{ef}{2d}$, so soon as we have by pure tractional, or better, flexural experiments, obtained the value of E_x; the values of E_y and E_z will then be known.

[284.] We may cite the following from Saint-Venant's concluding remarks (p. 785):

Au reste, si l'expérimentateur possède des moyens d'observation assez délicats pour mesurer aussi η_{xy}, η_{xz}, et par des extensions ou des *flexions* de petits prismes taillés transversalement, pour mesurer même

$$E_y,\ E_z,\ \eta_{yz},\ \eta_{yx},\ \eta_{zx},\ \eta_{zy},$$

les expressions en $a, b, c, d, \dots f'$ qu'on peut tirer de ces diverses quantités en résolvant les équations (i.e. those with nine coefficients: see our Art. 307) à second membre trinôme, en annulant deux à deux leurs premiers membres, donneront des moyens de contrôle des mesurages opérés, et même des suppositions (4), (8), (9) (of Art. 282), qui ne sont pas admises par tout le monde. C'est un contrôle de ce dernier genre qu'opère la principale expérience de 1869 de M. Cornu.......(See our discussion of his memoir *infra*.)

On n'a pas besoin d'ajouter qu'aux mesurages statiques des dilatations, flexions et torsions, on pourra substituer au besoin, comme ont fait MM. Wertheim et Chevandier, des observations des sons rendus par des vibrations longitudinales, transversales et tournantes.

Saint-Venant has forgotten to add that the kinetic values of the elastic coefficients thus obtained will probably differ from the statical values: see our Arts. 1301* (3) and 1404*.

[285.] *Sur la torsion des prismes à base mixtiligne, et sur une singularité que peuvent offrir certains emplois de la coordonnée logarithmique du système cylindrique isotherme de Lamé.* *Comptes* *rendus*, T. LXXXVII. 1878, pp. 849—54 and 893—9. There are additions in the off-print. This memoir was read on the 2nd and 9th of December.

Its object is explained in § 2 (pp. 850—1), after the solutions given in the memoir on *Torsion* (see our Art. 36, Nos. 4 and 5) have been cited:

Clebsch a remarqué, en 1862, qu'on obtient une variété de contours plus grande encore en se servant des coordonnées curvilignes isothermes orthogonales de Lamé (i.e. *conjugate functions*); et MM. Thomson et Tait dans leur beau livre *A Treatise on Natural Philosophy*, 1867, ont indiqué, sans le développer, leur emploi pour étendre les solutions telles que (3) {= (1) of our Art. 36}, relatives aux rectangles rectilignes, à des contours rectangulaires mixtilignes se composant d'un arc de cercle ou de deux arcs concentriques et des deux rayons qui les limitent, "ce qui est" disent-ils, "très-intéressant en théorie et d'une réelle utilité en Mécanique pratique."

Il m'a paru que la solution relative à ces sortes de sections pouvait être obtenue d'une manière simple et directe, sans substituer préalablement une certaine inconnue auxiliaire à l'inconnue géométrique u, et en s'en tenant aux coordonnées polaires ordinaires r, ϕ.

[286.] In § 3, Saint-Venant obtains the required solution in cylindrical coordinates. The fundamental equations (see our Art. 17, Eqn. vi.) become

$$\left.\begin{aligned} u_{rr} + u_r/r + u_{\phi\phi}/r^2 = 0 \\ \tau r dr + u_\phi dr/r - r u_r\, d\phi = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

If γ be the angle of the annular sector, r_0 and $r_1 (> r_0)$ its radii, then the second or surface equation reduces to the following conditions when the median line is taken as initial line:

$$\left.\begin{aligned} &\tau r^2 = -u_\phi \text{ for values of } r > r_0 < r_1, \text{ when } \phi = \pm\gamma/2, \\ &u_r = 0 \text{ when } r = r_0 \text{ or } r_1, \text{ for all values of } \phi \text{ between } \pm\gamma/2 \end{aligned}\right\}\ldots\text{(ii)}.$$

These conditions are found to be satisfied by the following value of u;

$$u = -\frac{\tau r^2}{2}\frac{\sin 2\phi}{\cos\gamma} - \frac{2\tau}{\pi}\frac{(-1)^n}{2n+1}\sum_0^\infty \frac{(r_1^{m+2} - r_0^{m+2})\, r^m - (r_0 r_1)^{m+2}(r_1^{m-2} - r_0^{m-2})\, r^{-m}}{r_1^{2m} - r_0^{2m}} \times \frac{\sin m\phi}{1 - m^2/4},$$

where
$$m = \frac{2n+1}{\gamma}\pi.$$

This result is practically obtained by assuming u to be of the form

$$Cr^2 \sin 2\phi + \Sigma\,(Ar^m + A'r^{-m}) \sin m\phi,$$

and determining the constants by the surface conditions (ii).

[287.] In the following section of the memoir, § 4, Saint-Venant treats precisely the same problem by the aid of the conjugate functions,

$$\beta = \tan^{-1}(z/y), \qquad \alpha = \log\sqrt{\frac{z^2+y^2}{a^2}}.$$

He obtains two solutions in terms of α, β for a function V,—related to u by the equations $V_z = u_y$, $V_y = -u_z$.

The first contains two infinite summations and is similar in character to those given by Lamé in the *Onzième Leçon* of his work on Curvilinear Coordinates (see his p. 184). The second is that of Thomson and Tait, (see § 707, p. 252, Part II. of the second edition of their treatise).

He remarks, however, (§ 5) that although the value of the function u, obtained from V, is quite determinate when $r_0 = 0$, yet that of V becomes *indeterminate*. In fact the series for V cease to be *convergent*, and at least for the case of $r_0 = 0$, we have reached the value of u by means of an expression for V, which has ceased to have any meaning. We are thus thrown back in this case on the value of u determined by the process indicated in our Art. 286. See on this point the footnote on p. 143 of the memoir of January, 1879, considered in our Art. 291.

[288.] In § 6 Saint-Venant expresses analytically the value of the torsional moment M and the slides, and in the following sections gives the results of numerical calculations made with these formulae.

We may cite the following for the torsional moment M:

(1) Full Sectors:

$\gamma=$	45°	60°	90°	120°	180°	270°	300°	360°
$\frac{M}{\mathfrak{M}}$	·0923	·1333	·2096	·2754	·3776	·4486	·5253	·5589
$\frac{M}{\mathfrak{M}'}$	·5921	·7036	·7499	·7023	·5902	·4876	·5429	·5589

Here $\mathfrak{M} = \mu\tau \times \frac{r_1^2\gamma}{2} \times \frac{r_1^2}{2}$, or the torsional moment about the *centre* of the sector on the old Coulomb theory; $\mathfrak{M}' = \mathfrak{M} \times \left(1 - \frac{16}{9}\frac{1-\cos\gamma}{\gamma^2}\right)$ = torsional moment about the *centroid* of the sector on the old Coulomb theory. As is well known (see our Art. 181, (*d*)) Saint-Venant's theory makes *both torsional moments equal*. It will be seen at once that for bodies of this kind the results of the old theory are most erroneous and very dangerous in practice. The reduction of the torsional resistance for a split section is well brought out by the result $M/\mathfrak{M} = \cdot 5589$ for $\gamma = 360^\circ$.

(2) Annular sectors when $r_1 = 2r_0$:

$\gamma=$	60°	120°	180°
$\frac{M}{\mathfrak{M}}$	·0800	·1068	·1160
$\frac{M}{\mathfrak{M}'}$	·6812	·3160	·1909

We see again that the errors, when the old theory is used, are simply enormous.

[289.] In §§ 9—10 Saint-Venant determines the points of the full sectors, $\gamma = 60^\circ$ and $\gamma = 120^\circ$, where the slide is zero. These points are at distances from the centre differing from those of the centroids by a small amount only. He then gives the values of the shift u for various points of the same two sectors:

Les plus grandes valeurs de u sont aux points de rencontre de l'arc avec les deux côtés rectilignes. La médiane $\phi = 0$ reste immobile, et les éléments de l'arc prennent, sur le plan primitif de la section, des inclinaisons croissantes avec les distances où ils sont du milieu de cet arc.

[290.] In § 11 Saint-Venant states the value and position of the maximum slides for the same two sectors, i.e. he finds the fail-points (*points dangereux*). In both cases the maxima lie upon the contour, but the maximum of the maxima upon the rectilinear sides.

For $\gamma = 60^\circ$ the fail-point is distant $\cdot 5622 r_1$ from the centre and $\sigma = \cdot 4900\, \tau r_1$,

„ $\gamma = 120^\circ$ the fail-point is distant $\cdot 3671 r_1$ from the centre and $\sigma = \cdot 6525\, \tau r_1$.

In a footnote Saint-Venant refers to the remark of Thomson and Tait (see their § 710) that for $\gamma > \pi$ the slide becomes infinite at the centre (i.e. when $r = 0$). This does not necessarily connote rupture, but only that the strain is greater than that to which we can apply the equations of mathematical elasticity. It suggests, however, the advisibility in practice of rounding off re-entering angles.

[291.] *Sur une formule donnant approximativement le moment de torsion. Comptes rendus,* T. LXXXVIII. pp. 142—7, 1879. This note was read on January 27, 1879.

This memoir has considerable practical value; it gives an *empirical* formula which embraces within narrow limits all Saint-Venant's torsional results; full sectors with re-entering angles alone excluded.

Starting with the formula for an elliptic section (see our Art. 18)

$$M = \frac{\pi b^3 c^3}{b^2 + c^2} \mu\tau,$$

we may write it

$$M = \kappa \frac{\alpha^4}{I_0} \mu\tau,$$

where I_0 is the moment of inertia of the cross-section about an axis perpendicular to the section through the *centroid* and α is the area. The quantity

$$\kappa = \frac{1}{4\pi^2} = \cdot 025330 = \frac{1}{39 \cdot 48}.$$

Now Saint-Venant finds that for the chief sections he has treated in his various memoirs κ varies only from ·0228 to ·026, while its mean value is very nearly $\cdot 025 = \frac{1}{40}$.

Hence we have very approximately for all sections the formula:

$$M = \frac{1}{40} \frac{\alpha^4}{I_0} \mu\tau.$$

It will be noted that the torsional moment varies *inversely* as the moment of inertia and not directly as in the old theory. Saint-Venant adds:

En y réfléchissant, on comprend qu'il en doit être généralement ainsi, car les sections allongées qui, à égale surface, ont le plus grand moment d'inertie polaire, sont aussi celles auxquelles la torsion fait prendre le plus de cette incurvation, de ce *gauchissement*, qui diminue l'inclinaison prise par les fibres sur les normales à leurs éléments, surtout aux points les plus éloignés du centre, et par conséquent, sont celles sur lesquelles les réactions élastiques développées ont le moment total M le plus petit (p. 142).

The final section of the memoir § 3 (pp. 143—7), is occupied with some general observations on the elasticity of rods whose axes are curves of double curvature. Their only relation to the preceding formula for torsion is the remark that the coefficient of torsional resistance used by some writers, namely $\mu\tau I_0$, must be replaced by $\frac{1}{40}\mu\tau\alpha^4/I_0$. Saint-Venant compares the results of his memoir of 1843 (see our Art. 1584*) with the more recent researches of Bresse and Resal: see our discussion of their memoirs below. There is nothing of importance to note; the footnote p. 145 should be cancelled.

[292.] *Analyse succincte des travaux de M. Boussinesq, professeur à la Faculté des sciences de Lille, faite par M. de Saint-Venant,* 1880. This report consists of 23 *lithographed* pages.

In April, 1880, Boussinesq had printed and presented to the members of the Academy a notice of his scientific writings. (Danel, Lille, in 4°.) Saint-Venant then drew up the above analysis, strongly recommending Boussinesq for membership of the Academy. Pp. 12—17 (§§ 6—9) treat of his contributions to the theory of elasticity ('*Les travaux de M. Boussinesq sur les corps solides et leur élasticité ne sont pas moins originaux et importants*'). Pp. 17—20 deal with his various mechanical and philosophical papers; pp. 20—23 with his contributions to the undulatory theory of light. We shall have occasion to return to Saint-Venant's essay when discussing Boussinesq's memoirs.

[293.] A second paper of Saint-Venant's dealing with the elastical researches of a contemporary may be noted here. It is entitled: *Sur le but théorique des principaux travaux de Henri Tresca. Comptes rendus,* T. CI., 1885, p. 119—22.

The influence on theory of Tresca's researches and the origin of the science of plasticity are sketched. The writer attributes to Tresca a keen appreciation of theory; he was no mere empiricist, as many have erroneously believed:

> Il importe de montrer, dans l'intérêt de sa mémoire comme dans celui de la vérité scientifique, que Tresca fut un esprit plus large, un homme de vraie Science et par conséquent de *théorie* dans la meilleure et la plus saine acceptation de ce mot si souvent mal compris, si fréquemment accusé, par légèreté ou en haine systématique de la Science, de n'exprimer que des chimères (p. 119).

[294.] Géométrie cinématique:—*Sur celle des déformations des corps soit élastiques, soit plastiques, soit fluides: Comptes rendus*, 1880, T. XC., pp. 53—56.

Saint-Venant draws attention to the importance of *pure kinematics* and notes how far it is possible to advance in physical problems without the aid of force or stress considerations. Saint-Venant may be legitimately looked upon as one of the forerunners of that reduction of all dynamics to kinematics, or the exclusion of the idea of force from physics, which is now probably only a matter of time. In a lithographed course of lectures given in 1851 (*Principes de Mécanique fondés sur la Cinématique*, delivered at Versailles to engineer-students) he had treated of great portions of mechanics on kinematic principles. In this direction he had been preceded by Grassmann and followed by Resal (*Cinématique pure*, 1862, and *Mécanique générale*, 1873). The present article points out how far we can advance in the geometry of strain or displacement without the conception of stress. Saint-Venant adduces the theorem of the distortion of a sphere into an ellipsoid, and speaks as if it were only true for small strains. That it is true for all strains was pointed out by Tissot (see a supplementary *Note*, p. 209 of same volume of *Comptes rendus*) who had given a demonstration of it in the *Nouvelles Annales de Mathématiques*, 1878, p. 152. Saint-Venant points out in this *Note* that his own proof of 1864 (*L'Institut*, No. 1614, p. 389) did not really introduce this restriction. The kinematics of strain had, moreover, been thoroughly considered in 1867 by Thomson and Tait in their *Treatise on Natural Philosophy*, pp. 98—124.

[295.] *Du choc longitudinal d'une barre élastique libre contre une barre élastique d'autre matière ou d'autre grosseur, fixée au bout non heurté; considération du cas extrême où la barre heurtante est très raide et très courte: Comptes rendus*, T. XCV., 1882, pp. 359—365, *Errata*, p. 422.

This is only an abstract of the memoir. It gives a solution in trigonometrical series for the case of one bar striking longitudinally a second with one end fixed.

If V be the initial uniform speed of the impelling bar, a_2 its length, a_1 that of the fixed bar, P_2, P_1 the weights of the two bars, x the abscissa measured along the common axis of the two bars from the end of the fixed bar, then the shifts u_2 and u_1 of either bar at any point *x during the impact* are:

$$u_2 = P_2 V \Sigma \frac{2 \cos \{m\tau_2 (a_1 + a_2 - x)/a_2\} \sin mt}{m \cos m\tau_2 \left(\dfrac{P_1}{\sin^2 m\tau_1} + \dfrac{P_2}{\cos^2 m\tau_2}\right)},$$

$$u_1 = P_2 V \Sigma \frac{2 \sin \{m\tau_1 x/a_1\} \sin mt}{m \sin m\tau_1 \left(\dfrac{P_1}{\sin^2 m\tau_1} + \dfrac{P_2}{\cos^2 m\tau_2}\right)},$$

where m is a root of the equation:

$$\frac{P_1}{\tau_1} \cot m\tau_1 - \frac{P_2}{\tau_2} \tan m\tau_2 = 0,$$

and $\tau_1 = a_1/k_1$, $\tau_2 = a_2/k_2$; k_1 and k_2 being the velocities of sound in the two bars.

[296.] Saint-Venant then considers the case when τ_2 is very small as compared with τ_1, and so deduces Navier and Poncelet's expression for the vibrations of a bar struck by a weight on its free terminal: see our Arts. 273*, and 991*. Saint-Venant does not enter into the question of the *time and manner* in which the bars separate. He goes on to remark that in the case of two *free* bars we may express the result in finite terms, as also in the case of one free bar and a weight moving with a definite velocity and striking it longitudinally on one terminal. The case of a bar fixed at one terminal and struck by a moving weight at the other, he does not in this memoir attempt to solve in finite terms. This, however, he proceeded to do in an article in the same volume of the *Comptes rendus*, on pp. 423—427, entitled:

[297.] *Solution, en termes finis et simples, du problème du choc longitudinal, par un corps quelconque, d'une barre élastique fixée à son extrémité non heurtée.*

This solution is very similar to the full treatment of the problem by Boussinesq referred to in our Art. 341. But it fails to determine the instant of separation, and so does not completely solve the problem. After Boussinesq had given his solution Saint-Venant with the aid of Flamant concluded the whole subject with a graphical investigation of the successive states of the bar and the impelling load for the whole duration of the impact: see our Arts. 401—7.

Section V.

The Annotated Clebsch.

[298.] *Théorie de l'élasticité des corps solides de Clebsch. Traduite par MM. Barré de Saint-Venant et Flamant, avec des Notes étendues de M. de Saint-Venant. Paris*, 1883, *pp.* 1—900 (but by means of subscripts the number of pages is much greater than thus appears, e.g. 480. *a*—480. *gg*).

This is Saint-Venant's last great and, we may say, most complete contribution to the theory of elasticity. By means of footnotes, section-notes and appendices he has almost trebled the matter given by Clebsch, and the result is a treatise on the theory of elasticity from the mathematico-physical standpoint which will long remain the standard work on this subject.

Au moyen de ces explications et annexes, auxquelles nous aurions pu donner plus d'étendue en rapportant d'autres résultats inédits de nos recherches déjà anciennes, nous espérons, si l'on veut bien y donner quelque attention, que la traduction offerte par nous aura une réelle utilité et que la belle et intéressante branche de physique mathématique ayant, avec l'art des constructions, des rapports si intimes, pourra être de mieux en mieux comprise, étudiée et appliquée (p. xxi).

With Clebsch's contributions to elasticity we shall busy ourselves later; so far as the text of his work is concerned, we have only to note here that his isotropic formulae are everywhere replaced by those for suitable distributions of homogeneity (see our Art. 114), and that various obscurities in his treatment are explained or corrected in copious footnotes. We shall occupy ourselves in the following articles with an analysis only of Saint-Venant's contributions to the volume.

[299.] Saint-Venant's first important note occurs on pp. 39—42. It is headed: *La preuve de la forme linéaire des expressions des composantes de tensions ne peut pas être purement mathématique.* This deals with the same matter as pp. 662—5 of the *Leçons de Navier*: see our Arts. 192 (*a*) and 928*, namely the futility of all purely mathematical deductions of the linearity of the stress-strain relations. Such deductions have been given by

Green, Clebsch, Thomson and others: see our Art. 928* and the footnote Vol. I., p. 625.

Généralement et philosophiquement aucune considération purement *mathématique* ne saurait révéler le mode de la dépendance mutuelle des forces agissant sur les éléments des corps, et des changements géométriques qui s'y opèrent, tels que ceux des longueurs et des angles de leurs côtés: la connaissance de ce mode ne peut être dérivée que des faits, ou de quelque loi *physique* exprimant un ensemble de faits constatés (p. 39).

Saint-Venant appeals to experiment and cites Stokes' adduction of the isochronism of sound vibrations with approval: see our Art. 928*. We have remarked elsewhere that the stress-strain relation cannot, however, be treated as linear for the slight elastic strains in many of the materials of practical structures: see Note D of our Vol. I., p. 891.

[300.] But Saint-Venant is not satisfied with appeal to experiment and observation; these give Keplerian laws, without the backbone of Newtonian gravitation:

En général, pour convaincre nos esprits, l'empirisme, qui ne rend compte de rien, ne suffit pas: il nous faut encore une explication, une raison scientifique, où la preuve que les formules qu'on nous propose dépendent de quelque loi assez générale, assez *grandiose*, c'est-à-dire simple, pour que nous puissions en raisonnant, comme faisait Leibnitz, quand ce ne serait que d'une manière instinctive, la regarder comme pouvant être celle à laquelle le souverain Législateur a soumis les phénomènes intimes dont les formules en question représentent et mesurent les manifestations extérieures (pp. 40–1).

Saint-Venant finds this *loi assez générale, assez grandiose* in the law of intermolecular central action, as a function only of the distance, and cites its acceptance by the leading physical mathematicians from Newton to Clausius. He then refers to Green and his followers, who, as we know, appealed to Taylor's Theorem, as a *loi assez grandiose.* Now behind this appeal for 21 independent constants to Taylor's Theorem, although unrecognised by Green, was the important conception that possibly intermolecular action depends not only on the individual molecules, but on the position of each pair of them in the universe relative to other molecules. For example, if intermolecular action arises from molecular pulsations in a fluid ether, we find intermolecular force is a function of molecular surface energy, which surface energy is itself a function of position relative to the totality of other molecules. It is true that the law of intermolecular force thus resulting is not *simple,* although with the knowledge we have of thermal and optical phenomena, it may tend to coordinate far better than any simpler law the total physical universe. Saint-Venant does not appear here to strengthen the arguments of the *Appendice* V (see our Art. 192 (a)) by the introduction of a *souverain Législateur,* for whom a *loi assez grandiose* must necessarily be *assez simple.* The assumption is, indeed, anthropomorphical in the extreme. When we regard thermal and optical phenomena,—and note the probable vibration of molecules and the existence of an ether—we may be quite certain that the law of intermolecular action whatever be its nature is far from being primary in the universe; it must be a result of the structure of molecule and ether; *grandiose* it certainly may be, but the addition *c'est-à-dire simple* is an anthropomorphical dogma, which recalls to our minds the *mundi fabrica est perfectissima* of Euler.

[301.] We must next consider the *Note finale du* § 16 which occupies pp. 63—111.

§§ 1—12 of the Note (pp. 65—75) are again concerned with the coefficient controversy, but take up a different line of argument from that of the *Appendice* V: see our Arts. 192—5. Saint-Venant here enquires how far Green's appeal to the principle of work and the impossibility of perpetual motion in itself involves the reduction of the elastic constants to 15.

[302.] He starts from the equation

$$\Sigma m V^2/2 + \psi(x, y, z, x', y', z', x'', y'', z''...) = \text{some constant } C...(a),$$

where V is the translational velocity of the molecule m, whose centroidal position is x, y, z, and the dashed letters give the positions of other molecules m', m'', etc. In other words he makes the total *translational* energy of the system a function of molecular position. He omits:

(1) from the kinetic energy a possible internal vibratory motion of the molecule due to pulsations in its atoms or to change in the relative motion of the atoms of the same molecule;

(2) possible factors in the potential energy due to strains in the molecule itself or to changes in its *aspect* with regard to other molecules.

Is he justified in thus making the translational energy of the molecular centroids a function solely of their position? He seems to think that both the omissions (1) and (2) are legitimate provided that there are no such changes of *temperature* as produce violent atomic vibrations, and that we take the mean of large numbers (see his § 12). But is it not within the bounds of possibility that the mean internal potential energy of the molecules may be changed by an elastic strain, although the mean internal kinetic energy on which the temperature may be supposed to depend remains unchanged? This change in the potential energy of the molecule will be a function of the relative molecular position, but it may be one of *aspect* as well as of centroidal position. If we accept, however, with both Green[1] and Saint-Venant that the former can only depend on the latter, we are thrown back, supposing no sensible thermal changes, on Equation (a).

[303.] Saint-Venant in § 5 proceeds to question whether the Equation (a) can give the form of ψ required by Green. He says that we can replace it by an equation of the form:

$$\Sigma m V^2/2 + \Psi_1 (r, r', r''\ldots) = C \ldots\ldots\ldots\ldots\ldots\ldots (b),$$

où Ψ_1 est une nouvelle fonction dont il importe peu que les variables r, r', r'' &c. soient ou ne soient pas, en partie, dépendantes les unes des autres,...r, r', r''...étant les distances des molécules du système tant entre elles qu'avec les centres d'action *fixes* extérieurs (p. 68).

Is this change legitimate? The form (a) retains the possibility of intermolecular action being a function of *aspect*. Is this lost in (b)?—It does not appear to be so if some of the variables r are the distances from *fixed external points*. From this equation we easily deduce for any molecule m, the typical equation:

$$m\ddot{x} = \Sigma \frac{d\Psi_1}{dr} \cos (rx) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c),$$

where Σ denotes a summation with regard to all values of r.

[1] Both Green and Sir William Thomson make the potential energy of the element a function only of the *change in shape*, i.e. of the relative position of molecular centroids. I think this assumes that the internal potential energy of the molecule can only be a function of centroidal position. It may, however, be that the internal potential energy of (either the molecule or) the element is a function of the relative *motion* of (the atoms or) the elements, in which case the velocities would appear in Ψ_1, and we should obtain by the Hamiltonian process totally different equations to those of Green for elasticity. These *generalised equations of elasticity* leading to the *Dissipative Function* etc., I propose to discuss elsewhere.

The rest of the investigation now turns upon the question whether $\frac{d\Psi_1}{dr}, \frac{d\Psi_1}{dr'}, \frac{d\Psi_1}{dr''} \ldots$ are solely functions of $r, r', r'' \ldots$ respectively. If they are, then the 36 coefficients reduce to 15. If they are not, then the action of one molecule on a second can depend: (1) upon mutual aspect, (2) upon the position of other molecules. The dependence of the mutual action of each molecular pair solely on their centroidal distance is the hypothesis, as Saint-Venant remarks, upon which most writers on mechanics have based their proofs of the conservation of energy (e.g. Helmholtz). At the same time it does not seem necessary to assume it for more than the atoms, and for the molecules *aspect* may really be important.

[304.] Saint-Venant now proceeds to investigate what consequences flow from rejecting this hypothesis. He remarks that the action between two molecules will now be a function of their distances from *other* molecules, and not only of their mutual distance. It appears to me that the action does not necessarily depend solely on their distances from other molecules, but perhaps also on their distances from imaginary molecules or fixed centres, which give the *aspect* influence. Saint-Venant tries to prove in the first place that the work done in a complete cycle cannot generally be zero, if the intermolecular force is a function of more than the single intermolecular distance. It is quite true, as he observes, that if we move two molecules from a mutual distance r, where the action is R_1 and bring them again to a mutual distance r, the action R_2 need not be equal to R_1, and so the elements of work $R_1 dr$ and $-R_2 dr$ need not be equal and opposite, provided the other intermolecular distances are not the same in the two positions. It is only necessary that the positive work created by one pair of molecules, shall be exactly equal to the negative work created by the action of the remaining pairs of molecules. Is there anything improbable in this? Saint-Venant seems to think so:

Or, quelle que soit la loi imaginable à laquelle on soumette les intensités des actions entre deux molécules, et leur mode de dépendance de la *simple présence* d'autres molécules, si une juste compensation, comme celle dont nous parlons, s'observe ainsi entre deux moitiés de certains systèmes parcourant certains cycles, elle cessera de s'observer en ajoutant à ces systèmes d'autres systèmes pouvant être pris infiniment variés, et en ajoutant aux parcours d'autres parcours quelconques arbitrairement choisis.

La nullité du travail total produit par un cycle ne peut donc être générale qu'autant qu'elle a lieu *pour chaque action individuelle*; ce qui oblige à admettre que la force que nous avons appelée R soit fonction *de la seule distance* que nous avons appelée r (p. 71).

I do not understand the argument which follows the words: *elle cessera de s'observer en ajoutant.* Suppose the molecules represented by electro-magnets then the total action during any motion of one such magnet A on another B would depend not only on the initial and final *relative* positions of A and B, but owing to the induced currents on the paths and positions of A and B with regard to the other bodies in the field. It seems to me that Saint-Venant's argument would compel us to assert that by introducing other magnets into the field or by moving them about in a proper manner, we could obtain perpetual motion.

[305.] Saint-Venant's second argument is of the following kind (see his § 9). If the intermolecular force depends on more than the particular centroidal distance, then the distances between astral molecules will affect the action between terrestrial. Here to start with, we have somewhat of an assumption; the action of A upon B may depend on the distance of both from C and D but not necessarily on the distance of C from D. For example such might be the case when we treat of *aspect* influence, as given by means of fixed centres having reference only to A and B. Saint-Venant continues: the influence of an astral intermolecular distance on a terrestrial must be absolutely insensible, for even when we are dealing with a small portion of terrestrial matter, the action of its molecules is sensibly independent of the state of other matter even at a visible distance.

Hence the form of Ψ_1, $(r, r', r''\ldots)$ ought to be such that for any small system $d\Psi_1/dr$ depends sensibly only on the molecules in the immediate neighbourhood of m. This condition of exclusion can be easily fulfilled for molecules at *sensible* distances by making Ψ_1 a function of the inverse powers of $r, r', r''\ldots.$ We will now cite Saint-Venant's actual words:

Mais cette ressource d'exclusion sensible est impuissante à l'égard des distances mutuelles de molécules appartenant en particulier à chacun de ces systèmes ou éléments non proches de celui dont on s'occupe.

Les distances mutuelles insensibles entre les molécules composant même chaque étoile auront une influence du même ordre sur la grandeur de $d\Psi_1/dr$, ou sur l'intensité de l'action mutuelle des deux molécules m, m' d'un corps terrestre que les petites distances des molécules qui les avoisinent dans le même corps, tant qu'on n'aura pas imposé à la forme de la fonction $\Psi_1(r, r', r''\ldots)$ une restriction ou particularisation plus grande (p. 72).

The reader will indeed find it difficult to discover a form of function in which the influence of A upon B, shall be affected by the distance between C and D, and yet shall vanish when C and D are both distant from A and B. Its discovery, however, does not seem impossible, and when we regard the ether as producing the action between A and B by its state of stress, it seems by no means improbable that the approach of C and D may affect the action of A on B.

If, however, we suppose that it is only the distances of A and B from C and D which influence the action of A on B, there is less difficulty in the matter. This case, of special importance, seems to have escaped Saint-Venant's notice. Thus let $\phi'(r)$ be a law of intermolecular action, which gives a zero action for sensibly large values of r, and a strong repulsive action for all values of r less than β, so that r is usually $>\beta$ and β/r a small quantity. Let $f(z_1, z_2, z_3,\ldots)$ be a function of the variables z_1, z_2, z_3,... which is practically independent of z_r, when z_r is small. Then the following form of Ψ_1 is suitable:

$$\Psi_1 = \Sigma\phi_{pq}(r_{pq})\{m_p m_q + \beta_{pq} f_{pq}(\beta_{pn}/r_{pn},\ \beta_{sq}/r_{sq}\ldots)\},$$

where in the variables of the function f_{pq} n and s are to take all values except p and q; finally we must sum the expression for all different values of p and q. Since $(\beta/r)^2$ is negligible, f' will not occur and thus $d\Psi_1/dr_{pq}$ will be independent of r_{ns} when n and s are both different from p and q; so that Saint-Venant's objection falls to the ground.

But we are not even compelled to suppose the action of A, B independent of the position of C, D. Let us take q_1, q_2, q_3...as either aspect or internal position coordinates of the molecules, for the purposes of illustration one for each molecule will suffice. Then it seems extremely probable that the potential energy of the system,—as a result of the stress in the ether—involves the generalized velocities $\dot{q}_1$, $\dot{q}_2$, $\dot{q}_3$, etc., so that we must write for Ψ_1 a function of $\dot{q}_1, \dot{q}_2, \dot{q}_3 \ldots r, r', r'' \ldots$ In this case our equation will be of the form:

$$\frac{\Sigma m V^2}{2} + \frac{\Sigma \alpha \dot{q}^2}{2} + \frac{\Sigma \gamma_{sn} \dot{q}_s \dot{q}_n}{2} + \Psi_1 (\dot{q}_1, \dot{q}_2, \dot{q}_3, \ldots r, r', r'' \ldots) = \text{const.} \ldots (d),$$

where α and γ are certain constants. We should have to apply the general dynamical equations to determine the V's and $\dot{q}$'s. Thus the intermolecular force between m_1 and m_2 might be a function of q_3, which in its turn might be found from the dynamical equations as a function of r' and r'', etc., distances, let us say, between m_3, m_4 etc., while r', r'' would have no *direct* influence on the action between m_1 and m_2: see Arts. 931*, 1529*.

The point is of very great physical interest, as it really concerns the *direct* application of the Second Law of Motion to the ultimate particles of bodies. Can we or can we not superpose the action of C on A to that of B on A, or does the action of C on A, affect that of B on A? See the footnotes to our pp. 183 and 185.

[306.] The strong points of the rari-constant argument seem to me to lie in: (i) the *probable* insignificance of the indirect action of C as compared with the direct action of A on B; (ii) the insufficiency of most of the experiments yet brought to bear against rari-constancy.

Be this as it may, I still feel it impossible to accept the following statements of Saint-Venant as satisfactory:

> j'affirme hardiment, et tout le monde, j'en suis convaincu, pensera comme moi, qu'il faudra absolument adopter la forme ou la particularisation indiquée ci-dessus:
>
> $$\Psi_1 (r, r', r'' \ldots) = f(r) + f_1(r') + f_2(r'') + \ldots\ldots$$
>
> ...Elle fait revenir à l'adoption, comme voulue ainsi par l'expérience même, de la loi des actions *fonctions des seules distances où elles s'exercent*, et non des autres distances; loi que le simple bon sens, aidé d'une observation générale des faits, a fait accepter pendant plus d'un siècle et demi. Et je suis convaincu que Green lui-même y croyait sans s'en rendre compte. Je ne peux, en effet, interpréter d'une autre manière cet instinct de physicien et de géomètre, ce sentiment "que les forces, dans l'univers, sont disposées *de manière à faire, du mouvement perpétuel, une naturelle impossibilité.*" Green, sans aucun doute, refusait ainsi, *à chaque action moléculaire mutuelle en particulier*, la possibilité contraire...(pp. 72 and 73).

I doubt whether Green had thoroughly seen the important physical consequences which flow from multi-constancy, but I do not see why he should have objected to two molecules having done work on their return to the same distance at a different point of the field. In § 12 (pp. 74–5) Saint-Venant recognises a distinction between atomic actions and their resultant, or molecular action. At the same time, however, he holds that if the latter be indeed a function of *aspect*, it will not produce on the principle of averages any great inequality in the coefficients of type $|xxyy|$ and $|xyxy|$. Notwithstanding these rari-constant views, he wisely adopts in the *Clebsch* for the equal coefficients of rari-constancy letters distinguished by a dash.

[307.] On pp. 75—84 (§§ 13—16) Saint-Venant reproduces the results of the memoirs of 1863 and 1878, or of the *Leçons de Navier*, p. 808 *et seq.*: see our Arts. 151, and 198 (*e*). The results given in § 15 are precisely those obtained by Neumann in 1834: see our Art. 796*. In the notation of our work, if a, b, c are the direct-stretch, d, e, f the direct-slide and d', e', f' the cross-stretch coefficients, for a material with three planes of elastic symmetry, then:

$$\frac{(bc-d'^2)}{1/E_x}=\frac{(ca-e'^2)}{1/E_y}=\frac{ab-f'^2}{1/E_z}=\frac{ad'-e'f'}{1/F_x}=\frac{be'-f'd'}{1/F_y}$$
$$=\frac{cf'-d'e'}{1/F_z}=\Delta=\begin{vmatrix} a & e' & f' \\ e' & b & d' \\ f' & d' & c \end{vmatrix}.$$

Further as a typical strain-stress equation we have:

$$s_x=\widehat{xx}/E_x-\widehat{yy}/F_z-\widehat{zz}/F_y,$$

so that $1/E_x$, $-1/F_z$, $-1/F_y$ etc., are Rankine's *thlipsinomic* coefficients: see our Chapter XI.

In addition we have for the stretch-squeeze ratios equations of the type:

$$\eta_{yz}/E_y=\eta_{zy}/E_z=1/F_x.$$

[308.] In § 17 Saint-Venant deals with *amorphic* bodies, or those for which the following relations hold:

$$2d+d'=\sqrt{bc},\quad 2e+e'=\sqrt{ca},\quad 2f+f'=\sqrt{ab}\ldots\ldots\text{(i)}.$$

If the quantities $\frac{1}{2}(\sqrt{b}-\sqrt{c})^2$, $\frac{1}{2}(\sqrt{c}-\sqrt{a})^2$, $\frac{1}{2}(\sqrt{a}-\sqrt{b})^2$ are small we may write these relations:

$$2d+d'=\frac{b+c}{2},\quad 2e+e'=\frac{c+a}{2},\quad 2f+f'=\frac{a+b}{2}\ldots\text{(ii)}.$$

See the memoirs of 1863 and 1868; or our Arts. 139, 142—4 and 281.

Saint-Venant holds that for a feeble degree of aeolotropy produced by permanent compressions as, for example, in drawn or rolled metal and in some kinds of stone the relations (i) or (ii) suffice. For wood however some other conditions must hold. For let us suppose:

(*a*) The relations (ii) to hold with equal transverse elasticity (or $a=b$) and rari-constancy then:

$$3f=a, \quad 6e=c+a \text{ and } c=6e-3f.$$

We easily find from the formulae of Art. 307, that:

$$\eta_{zx}=\tfrac{1}{4}e/f, \quad E_z/E_x=\tfrac{1}{8}\left(18\, e/f-\frac{e^2}{f^2}-9\right),$$

whence $$\eta_{zx}=9/4-\tfrac{1}{2}\sqrt{18-2\, E_z/E_x},$$

or, in order that the stretch-squeeze ratio be real we must have $E_z/E_x<9$.

This result is contradicted by Hagen's experiments (see our Art. 1229*). Hagen found:

$$E_z/E_x=\begin{cases} 15 & \text{for oak,} \\ 22\cdot 5 & \text{for beech,} \\ 48 & \text{for pine,} \\ 83 & \text{for fir.}\end{cases}$$

(*b*) The relations (i) to hold together with $a=b$, $d=e=d'=e'$.

It follows that $\eta_{zx}=\frac{1}{4}\, e/f$, $E_z/E_x=e^2/f^2$,

whence $$\eta_{zx}=\tfrac{1}{4}\sqrt{E_z/E_x}, \quad E_z/G=\tfrac{5}{2}\sqrt{E_z/E_x}.$$

These expressions are never imaginary and give reasonable values for η_{zx} up to $E_z/E_x=4$. After this η_{zx} begins to take unsuitable values till for $E_z/E_x=80$, we have η_{zx} so large as $2\cdot 236$.

Clebsch (p. 8, § 2) and at one time Saint-Venant (see our Art. 169 (*d*)) had held that η must necessarily be $<\frac{1}{2}$. This error the latter had recognised in the *Appendice complémentaire* to the *Leçons de Navier*, and he now adds:

Cette opinion n'est fondée sur aucun fait; il ne l'exprime même que *pour les corps isotropes*, et quelques expériences de Wertheim ont montré qu'aux approches de la rupture d'une tige métallique, c'est-à-dire au moment où sa matière est arrivée à un état très fibreux, comparable à celui des bois, une extension de plus diminue le volume; en sorte que, sans pouvoir aller jusqu'à $\eta=2\cdot 236$, rien n'empêcherait de porter η jusqu'à 1 pour les bois tendres (p. 89).

Saint-Venant now seeks some correction of the amorphic formulae (i) which will give better results than this for η_{zx} when E_z/E_x is large.

[309.] He first proceeds on pp. 89—95 to determine Neumann's stretch-modulus quartic; he obtains it in the form:

$$\frac{1}{E_r}=\frac{c_x^4}{E_x}+\frac{c_y^4}{E_y}+\frac{c_z^4}{E_z}+2\frac{c_y^2c_z^2}{F_1}+2\frac{c_z^2c_x^2}{F_2}+2\frac{c_x^2c_y^2}{F_3}\ \ldots\ldots\ldots \text{(iii)},$$

where c_x, c_y, c_z are the direction-cosines of the line r, and

$$\left.\begin{aligned} 1/F_1 &= 1/(2d) - 1/F_x \\ 1/F_2 &= 1/(2e) - 1/F_y \\ 1/F_3 &= 1/(2f) - 1/F_z \end{aligned}\right\}.$$

This agrees with Neumann's result (see our Art. 799*) if we note that his N_a, N_c, M_b, M_c, P_a, P_b are really *cross*-stretches and therefore of negative sign[1].

By taking $\quad x = c_x \sqrt[4]{E_r}, \quad y = c_y \sqrt[4]{E_r}, \quad z = c_z \sqrt[4]{E_r},$

we have a surface of the fourth order, whose ray measures $\sqrt[4]{E_r}$ in the same direction.

[310.] In § 21 (pp. 95—8), Saint-Venant enters upon a lengthy calculation of the maxima and minima values of E for different directions. If three relations of the type

$$F_3 = \sqrt{E_x E_y} \quad \text{..............................(iv)}$$

hold, then (iii) reduces to an ellipsoid and we have the ellipsoidal distribution of elasticity. This gives only three maxima and minima for E_r. Saint-Venant seeks conditions under which there shall only be three maxima for the surface (iii) when the relations of type (iv) are not fulfilled; in other words, he seeks when there will be, as he expresses it, a *variation simple et graduelle des élasticités*.

The conditions are

(1) that F_1 lie between E_y and E_z, and two others of the same type;

(2) that the three expressions whose type is

$$(1/E_y - 1/F_3)(1/E_z - 1/F_2) + (1/F_3 - 1/F_1)(1/F_1 - 1/F_2),$$

shall not all be of the same sign.

[311.] In § 22 Saint-Venant shows that the three ellipsoidal conditions of type $F_3 = \sqrt{E_x E_y}$ are identical with the three of type $2f + f' = \sqrt{ab}$, provided either $\frac{d}{d'} = \frac{e}{e'} = \frac{f}{f'}$, or again that rari-constancy is assumed to hold.

[312.] He next seeks for some non-ellipsoidal distribution which shall satisfy the conditions for *variation simple* of our Art. 310. He takes as a probable solution: (1) rari-constancy, and (2) two of the ellipsoidal relations, i.e. he writes:

$$a = 3ef/d, \quad b = 3fd/e,$$

and searches for a value of n, where

$$c = 3de/(fn),$$

[1] Unfortunately the wrong signs are given in Art. 796* to all the quantities M, N, P. If these are corrected, a negative sign must be inserted in the second table of Art. 795* before the $1/F$'s. The value of $1/E_r$ in Art. 799* is then accurate. I regret that this slip of Neumann's escaped me.

which shall satisfy those conditions. After some rather complex analysis the necessary and sufficient conditions are found to reduce to

$$\frac{9+12\,E_z/E_x-\sqrt{81+144\,E_z/E_x}}{2+4\,E_z/E_x} > n > \frac{12+9\,E_z/E_x-\sqrt{144\,E_z/E_x+81\,(E_z/E_x)^2}}{4+2\,E_z/E_x},$$

where we suppose $E_z > E_x > E_y$.

Saint-Venant then gives a table of the limiting values of n and of $\eta_{zx}=\frac{1}{4}\frac{d}{f}=\sqrt{\frac{n}{18-2n}\cdot\frac{E_z}{E_x}}$, for various values of E_z/E_x from 1 to 80 and also for ∞.

The values of η_{zx} are now found to be possible, provided a suitable value of n be chosen. What shall this be?

[313.] The empirical formula for n

$$1/n = 1 + \frac{1}{\gamma}\,(E_z/E_x - 1)\text{..........................(v)},$$

is suggested on p. 104. On p. 105 Saint-Venant tabulates the values of n and η_{zx} for the parameter E_z/E_x (= 1 to 80) when γ has the numerical values 9 and 22·22. These values for γ are chosen because, for $E_z/E_x = 80$, they give respectively η_{zx} = about 2/3 and 1. The Table also contains the corresponding values of E_z/e (= E/μ with transverse isotropy). These values vary on the first supposition ($\gamma = 9$) from 2·5 to 78·2, and on the second ($\gamma = 22·22$) from 2·5 to 52·67. The ratio E/μ can thus be very great, but for E_z/E_x very great, this does not seem at all improbable, at least we have at present no experiments to contradict it. As for the value of γ we need not confine it to 9 or 22·22, but in general we may take it from 7 or 8 to 30 (p. 108). *Nous pensons qu'on ne courra guère risque de se tromper en faisant* $\gamma = 16$ (p. 108).

As Saint-Venant observes there is a great need of new experiments to determine E_z and E_x (by flexure), μ (by torsion) and η ($= -s_x/s_z$, by delicate measurements of the transverse dimensions of bars under traction).

[314.] In default of experiment we may finally adopt as formulae most probably sufficient for elastic problems concerning amorphic aeolotropic solids, such as stone, wood, and the metals employed in the construction of bridges and machines:

$$\left.\begin{aligned}
\widehat{xx} &= \frac{3ef}{d}\,s_x + fs_y + es_z, & \widehat{yz} &= d\sigma_{yz}\\
\widehat{yy} &= fs_x + \frac{3fd}{e}\,s_y + ds_z, & \widehat{zx} &= e\sigma_{zx}\\
\widehat{zz} &= es_x + ds_y + \frac{3de}{nf}\,s_z, & \widehat{xy} &= f\sigma_{xy}
\end{aligned}\right\}\text{.............(vi)},$$

where at each point yz, zx, xy are three rectangular planes of elastic symmetry and z is the direction of greatest elastic resistance, generally 'longitudinal,' that is, in the case of wood in the direction of the fibre, or in a metal bar in the direction of the prismatic axis. In a metal plate it will be perpendicular generally to the plane of the plate.

The quantity n is to be determined by Equation (v), where γ may be taken $= 16$, when we have no further experimental data to suggest a better value.

Since $E_z = \frac{de}{f}\frac{6-n}{2n}$, it is obvious that three torsional experiments and one tractional experiment will give d, e, f and n, or all the constants of the stress-strain relations (vi).

Indeed we may write the value of $\widehat{zz}$

$$\widehat{zz} = es_x + ds_y + \left(E_z + \frac{1}{2}\frac{de}{f}\right)s_z,$$

and so get rid of n altogether.

For the case of transverse isotropy, if $E_z = E$, $d = e = \mu$, $f = \mu'$, we have:

$$\left.\begin{array}{ll} \widehat{xx} = \mu'(3s_x + s_y) + \mu s_z & \widehat{yz} = \mu\sigma_{yz} \\ \widehat{yy} = \mu'(s_x + 3s_y) + \mu s_z & \widehat{zx} = \mu\sigma_{zx} \\ \widehat{zz} = \mu(s_x + s_y) + \left(E + \frac{\mu^2}{2\mu'}\right)s_z & \widehat{xy} = \mu'\sigma_{xy} \end{array}\right\} \ldots\ldots\ldots\ldots\text{(vii)}.$$

Here μ and E are easy to determine experimentally, but μ' far more difficult.

Saint-Venant gives the following empirical formula for μ' which he considers very probably exact enough in practice:

$$\frac{\mu'}{\mu} = 1 - \beta\left(\frac{2}{5} - \frac{\mu}{E}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(viii)}.$$

When γ of (v) is taken $= 9$, then $\beta = \frac{5}{3}$, or $\frac{\mu'}{\mu} = \frac{1}{3} + \frac{5}{3}\frac{\mu}{E}$,

" " " " $= 22{\cdot}22$, then $\beta = 2$, or $\frac{\mu'}{\mu} = \frac{1}{5} + 2\frac{\mu}{E}$.

For these values of β, the corresponding values of μ'/μ and μ/E differ by only 1/16, from those obtained from equation (v).

We have reproduced these results because they supply, although to some extent empirically, the most probable formulae yet suggested for technical materials. Such formulae have been much needed, and Saint-Venant, as usual, has been the first to recognise the wants of practice.

[315.] A note of Saint-Venant to § 22 (see pp. 142—5) deals briefly with the history of the flexure and torsion of prisms. It contributes nothing to the section on the same subject in the *Historique Abrégé*. We pass on to the longer note attached to § 28 which occupies pp. 174—90.

[316.] This note is concerned with the applicability of Saint-Venant's torsion and flexure solutions to such cases as occur in practice. The first four sections (pp. 174—7) reproduce arguments already given in the memoir on *Torsion* or the *Leçons de Navier* for the approximate elastic equivalence of statically equipollent loads: see our Arts. 8, 9 and 170. The remaining sections (§§ 5—17) seek arguments in favour of the legitimacy of the assumptions

$$\widehat{xx} = \widehat{yy} = \widehat{xy} = 0 \qquad\qquad (a),$$

taken by Saint-Venant as the basis of his solutions. In other words, is it legitimate to assume that for all practical loadings there is little or no mutual action *parallel* to the prismatic cross-section between adjacent longitudinal fibres?

After referring to the labours of Poisson and Cauchy on the subject of rods (see our Arts. 466* and 618*) as involving arbitrary assumptions only true for rods of length great as compared with the linear dimensions of the cross-section, Saint-Venant enquires whether the investigations of Kirchhoff give any better validity to the assumptions (a). He points out that Kirchhoff proves only the possibility, not the necessity of these questionable relations (p. 181): see my footnote, p. 266.

[317]. Saint-Venant next turns to Boussinesq's memoirs of 1871 and 1879: see later our discussion of that author's researches. Saint-Venant applies the method of those memoirs to the simple case of a bar of homogeneous material with three planes of elastic symmetry.

Instead of setting out from the assumptions (a) our author supposes the following conditions to hold, z being the direction of the prismatic axis:

$$\frac{d^2 s_x}{dz^2} = \frac{d^2 s_y}{dz^2} = \frac{d^2 s_z}{dz^2} = \frac{d^2 \sigma_{xy}}{dz^2} = \frac{d\sigma_{xz}}{dz} = \frac{d\sigma_{zy}}{dz} = 0 \qquad\qquad (b).$$

These are described as *fort approchées, quand elles ne sont pas rigoureuses.*

From the conditions (b) the conditions (a) are deduced by the principle of elastic work. The proof holds only for *rods*, i.e. prisms the length of which is great as compared with the linear dimensions of the cross-section; the cross-section may, however, be supposed to vary slightly, and the terminal load as well as the distribution of body force are perfectly general, provided only the body force on any element of length of the rod does not exceed the surface stresses or the loads on the terminal cross-sections of the element.

[318.] We may ask: whether the conditions (b) do not assume as much as conditions (a)? We reproduce the arguments by which Saint-Venant reaches (b). It does not seem to me that the condition $d^2s_z/dz^2 = 0$ would be true when the flexure was due to *buckling*, which in the case of a long rod does not seem excluded by the load distributions referred to: see our Art. 911*.

Prenons pour axe des z, en chaque endroit, la ligne des centres de gravité des sections transversales, et les axes des x, y, rectangulaires entre eux et à cette ligne sur une des sections. Dans une quelconque des portions dont nous parlons, que nous appelons longues parce qu'elles sont supposées l'être beaucoup par rapport aux dimensions transversales, il est facile de reconnaître que les composantes de tension et les dilatations ou glissements s, σ, varient d'une manière incomparablement moins rapide dans le sens longitudinal z que dans les sens x et y; de sorte que, si nous exceptons de petites portions de tige avoisinant les extrémités, où se trouvent les points d'application des forces locales ou discontinues, les dérivées de ces déformations s, σ, par rapport à z seront, de nécessité, considérablement moindres que ce que sont ou peuvent être leurs dérivées par rapport à x et à y. En effet, pour σ_{zx}, par exemple, $d\sigma_{zx}/dz$ sera de l'ordre de grandeur du quotient, par la longueur de la tige ou de la longue portion de la tige considérée, de cette déformation σ_{zx}, ou de la différence des valeurs qu'elle a aux extrémités; tandis que $d\sigma_{zx}/dx$ pourra être de l'ordre de grandeur du quotient de σ_{zx} par la demi-épaisseur, qui n'est, disons-nous, qu'une fort petite fraction de la longueur. Autrement dit, si pour fixer les idées nous divisons la tige, par la pensée, en tronçons dont la longueur soit de l'ordre de grandeur de la dimension transversale moyenne, les s, σ auront des valeurs extrêmement peu différentes en deux points homologues des bases de chaque tronçon, tandis qu'ils pourront avoir, du centre au périmètre des sections, des différences de valeur aussi considérables que d'une extrémité à l'autre de la tige. Nous pouvons donc comme approximation, déterminer la loi de variation des déformations s, σ, transversalement, ou en fonction de x et y, *comme si leurs dérivées par rapport à z étaient nulles.* Cette hypothèse, ou ce point de départ, n'est que comme une traduction analytique de l'énoncé de la question même qui nous occupe, et qui est de déterminer ce qui se passe dans une tige allongée et très mince sollicitée de la manière continue que nous venons de supposer (pp. 184—5).

This reasoning does not appear to me wholly satisfactory, and can at best only apply to *rods* and not the prisms of Saint-Venant's problems. It may, however, still be the method

la meilleure et la plus complète qui en ait été théoriquement donnée (p. 190).

Perhaps on the whole the appeal to experiment referred to in our Arts. 8—10 is more satisfactory.

[319.] In a note pp. 195—7 Saint-Venant proves for the case of flexure the results

$$\int \widehat{zx}\, d\omega = \frac{d}{dz}\int \widehat{zz}\, x d\omega\ ; \quad \int \widehat{zy}\, d\omega = \frac{d}{dz}\int \widehat{zz}\, y d\omega,$$

where z is an axis in direction of the prismatic axis, and x, y are any rectangular axes in the cross-section of which $d\omega$ is an element of area. These formulae express analytically :

ce théorème connu et très utile, que l'*effort tranchant*, pour une section quelconque, ou la force tangentielle totale dans une direction transversale *aussi quelconque*, est égale à la dérivée, par rapport à la coordonnée longitudinale, du *moment de flexion* autour d'une droite tracée sur la section perpendiculairement à cette direction (p. 197).

See pp. 389—9 etc. of the *Leçons de Navier*.

[320.] The following *Note*, pp. 210—20, reproduces only portions of the great or the subsidiary memoirs on *Torsion*: see our Arts. 1, 285 and 291 ; and the *Note*, pp. 240—2, some results from Chapter XI. of the *Torsion*: see our Art. 49.

The *Note finale du* § 37 (pp. 252—82) corrects Clebsch's erroneous assumption of a stress-limit by the proper stretch-conditions. Its contents are extracted from the memoir on *Torsion* and the *Leçons de Navier*: see our Arts. 5, (*b*)—(*f*), and 180.

[321.] We may refer to one or two points in this last *Note* :

(*a*) Saint-Venant takes two simple cases for an isotropic material and compares the stress and stretch-conditions for safe loading. First take the case when only the stresses $\widehat{xx}$, $\widehat{xz}$, $\widehat{xy}$ have values differing from zero, we easily find from the equation of our Art. 53, Case (i), that we must have

$$T_0 = \text{or} > (1-\eta)\,\widehat{xx}/2 + (1+\eta)\sqrt{\widehat{xx}^2/4 + \widehat{xy}^2 + \widehat{xz}^2},$$

while Clebsch obtains from the stress condition

$$T_0 = \text{or} > \widehat{xx}/2 + \sqrt{\widehat{xx}^2/4 + \widehat{xy}^2 + \widehat{xz}^2}.$$

In the second case suppose the traction $\widehat{xx}$ zero, then we have:

from the stretch condition,

$$\sqrt{\widehat{xy}^2 + \widehat{xz}^2} = \text{or} < T_0/(1+\eta),$$

from the stress condition,

$$\sqrt{\widehat{xy}^2 + \widehat{xz}^2} = \text{or} < T_0.$$

When the shears are zero the conditions agree. As a rule *safety is on the side of the stretch-condition.*

(*b*) Some remarks confirmatory of Poncelet's theory of rupture (better *elastic failure*) under compression by transverse stretch are given on p. 270 and may be cited[1]. The theory leads, as we have seen, in isotropic material to the relation $T_0/T_0' = 1/\eta$: see our Arts. 164, and 175.

1°. Les petits prismes de pierre dure, lors de leur écrasement, se séparent d'abord en aiguilles verticales, ce qui prouve bien une extension dans le sens transversal.

2°. Lors de l'écrasement des bois par compression dans le sens de leurs fibres, celles-ci se séparent d'abord, et ensuite ploient sans résistance.

3°. Les petits cylindres de fonte douce ou malléables, écrasés, se gercent sur les bords de manière à former une rosette, ce qui prouve qu'il y a eu, tout autour, rupture par dilatation transversale vers la circonférence.

4°. Dans beaucoup d'expériences de rupture de pièces de fonte par flexion, il s'est détaché latéralement une sorte de coin du côté devenu concave ou comprimé.

5°. La puissante machine de M. Blanchard, de Boston, à courber les pièces de bois, contenues de manière à ne pouvoir se dilater du côté convexe ni se boursoufler latéralement du côté concave, comprime violemment ce dernier côté sans le désorganiser aucunement.

6°. Le rapport des coefficients T_1' et T_1 de rupture immédiate par écrasement et par traction, ou des forces capables de produire, pour une base $=1$, ces deux sortes d'effet, a été trouvé le plus souvent, pour la fonte, entre 4 : 1 et 6 : 1; et il devait, en effet, excéder $1/\eta$ qui est 4 pour les corps isotropes. Car lorsqu'on opère la compression d'un prisme court, entre deux plans durs où ses bases s'appliquent, celles-ci sont empêchées de se dilater, en sorte que le renflement latéral n'acquiert toute sa grandeur que vers le milieu de la hauteur du prisme.

Saint-Venant remarks that the limits T_0, T_0' must be based directly on experiment; but experiment only gives such limits as

[1] Professor A. B. W. Kennedy has kindly made some experiments for me on lateral stretch in which *three* short cast-iron prisms placed end to end were subjected to contractive load. The load terminals of the outer prisms were found to have expanded somewhat, but not to the same extent as their other terminal sections or those of the mid-prism. Rupture took place by portions of the end prisms shearing off. The mid-prism was then cut open longitudinally and acid applied to the face, the openings thus brought to sight were more or less longitudinal, but not very definite. Indeed the condition marked rather a plastic than a ruptural change.

the T_1 and T_1' of immediate rupture. A constant ratio between T_1 and T_0 is usually assumed:

rapport qu'on prend généralement d'un dixième en France, d'après l'exemple des colonnes légères d'une ancienne église d'Angers, mais que des ingénieurs anglais portent à un sixième (p. 271).

(*c*) We may note that Saint-Venant on pp. 274—5 in repeating case 3° of Art. 122 of the *Torsion*: see our Art. 53, Case (iii), now replaces the $s_y/\bar{s}_y$ and $s_z/\bar{s}_z$ of the notation of that article by *their mean*, so that he appears to have been dissatisfied with the value adopted in the memoir. He does not, however, work out the value of η_2 of our Art. 53, Case (ii) ($=\eta_1$ of his notation).

(*d*) A very good example of Saint-Venant's fail-point method is given on pp. 279—82 (§ 17). It brings out well the influence which want of isotropy and slide have on the condition for safety.

Let us take the case of a beam of length l, of cross-section ω, and of transverse elastic isotropy denoted by E, μ and η. Suppose it built-in at one end and loaded with P at the other, or of length $2l$ with a load $2P$ in the centre. Then if κ be the swing-radius of the section about the neutral axis and h the distance from that axis of the farthest 'fibre', we see that the fail-point will be at the built-in section which remains plane. Here the maximum stretch and the uniform slide are given by:

$$s = Plh/(E\omega\kappa^2), \quad \sigma = P/(\mu\omega).$$

Whence the condition of our Art. 53, (i), becomes with slightly modified notation:

$$T_0 = \text{or} > \frac{1-\eta}{2}\frac{Plh}{\omega\kappa^2} + \sqrt{\left(\frac{1+\eta}{2}\right)^2\left(\frac{Plh}{\omega\kappa^2}\right)^2 + \left(\frac{P}{\omega}\right)^2\left(\frac{T_0}{S_0}\right)^2},$$

$$= \text{or} > \frac{Plh}{\omega\kappa^2}\left\{\frac{1-\eta}{2} + \frac{1+\eta}{2}\sqrt{1 + \left(\frac{E}{2(1+\eta)\mu}\right)^2\left(\frac{2\kappa^2}{lh}\right)^2}\right\} \ldots\text{(i)},$$

since $S_0/\mu = 2T_0/E$ by our Art. 5, (*d*).

In the case of the rectangular cross section $b \times c$, with c parallel to the load-plane, we have $\kappa^2 = c^2/12$, $\omega = bc$, $h = c/2$ and the condition becomes:

$$T_0 = \text{or} > \frac{6Pl}{bc^2}\left\{\frac{1-\eta}{2} + \frac{1+\eta}{2}\sqrt{1 + \left(\frac{E}{2(1+\eta)\mu}\right)^2\left(\frac{c}{3l}\right)^2}\right\} \ldots\text{(ii)},$$

or,

$$T_0 = \text{or} > \frac{6Pl}{bc^2}\left\{1 + \frac{1}{1+\eta}\left(\frac{E}{12\mu}\right)^2\left(\frac{c}{l}\right)^2\right\} \ldots\ldots\ldots\ldots \text{(iii)},$$

if the second term under the radical is, as usual, small.

Saint-Venant now introduces the following suggestive table determined by the method of our Arts. 312–4, z being the direction of the prismatic axis:

For									
	$E_z/E_x=1$	1·5	2	5	10	15	20	40	80
	$\eta=\cdot25$	·30	·34	·48	·60	·65	·70	·80	·85
	$E/\mu=2\cdot5$	3	3·8	6·6	11	15	18	36	66
Whence	$\left\{\dfrac{E}{2(1+\eta)\mu}\right\}^2=1$	1·331	2·011	4·973	11·816	20·657	28·026	100	318·27
	$\dfrac{1}{1+\eta}\left(\dfrac{E}{12\mu}\right)^2=\cdot035$	·048	·075	·204	·525	·947	1·323	5	16·35

Now the value given by the old theory was

$$T_0 = \text{or} > \frac{6Pl}{bc^2}.$$

Whence we see that for certain kinds of wood when $E_z/E_x=80$, for short lengths only double of the diameter, the value of P obtained from the old theory may be *double* what is given by the true theory. Indeed these numbers are most suggestive and valuable for the problem of flexure.

[322.] To Clebsch's §§ 39—46 dealing with thick plates, Saint-Venant contributes two long notes. The first occupies pp. 337—367 and treats of various *rigorous* solutions for the bending of plates by an analysis which for simplicity compares favorably with that of Clebsch. Indeed we have here the most complete account yet given of the bending of *thick circular* plates, and as usual Saint-Venant keeps in view practical cases. The results are all given in terms of the 5-constant formulae (see our Art. 282, (2)), or for a material with transverse isotropy on the multi-constant hypothesis. Many of the results are new and the method seems to me novel; some of the formulae are apparently due to Saint-Venant's old pupil M. Boussinesq, who investigated the matter at his request. The following problems are investigated: (i) case of simple cylindrical flexure; (ii) case of combined cylindrical flexure; (iii) cases of shearing load on the lateral sides of a plate; (iv) general case of circular plate with a great variety of special cases of contour and load conditions.

[323.] *Case of simple cylindrical flexure.* Let 2ϵ be the thickness of the plate; let the axis of z be perpendicular to the initial plane of the plate; and let those of x, y, lie in the plane of the plate. Suppose the plate to be infinite in the direction of y, but of any length in the direction of x. Then consider the following shifts, where ρ is a constant:

$$u = -\frac{xz}{\rho}, \quad v = 0, \quad w = \frac{1}{2\rho}\left(x^2 + \frac{d'}{c}z^2\right) \ldots\ldots\ldots\ldots\ldots (\text{i}).$$

We find at once :

$$\left.\begin{array}{l} s_x = -z/\rho, \quad s_y = 0, \quad s_z = \dfrac{d'}{c}\dfrac{z}{\rho}, \\ \sigma_{yz} = \sigma_{zx} = \sigma_{xy} = 0 \end{array}\right\} \text{.....................(ii).}$$

Substitute in the formulae of Art. 117, (*a*) and (*b*), and we have :

$$\left.\begin{array}{l} \widehat{yz} = \widehat{zx} = \widehat{xy} = 0\,; \quad \widehat{zz} = 0\,; \\ \widehat{xx} = \left\{\dfrac{d'^2}{c} - (2f+f')\right\}\dfrac{z}{\rho}; \quad \widehat{yy} = \left\{\dfrac{d'^2}{c} - f'\right\}\dfrac{z}{\rho} \end{array}\right\} \text{............(iii).}$$

Here the quantity $2f+f'-d'^2/c$ corresponds for the case of plates to the stretch-modulus in the simple flexure of a bar. We shall denote it by H, where in the case of isotropy, $H = \dfrac{4\mu(\mu+\lambda)}{2\mu+\lambda}$.

We easily see that (iii) satisfy the body-stress equations.

The load reduces to

$$\widehat{xx} = -\frac{Hz}{\rho}$$

over the sides perpendicular to x, and we can see that this gives a couple round the axis of y for each element $2\epsilon\delta y$ of the side $= M_y\delta y$, where

$$M_y = \int_{-\epsilon}^{+\epsilon} \widehat{xx} \,.\, z\,dz = -2H\epsilon^3/(3\rho).$$

We can cut away a portion of the plate by planes perpendicular to the axis of y if we impose a load at each point of the new sides given by

$$\widehat{yy} = -(H-2f)\,z/\rho.$$

Obviously $1/\rho$ must be very small, and the plate then takes a cylindrical curvature of radius ρ.

[324.] *Case of two combined cylindrical flexures.* In § 3 Saint-Venant first combines two solutions such as that of our Art. 323, the value of ρ being the same for both. He transfers to cylindrical coordinates r, ϕ, and thus obtains with the notation of our p. 79 the results :

$$\left.\begin{array}{l} u = -rz/\rho, \quad v = 0, \quad w = \left(r^2/2 + \dfrac{d'}{c}z^2\right)\Big/\rho \\ \widehat{zz} = 0, \quad \widehat{rr} = \widehat{\phi\phi} = -2(H-f)\,z/\rho \end{array}\right\} \text{............(iv).}$$

This is the case of *spherical* curvature. The proper distribution of side load must be obtained by compounding $\widehat{rr}$ and $\widehat{\phi\phi}$, the shears being all zero. The corresponding total couples are

$$M_r = M_\phi = -\frac{4\epsilon^3}{3}\frac{H-f}{\rho} \text{.....................(v).}$$

Saint-Venant remarks:

Ils ont un intérêt pratique bien que l'application, au contour, de forces normales distribuées comme l'exigent les expressions ci-dessus $\widehat{rr}$, $\widehat{\phi\phi}$ soit irréalisable; car si à leur place, il y a [see our Arts. 8 and 170] tout auprès des bords d'une plaque mince, d'autres forces appliquées par exemple sur les faces supérieure et inférieure de manière à n'avoir pas de résultante et à produire des couples dont les moments fléchissants aient par unité de longueur la valeur (v), la plaque soit rectangle, soit circulaire, éprouvera *très approximativement* la déformation sphérique indiquée, *partout* sauf de très petites zones auprès des bords, par les raisons que nous avons données précédemment en traitant des tiges (p. 343).

[325.] The second case of combined flexure given by Saint-Venant is obtained by taking for u and v two expressions like that given for simple cylindrical flexure, with ρ different; we have at once:

$$u = -xz/\rho, \quad v = -yz/\rho', \quad w = x^2/(2\rho) + y^2/(2\rho') + (1/\rho + 1/\rho')\frac{d'}{2c}z^2,$$

$$\widehat{zz} = \widehat{yz} = \widehat{zx} = \widehat{xy} = 0, \quad \widehat{xx} = -\left(\frac{H}{\rho} + \frac{H-2f}{\rho'}\right)z, \quad \widehat{yy} = -\left(\frac{H-2f}{\rho} + \frac{H}{\rho'}\right)z.$$

Here the curvature is elliptic or hyperbolic according as ρ and ρ' are of the same or different signs. If $\rho = -\rho'$:

le feuillet moyen devient une de ces surfaces à courbures principales égales et opposées, appelées *anticlastiques* par MM. Thomson et Tait dans leur grand *A Treatise of Natural Philosophy*, de 1867, dont un seul exemplaire existe en France, et dont il n'a encore été réédité que le premier volume (p. 344).

As is well-known the distinguished scientists gave up in their second edition the idea of proceeding further. How Saint-Venant formed his conclusion as to the existence of a *seul exemplaire,* we cannot say, as with few exceptions French scientists refrain when citing from giving exact references to the sources of their information.

[326.] *Plates subjected laterally to shearing load.* Saint-Venant first takes the case of a rectangular plate *infinitely* long in the direction of y but bounded in the direction of x by the planes $x = \pm a$.

Let $P\delta y$ be the total shearing-load parallel to z, on the strip $2\epsilon dy$, then we have for a section of the plate by a plane at distance x from the origin:

$$\int_{-\epsilon}^{+\epsilon} \widehat{zx}\, dz = P; \quad \int_{-\epsilon}^{+\epsilon} \widehat{xx}\, z dz = P(a - x).$$

Boussinesq had found at Saint-Venant's request the following suitable values for the shifts:

$$\left.\begin{aligned} u &= \frac{3P}{2H\epsilon^3}\left[-z\left(ax - \frac{x^2}{2}\right) + \frac{d'}{c}\frac{z^3}{6}\right] + \frac{3P}{4e}\left(\frac{z}{\epsilon} - \frac{z^3}{3\epsilon^3}\right), \\ v &= 0, \quad u = \frac{3P}{2H\epsilon^3}\left[\frac{ax^2}{2} - \frac{x^3}{6} + \frac{d'}{c}(a - x)\frac{z^2}{2}\right]. \end{aligned}\right\} \ldots\ldots\text{(vi)}.$$

Hence we find for the stresses:

$$\left.\begin{aligned} &\widehat{zz} = \widehat{yz} = \widehat{xy} = 0, \\ &\widehat{xx} = -\frac{3Pz}{2\epsilon^3}(a-x), \quad \widehat{xz} = \frac{3P}{4\epsilon}\left(1-\frac{z^2}{\epsilon^2}\right), \\ &\widehat{yy} = -\frac{H-2f}{H}\,\frac{3Pz}{2\epsilon^3}(a-x) \end{aligned}\right\} \text{..........(vii).}$$

The deflection of the central plane is given by the cubical parabola

$$w_0 = \frac{Pa^3}{2H\epsilon^3}\left(\frac{3}{2}\frac{x^2}{a^2} - \frac{1}{2}\frac{x^3}{a^3}\right) \text{....................(viii).}$$

This agrees with the case of a rod of length $2a$ and depth 2ϵ, terminally supported and loaded with $2P$ at the centre if the plate-modulus H be replaced by the stretch-modulus E.

[327.] We can cut out a definite portion of the plate by planes perpendicular to y, if we impose the tractive loads given by $\widehat{yy}$ of equations (vii).

Suppose we try to combine two sets of solutions such as (vi) of the previous Article, giving the plate now a flexure parallel to y. Then we find, if Q corresponding to P, and b to a, from (viii):

$$w_0 = \frac{3}{2H\epsilon^3}\left(\frac{Pax^2 + Qby^2}{2} - \frac{Px^3 + Qy^3}{6}\right).$$

Hence although we combine this with a solution of the form given in Art. 325, we can make only the square not the cubic terms in x and y vanish. In other words for $x = \pm a$, together with $y =$ any value from b to $-b$, and for $y = \pm b$, together with $x =$ any value from a to $-a$, we cannot make $w_0 = 0$. Thus the contour of the mid-plane of the rectangular plate cannot be treated as *fixed*.

Le problème de la flexion de la plaque rectangulaire posée de niveau tout autour ne peut probablement recevoir que des solutions approximatives.... (p. 346).

[328.] *Problem of the thick circular plate.* This can be solved accurately for flexure whatever the thickness, if the plate be *symmetrically* loaded in all directions round its axis of figure by forces applied to its cylindrical boundary. Just as in the case of torsion or flexure, these forces will be supposed distributed in a definite manner, but the resultant shearing force and couple about the tangent to the contour of the mid plane will be arbitrary. In practical applications we must appeal to the principle of the elastic equivalence of statically equipollent load-systems: see our Art. 8. We shall suppose that there is no tendency to extension in the plate and that it is bounded by two coaxial cylinders of radii r_1 and r_0 ($r_1 > r_0$).

We shall find that the magnitude of the central shift can be determined for any load whatever, not necessarily symmetrical.

[329.] *The general solution.* Let 2ϵ be the thickness; P the shearing load parallel to the axis per unit of length of contour of the plate; $Q = 2\pi a P$ the total shearing load on the whole lateral area $2\pi a \times 2\epsilon$ of the plate; M_r the moment of the couple, per unit of length, on a vertical strip of the cylindrical surface of radius r about the tangent to the contour of the mid-plane; M_{r_1} will then denote the corresponding load couple on the outer bounding cylinder. We shall suppose the mid-circle of the inner cylindrical boundary fixed.

The strains are given in the footnote to our p. 79, except that on account of the symmetry we put $v = 0$, and the variation with regard to ϕ zero for all quantities. The stresses then become on the hypothesis of elastic isotropy in the plane of the plate [see Art. 117 (b)]:

$$\left.\begin{array}{ll} \widehat{rr} = (2f + f')\, u_r + f'u/r + d'w_z, & \widehat{rz} = e\,(u_z + w_r) \\ \widehat{\phi\phi} = f'u_r + (2f + f')\, u/r + d'w_z, & \widehat{r\phi} = \widehat{z\phi} = 0 \\ \widehat{zz} = d'\,(u_r + u/r) + cw_z & \end{array}\right\}\ldots\ldots\ldots\text{(i)}.$$

Further we have $M_r = \int_{-\epsilon}^{+\epsilon} \widehat{rr}\, z dz.$

The body stress-equations reduce to:

$$\frac{d\widehat{rr}}{dr} + \frac{d\widehat{rz}}{dz} + \frac{\widehat{rr} - \widehat{\phi\phi}}{r} = 0\,; \quad \frac{d\widehat{rz}}{dr} + \frac{d\widehat{zz}}{dz} + \frac{\widehat{zr}}{r} = 0 \ldots\ldots\ldots \text{(ii)}.$$

The surface or load conditions are:

$$\left.\begin{array}{l} \text{for } z = \pm\epsilon, \quad \widehat{zz} = \widehat{rz} = 0 \text{ for all values of } r, \\ \text{for } r = r_1, \quad \int_{-\epsilon}^{+\epsilon} \widehat{rr}\, dz = 0, \quad \int_{-\epsilon}^{+\epsilon} \widehat{rz}\, dz = P, \\ \qquad\qquad M_r = M_{r_1} \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}.$$

[330.] Saint-Venant's mode of solution is the following. He assumes $\widehat{rz}$ to be of the form $\frac{3P}{4\epsilon^3} \cdot \frac{f(r)}{f(r_1)} (\epsilon^2 - z^2)$, and also that, $\widehat{zz} = 0$ throughout the plate. He thus satisfies the load conditions.

These assumptions of the *semi-inverse* method were undoubtedly suggested by equations (vii) of our Art. 326.

The second body-stress equation at once gives us $f(r) = \frac{\text{constant}}{r}$;

so that
$$\widehat{rz} = \frac{3P}{4\epsilon^3} \frac{r_1}{r} (\epsilon^2 - z^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)}.$$

Straight-forward substitution, remembering $\widehat{zz} = 0$, or $w_z = -\frac{d'}{c} \frac{1}{r} \frac{d(ru)}{dr}$, leads to the following form of the first body-stress equation (ii):

$$\frac{d}{dr}\left(\frac{1}{r} \frac{d(ru)}{dr}\right) = \frac{4z}{Ir},$$

where
$$I = \frac{8H\epsilon^3}{3Pr_1}.$$

Integrating we find, if A be an arbitrary constant:

$$\frac{1}{r}\frac{d(ru)}{dr} = z\left(\frac{4}{I}\log(r/r_1) - \frac{2}{A}\right) = -\frac{c}{d'}w_z \text{ .... (v).}$$

Integrating again we have:

$$\left.\begin{aligned} u &= \frac{z}{I}\{2r\log(r/r_1) - r\} + \frac{Bz}{r} - \frac{rz}{A} + \frac{\phi(z)}{r} \\ w &= -\frac{d'}{c}\left\{\frac{2z^2}{I}\log(r/r_1) - \frac{z^2}{A}\right\} + \chi(r) \end{aligned}\right\} \text{.........(vi).}$$

Here B is another arbitrary constant and χ, ϕ arbitrary functions of r and z respectively.

Now we have $\widehat{rz} = e(u_z + w_r) = \frac{3P}{4\epsilon^3}\frac{r_1}{r}(\epsilon^2 - z^2)$; substituting for u and w from (vi) we find the following relation between χ and ϕ:

$$\frac{2r}{I}\log\frac{r}{r_1} - \frac{r}{I} + \frac{B}{r} - \frac{r}{A} + \frac{d\chi}{dr} = \frac{1}{r}\left\{\frac{2}{I}\frac{d'}{c}z^2 + \frac{2}{I}\frac{H}{e}(\epsilon^2 - z^2) - \frac{d\phi}{dz}\right\}.$$

Saint-Venant remarks that we can satisfy this relation in several ways (p. 350), but the proper method seems to me to equate either side multiplied by r to the same constant. He *takes this constant to be zero.* If this constant be retained, however, it only alters the value of the constant B in the expressions for the shifts we are about to give, and so may be neglected. We ought to add a constant C' to the value of $\chi(r)$; but this leads to a term in $u = C'/r$, or in $\widehat{rr} = -2fC'/r^2$, which, not containing an odd power of z, would prevent us from fulfilling the condition

$$\int_{-\epsilon}^{+\epsilon} \widehat{rr}\,dz = 0 \quad \text{for} \quad r = r_1.$$

Substituting the values obtained by integration for ϕ and χ in (vi), we have:

$$\left.\begin{aligned} u &= -\frac{rz}{A} + \frac{1}{I}\left\{2rz\log\frac{r}{r_1} - rz + \frac{2d'}{c}\frac{z^3}{3r} + \frac{2}{r}\frac{H}{e}\left(\epsilon^2 z - \frac{z^3}{3}\right)\right\} + \frac{Bz}{r} \\ w &= \frac{1}{A}\left(\frac{r^2}{2} + \frac{d'z^2}{c}\right) + \frac{1}{I}\left\{r^2 - \left(r^2 + \frac{2d'z^2}{c}\right)\log\frac{r}{r_1}\right\} - C - B\log\frac{r}{r_1} \end{aligned}\right\} \text{...(vii).}$$

The values of $\widehat{rr}$ and M_r may then be easily deduced. Saint-Venant gives expressions for them on pp. 351—2. By putting $r = r_1$, we obtain:

$$M_{r_1} = -\frac{4\epsilon^3}{3}\frac{H-f}{A} + \frac{4\epsilon^3}{3}\frac{f - H\gamma^2}{I} - \frac{4\epsilon^3}{3}f\frac{B}{r_1^2} \text{ ....(viii),}$$

where γ^2 is given by:

$$\frac{2}{5}\frac{f}{H}\left(\frac{d'}{c} + 4\frac{H}{e}\right)\frac{\epsilon^2}{r_1^2} = \gamma^2 \text{ .... (ix),}$$

and may be neglected when ϵ/r_1 is small.

If P or $1/I = 0$ and we put $B = 0$, this value of M_{r_1}, agrees with that of M_r in equation (v) of our Art. 324. We shall then write for simplification

$$\frac{1}{\rho} = -\frac{3}{4\epsilon^3}\frac{M_{r_1}}{H-f},$$

and we find
$$\frac{1}{A} = \frac{1}{\rho} + \frac{1}{I}\frac{f-H\gamma^2}{H-f} - \frac{f}{H-f}\frac{B}{r_1^2} \text{ .... (x).}$$

Substituting this value of $\frac{1}{A}$ in equations (vii) we note the following final results given on p. 354 and attributed by Saint-Venant to Boussinesq (*'que M. Boussinesq a cherchées et trouvées à ma prière'*):

$$\left.\begin{aligned}
u &= -\frac{rz}{\rho} + \frac{1}{I}\left\{rz\left(2\log\frac{r}{r_1} - H\frac{1-\gamma^2}{H-f}\right) + \frac{2}{r}\frac{d'}{c}\frac{z^3}{3} + \frac{2}{r}\frac{H}{e}\left(\epsilon^2 z - \frac{z^3}{3}\right)\right\} \\
&\qquad + \frac{Bz}{r} + \frac{f}{H-f}\frac{Brz}{r_1^2}, \\
w &= \frac{1}{\rho}\left(\frac{r^2}{2} + \frac{d'}{c}z^2\right) + \frac{1}{I}\left\{r^2 + \left(\frac{r^2}{2} + \frac{d'}{c}z^2\right)\left(\frac{f-H\gamma^2}{H-f} - 2\log\frac{r}{r_1}\right)\right\} \\
&\qquad - C - B\log\frac{r}{r_1} - \frac{f}{H-f}\cdot\frac{B}{r_1^2}\left(\frac{r^2}{2} + \frac{d'}{c}z^2\right), \\
M_r &= \frac{4\epsilon^3}{3}\left\{-\frac{H-f}{\rho} + \frac{1}{I}\left[2(H-f)\log\frac{r}{r_1} - H\gamma^2\left(\frac{r_1^2}{r^2} - 1\right)\right]\right. \\
&\qquad \left. - fB\left(\frac{1}{r^2} - \frac{1}{r_1^2}\right)\right\}; \\
\text{where}\quad \frac{1}{I} &= \frac{3Pr_1}{8H\epsilon^3},\quad \frac{1}{\rho} = -\frac{3}{4\epsilon^3}\frac{M_{r_1}}{H-f},\quad \gamma^2 = \frac{2}{5}\frac{f}{H}\left(\frac{d'}{c} + 4\frac{H}{e}\right)\frac{\epsilon^2}{r_1^2}
\end{aligned}\right\} \text{...(xi).}$$

For the vertical shift and shift-fluxion of the mid-plane we have when $B = C = 0$:

$$\left.\begin{aligned}
w_0 &= \frac{r^2}{2\rho} + \frac{1}{I}\left(r^2 + \frac{r^2}{2}\frac{f-H\gamma^2}{H-f} - r^2\log\frac{r}{r_1}\right) \\
\frac{dw_0}{dr} &= \frac{r}{\rho} + \frac{1}{I}\left(H\frac{1-\gamma^2}{H-f}r - 2r\log\frac{r}{r_1}\right)
\end{aligned}\right\} \text{ ....(xii).}$$

These very important results can be applied to a great number of special examples. They include the solutions of Poisson given for *thin* circular plates, and various other particular cases (as of isotropy, etc.) treated by diverse writers: see our Arts. 494*—504*[1].

[331.] *Special cases.*

(*a*) Suppose the plate not to be *annular*, but to rest on the rim of a disc of radius r_0 in such a manner that its bending is not interfered

[1] To obtain Saint-Venant's notation we must replace, u, w by capitals, H by a_1, I by H, r_1 by a, and ρ by R.

with (§ 14). The plate may now be dealt with as consisting of an 'inner disc' and 'outer annulus.' Then evidently $dw_0/dr=0$ when $r=0$ because the tangent plane to the mid-section at $z=0$, $r=0$, must be horizontal; further round the ring $r=r_0$ the shearing stress must vanish for the inner disc which can thus only be acted upon by couples and will take a spherical curvature $(1/\rho_0)$ as in our Art. 324. Thus for the inner disc

$$u=-\frac{rz}{\rho_0},\quad w=\frac{1}{\rho_0}\left(\frac{r^2-r_0^2}{2}+\frac{d'}{c}z^2\right),\quad M_r=-\frac{4\epsilon^3}{3}\frac{H-f}{\rho_0},$$

and for the conditions at $r=r_0$

$$w_0=0,\quad \frac{dw_0}{dr}=\frac{r_0}{\rho_0},\quad M_{r_0}=-\frac{4\epsilon^3}{3}\frac{H-f}{\rho_0}.$$

Three equations to determine the three constants ρ_0, B and C (ρ is known from M_{r_1}) of the problem are then obtainable by putting $r=r_0$ in the equations (xi) which hold for the outer annulus. Saint-Venant finds:

$$\left.\begin{aligned}B=\frac{r_0-\gamma^2r_1^2}{I},\quad C=\frac{r_0^2}{2\rho}+\frac{1}{I}\Big\{r_0^2\left(1-\frac{\gamma^2}{2}\right)+\frac{f}{H-f}\frac{r_0^2}{2}\left(1-\frac{r_0^2}{r_1^2}\right)&\\ -(2r_0^2-\gamma^2r_1^2)\log\frac{r_0}{r_1}\Big\},&\\ \frac{1}{\rho_0}=\frac{1}{\rho}+\frac{1}{I}\left\{\left(\frac{f}{H-f}+\frac{\gamma^2r_1^2}{r_0^2}\right)\left(1-\frac{r_0^2}{r_1^2}\right)-2\log\frac{r_0}{r_1}\right\}&\end{aligned}\right\}\ \ldots\text{(xiii)},$$

whence the values of u, and w for $r>r_0<r_1$, can be at once found.

The solutions obtained by Saint-Venant in this first case are, as he himself observes, hardly satisfactory except for the case of a very *thin* plate. What he does is to make the vertical shifts of the mid-plane zero for the disc and the annulus when $r=r_0$; then the slopes of the tangent planes for both are equated, and finally the total couples along the same circle $r=r_0$. In the solutions he gives for the shifts the u and w for the annulus are not equal to the u and w for the disc when $r=r_0$, except for the mid-plane. In particular u when $r=r_0$ is a function of z only for the disc, but of z^3 as well for the annulus. In other words we have theoretical separation of the material at $r=r_0$. Thus the solutions are at best only approximate, and cannot be considered to hold at all in the neighbourhood of the rim itself. But shall we assume they hold accurately at points not in the neighbourhood of this rim? If the stresses acting at this rim were really confined to a line, they would certainly produce permanent alterations in the material; are we then justified in assuming that equating the vertical shifts and the tangent plane slopes (w and dw/dr) for

$r = r_0$ will give us the best values of the constants? I am inclined to doubt at least the presumed equality of the tangent-plane slopes: see our Art. 1572*, and p. 23 of the *Leçons de Navier*. The results become of course exact when we may neglect ϵ^2.

[332.] (*b*) Suppose the centre of the plate to rest upon a fixed circle of very small radius (p. 357). Then equations (xii) give the total deflection δ by putting $r = r_1$:

$$\delta = \frac{r_1^2}{2\rho} + \frac{3Qr_1^2}{16\pi H\epsilon^3} \cdot \frac{2H - f - H\gamma^2}{2H - 2f} \quad\text{................(xiv)},$$

where $$Q = 2\pi r_1 P, \quad 1/\rho = -\frac{3}{4\epsilon^3}\frac{M_{r_1}}{H - f}.$$

Sub-cases are:

(i) $M_{r_1} = 0$, or $1/\rho = 0$; this is the simple case of only shearing load on the cylindrical sides. Such might happen if the mid-plane contour were fixed to a ring.

(ii) The cylindrical faces of the plate are fixed (see our footnote p. 231) and a normal load Q applied at the centre by means of a circle of very small radius. Here dw_0/dr of equation (xii) must be zero for $r = r_1$ or:

$$\frac{1}{\rho} = -\frac{H}{I}\frac{1 - \gamma^2}{H - f}.$$

This gives $$M_{r_1} = \frac{Pr_1}{2}(1 - \gamma^2),$$

and $$\delta = \frac{3Qr_1^2}{32\pi H\epsilon^3}.$$

(iii) Elastic isotropy (p. 358, § 16). We have only to put

$$f = e = \mu, \quad d' = f' = \lambda, \quad c = 2\mu + \lambda,$$

$$H = 2f + f' - \frac{d'^2}{c} = \frac{4(\mu + \lambda)\mu}{2\mu + \lambda},$$

$$\gamma^2 = \frac{2}{5}\frac{f}{H}\left(\frac{d'}{c} + \frac{4H}{e}\right)\frac{\epsilon^2}{r_1^2} = \frac{16\mu + 17\lambda}{10(\lambda + \mu)}\frac{\epsilon^2}{r_1^2}.$$

[333.] (*c*) Saint-Venant now returns to the case of a complete plate resting on a circular rim (of radius r_0) as given in our Art. 331, (*a*), and determines the deflections when the contour (of radius r_1) is (i) fixed, (ii) built-in (see his p. 360).

[334.] (*d*) In § 18 we have the remark that the force exerted on the ring $r = r_0$ must be equal and opposite to the force exerted on the ring $r = r_1$, or it must equal Pr_1/r_0 per unit of length of the arc. Thus the solutions of (c) are applicable to the case of a plate either fixed or built-in at its contour and loaded with Q uniformly distributed round the ring $r = r_0$. The deflections obtained by Saint-Venant are (p. 362):

(i) Mid-plane contour simply supported or fixed

$$\delta = \frac{3Qr_1^2}{32\pi H\epsilon^3}\left\{\frac{2H-f}{H-f}\left(1-\frac{r_0^2}{r_1^2}\right)+\left(\frac{r_0^2}{r_1^2}-\gamma^2\right)\log\frac{r_0^2}{r_1^2}\right\}.$$

(ii) Cylindrical face built-in

$$\delta' = \frac{3Qr_1^2}{32\pi H\epsilon^3}\left\{1-\frac{r_0^2}{r_1^2}+\left(\frac{r_0^2}{r_1^2}-\gamma^2\right)\log\frac{r_0^2}{r_1^2}\right\}.$$

For the reasons given in my Art. 331, I am doubtful as to the validity of these results except in the case when we may neglect γ^2.

[335.] (e) In § 19, p. 362, Saint-Venant explains how we may treat the problem of a thick circular plate subjected to any symmetrical load continuous or discontinuous on a plane face. We have in the case of a continuous load to substitute $\phi(r_0)\, 2\pi r_0 dr_0$ for Q in the equations of (d) and integrate between the limits 0 and r_0, to find the total deflection. If we integrate from 0 to r_0, we shall obtain the deflection of the centre below any ring r_0 and so the form of the surface taken by the mid-plane. Saint-Venant seems to think this process more rigorous than that for thin plates dependent on Lagrange's equation and used by Poisson: see our Arts. 284*, 496*—504*. But I cannot get over the difficulty suggested in my Art. 331. The results are not true for the ring in consideration unless γ^2 may be neglected, but Saint-Venant practically divides his whole plate up into such rings, when thus integrating. It appears to me possible that he may thus be really introducing an important sum of small errors.

In § 21, p. 365, he treats by this method the case of a thick plate uniformly loaded and finds from the results in (d):

$$\delta = \frac{3Qr_1^2}{128\pi H\epsilon^3}\left(\frac{3H-f}{H-f}+4\gamma^2\right),$$

$$\delta' = \frac{3Qr_1^2}{128\pi H\epsilon^3}(1+4\gamma^2),$$

where Q is the total load.

These results, first given by Boussinesq, agree in the case of uni-constant isotropy and neglect of γ^2 with those of Poisson: see our Art. 502*.

[336.] (f) This case is the most general possible and is thus stated by Saint-Venant:

Mais, lorsqu'on se propose *d'avoir seulement la flèche centrale*, sans chercher la forme que prend la plaque en ses divers points, une remarque bien simple montre que les expressions en r_1 et r_0 suffisent au calcul de cette flèche pour toutes les distributions possibles, même non symétriques, même discontinues et irrégulières, des charges que supporte la plaque soutenue en haut (p. 363).

We note that if we have a single load P at any point of a rim-supported plate, it must produce the same central deflection as if it were at *any other point at the same distance from the centre.* Hence by the principle of super-position of displacements in the case of elastic strain, a load P at an isolated point distant r from the centre, must produce the same central deflection as if it were uniformly distributed round the ring of radius r. Thus the formulae of (d) hold if the load Q be concentrated at a distance r_0 from the centre. This result seems first to have been stated by Lévy in a memoir of 1877, although it was involved in the results of § 76 of Clebsch's treatise.

[337.] Saint-Venant concludes this *Note* on thick plates with the following words:

Nous avons démontré, dans la présente Note, comme on a vu, nos formules d'une manière rigoureuse, ou sans annulations de termes. Leur parfaite rigueur est subordonnée, il est vrai, comme est celle de toutes les formules ci-dessus d'extension, flexion, torsion des tiges, à ce que les forces ou les réactions d'appuis et d'encastrements agissent exclusivement sur une certaine surface qui est, pour les plaques, leur cylindre contournant, en s'y distribuant des manières qui sont exprimées en z par les formules du deuxième et du troisième degré donnant $\widehat{rz}$ et $\widehat{rr}$, et spécifiées pour $r = r_1$. Mais, ainsi que nous avons eu bien des fois occasion de le dire, elles donnent des résultats très suffisamment approchés quel que soit le mode d'application et de distribution si la plaque est peu épaisse; et, en tous cas, notre analyse actuelle, outre qu'elle tient compte de termes (ceux en γ^2 ou ϵ^2/r_1^2) dont il n'est nulle question dans l'analyse connue, a l'avantage de ne donner que les résultats où tout a été mis en compte dès le commencement ou sans suppressions faites de prime abord, et dont on n'aperçoit pas *a priori* la portée et le degré d'influence sur les résultats lorsqu'on en opère de ce genre (p. 367).

But does this paragraph explain all the assumptions? I think not: see our Art. 331.

[338.] The second *Note* inserted by Saint-Venant in Clebsch's third chapter is due to Boussinesq. It is a résumé of the results obtained by the latter in a series of memoirs during the years 1878—9, and afterwards published separately under the title: *Recherches sur l'application des potentiels à la théorie de l'équilibre intérieur des solides élastiques;* see our detailed account of this important work below. The *Note* itself is entitled: *Sur l'équilibre des corps massifs sollicités en un point superficiel ou intérieur.* It occupies pp. 374—405; an addition occupies pp. 405[a]—407[a]; while some consideration, also due to Boussinesq, of Cerruti's

Memoir of 1882 on the same subject, will be found on pp. 881—8 (*Complément à la Note finale du* § 46). As these contributions are not due to Saint-Venant we postpone the discussion of their contents until we are dealing with the special researches of Boussinesq and Cerruti.

[339.] The next important addition of Saint-Venant is the *Note finale du* § 60. It is entitled: *Théorie de l'impulsion longitudinale d'une barre élastique par un corps massif qui vient heurter une de ses deux extrémités; et de la résistance de la matière de la barre à un pareil choc;* it occupies pp. 480 a—480 gg. The numerical results of this note together with their graphical representation will be considered in our account of the Memoir of 1883: see our Arts. 401—7.

[340.] The first seven sections (pp. 480 a—480 k) give an account of the various tentative stages in the history of the theory. We have first two theorems of Young, which as first approximations may be cited. Let the bar be of weight P, density ρ, section ω, length l and stretch-modulus E; let Q be the weight and V the velocity of the body which strikes it at the free end, the other end being fixed.

Then if u be the total shift of the free end, g gravitational acceleration, and we suppose the stretch uniformly distributed, we have from the principle of work:

(i) *Bar horizontal:*

$$\frac{E\omega u^2}{2l} = \frac{Q}{g}\frac{V^2}{2}, \quad \text{or if } u_0 = \frac{Ql}{E\omega} \text{ be the statical shift,}$$

$$u = V\sqrt{u_0/g}.$$

(ii) *Bar vertical:*

$$\frac{E\omega u^2}{2l} = \frac{Q}{g}\frac{V^2}{2} + Qu \quad \therefore\ u = u_0 + \sqrt{u_0^2 + V^2\frac{u_0}{g}}.$$

Let $u/l = T_0/E$ the greatest safe stretch within the elastic limit, then in Case (i):

$$\frac{E\omega l}{2}\left(\frac{T_0}{E}\right)^2 = \frac{QV^2}{2g}.$$

Now $\frac{QV^2}{2g}$ is the work necessary to destroy the efficiency of the bar, or its resilience. Hence the resilience of a bar varies as its volume ωl, multiplied by $\frac{T_0^2}{2E}$, a quantity depending only on its elasticity. In this form of Young's theorem, the quantity T_0^2/E has been termed by Tredgold the *modulus of resilience:* see our Arts. 999* and 982* and Vol. I., p. 875.

[341.] The next stage in the history of longitudinal impact was due to Navier (see our Arts. 272*—4*). He expressed the complete analytical solution of the problem for the case of the horizontal bar in a Fourier's series. Poncelet added to this solution the effect of gravity and the statical action of the weight, supposed to strike the bar in a vertical position: see our Art. 990*. Neither Navier nor Poncelet developed this analytical solution, except for the special case of P/Q being very small when the results agree with those of the preceding article. Saint-Venant undertook this development, so far as ascertaining the *shift* is concerned, in 1865 and 1868 (see our Arts. 200 and 201) for certain common values of P/Q, i.e. $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4. He found it possible to determine the shift of the end struck, but the series gave no prospect, however far the numerical calculations were carried, of ascertaining the maximum stretch or squeeze (p. 480 g). It became necessary then to find a solution in finite terms. The form of these finite terms seems to have been suggested by Saint-Venant's course of memoirs lasting from 1865—1882 on the impact of two bars: see our Arts. 203 and 221. The next stage was Boussinesq's solution in terms of a single exponential for the shift at a time not greater than $2l/\alpha$: see our Art. 403 and the account later of his paper of 1882. Later in the same year two officers of the French marine artillery, Sébert and Hugoniot, obtained an exponential solution in finite terms for a vibrating bar fixed at one end and subjected at the other to a force varying with the time. This solution really covers that of Boussinesq, who hearing only of the method of Sébert and Hugoniot, sent to Saint-Venant in the summer vacation of 1882 a direct and complete solution of the problem of longitudinal impact. Judging from the communications of M. Hugoniot to Saint-Venant (see pp. 480 j—480 k) the merit of the solution must be divided between the two naval officers and the professor of Lille.

The reader will find an account of Boussinesq's solution in the chapter devoted to that elastician.

[342.] The next insertion of Saint-Venant is the *Note finale du* § 61. It occupies no less than 138 pages (pp. 490—627) and contains the complete theory of the transverse impulse of bars, including results of Saint-Venant's not hitherto published: see our Arts. 104–5, 200–1, and *Notice* II. p. 20, 2°. The *Note* is entitled: *De l'Impulsion transversale des barres élastiques, et de leur vibration avec le corps qui les aura mises en mouvement. Détermination de leur flexion ainsi que des conditions de leur résistance vive ou dynamique.*

[343.] The first 51 sections (pp. 490—597) are devoted to the analytical and numerical solution of various problems of bars vibrating transversely with a load attached:

ces pièces sont supposées *vibrer* non pas *seules* comme le supposent les solutions données par Clebsch, mais *unies avec le corps étranger* dont l'impulsion, ou brusque, ou graduée, les a fait sortir de leur état d'équilibre; car c'est pendant cette union, ne durât-elle que le temps d'une demi-période oscillatoire, que les déplacements relatifs des parties de ces pièces atteignent leur maximum et qu'elles courent le plus grand danger de rupture ou d'énervation dont les calculs de résistance ont pour objet de les sauver (p. 490).

Saint-Venant's method is simply to solve in 'normal' functions or coordinates the equation:

$$\frac{d^2}{dz^2}\left(E\omega\kappa^2\frac{d^2u}{dz^2}\right)+\frac{p}{g}\left(\frac{d^2u}{dt^2}-\mathfrak{g}\right)=0\dots\dots\dots\dots\dots\text{(i)},$$

where u is the transverse shift of the point in the axis at distance z from one end of the bar, p/g is the mass per unit length of the bar and of any permanent load at the same point, $\mathfrak{g}$ the body acceleration (usually only gravity) on the same length, and $E\omega\kappa^2$ with our usual notation the rigidity, which may vary from point to point. The bar is supposed to be loaded and to receive displacement in a plane which passes through a principal axis of each cross-section. The terminal and initial conditions determine the constants of the normal functions while the conditions at the impelled point select the normal functions required and determine the notes.

[344.] The process of solution and the calculation of the dynamical deflection are generally long, even if we keep only one term of the series, but:

cette expression simple de la flèche dynamique peut, comme je l'ai reconnu dans une multitude d'exemples, être identiquement obtenue sans poser d'équations différentielles, en s'aidant d'une hypothèse plausible sur les rapports mutuels des déplacements, et en y appliquant d'une manière tout élémentaire, le théorème des vitesses virtuelles ou celui des pertes brusques de force vive; en sorte que rien n'empêchera d'introduire dans les cours, même industriels, cette méthode que j'appelle de *deuxième approximation*, tenant suffisamment compte de l'*inertie* des systèmes heurtés, et d'en substituer l'enseignement général à celui qui y est quelquefois donné, pour deux cas particuliers, de la méthode dans laquelle, en abstrayant tout à fait ou en supposant infiniment petite la masse de ces systèmes, on s'éloigne généralement beaucoup de la réalité et des faits (p. 491).

The *hypothèse plausible* which Saint-Venant makes is precisely that of Cox (see his p. 584, § 46) and his results, pp. 584—597, are those of Cox (see our Arts. 1435–7*), or those I had obtained by Cox's method before examining Saint-Venant's work (see Vol. I. pp. 894—6). Thus the merit of this elementary treatment of the problem is entirely Cox's, but Saint-Venant's work, taking first into account the vibratory terms is really the justification of the hypothesis. I am somewhat surprised that Cox's paper escaped Saint-Venant, as he is usually very careful in his historical notices, and he had certainly read Stokes' papers in the volumes of the *Cambridge Transactions*.

The Note terminates with a consideration of Willis's problem and a discussion of the numerical results of the *Iron Commissioners' Report*: see our Arts. 1276*, 1406* and 1417*.

[345.] I propose to describe in one case Saint-Venant's method of solution, and then to record the other problems with which he has dealt in this Note. The following conditions are easily seen to hold:

(i) at a free end:

Bending moment $= E\omega\kappa^2 \dfrac{d^2u}{dz^2} = 0$; shear $= -\dfrac{d}{dz}\left(E\omega\kappa^2 \dfrac{d^2u}{dz^2}\right) = 0$.

(ii) at a fixed end[1]: $u = 0$, $E\omega\kappa^2 \dfrac{d^2u}{dz^2} = 0$. (We retain the $E\omega\kappa^2$ in both cases as $\omega\kappa^2$ may be the vanishing factor in certain systems.)

(iii) at a built-in end: $u = 0$, $\dfrac{du}{dz} = 0$.

[1] At a *fixed* end the terminal direction is free; the word *supported* should also be interpreted as equivalent to *fixed*, i.e. allowing only of shearing force, but this in either sense: see footnote Vol. I., p. 52.

(iv) At the join of two bars:

$$u = u_1, \quad \frac{du}{dz} = \frac{du_1}{dz}, \quad E\omega\kappa^2 \frac{d^2u}{dz^2} = E_1\omega_1\kappa_1^2 \frac{d^2u_1}{dz^2}.$$

(v) At the join of two bars where there is a weight of mass Q/g we must have:

$$\frac{Q}{g}\frac{d^2u}{dt^2} - \frac{d}{dz}\left(E\omega\kappa^2 \frac{d^2u}{dz^2}\right) + \frac{d}{dz}\left(E_1\omega_1\kappa_1^2 \frac{d^2u_1}{dz^2}\right) = 0,$$

together with the relations (iv): see Saint-Venant's pp. 494-5.

[346.] Let us apply these results to the simple case of a prismatic bar supported terminally and struck by a weight Q with velocity V at its mid-point. Let the length of the bar be $2l$, its weight P, and $\tau^2 = Pl^3/(2gE\omega\kappa^2)$. We shall suppose the bar so placed that the impact is horizontal, or $\mathfrak{g}$ may be put zero. Equation (i) of Art. 343, thus becomes:

$$\tau^2 \frac{d^2u}{dt^2} + l^4 \frac{d^4u}{dz^4} = 0 \quad \text{(i)},$$

together with the conditions:

$$u = 0, \quad \frac{d^2u}{dz^2} = 0, \text{ when } z = 0 \quad \text{(ii)},$$

$$\tau^2 \frac{d^2u}{dt^2} = \frac{Pl^3}{Q}\frac{d^3u}{dz^3}, \quad \frac{du}{dz} = 0, \text{ when } z = l \quad \text{(iii)}.$$

Take as a particular integral:

$$z = Z_m \left\{A'_m \frac{\tau}{m^2} \sin \frac{m^2t}{\tau} + B'_m \cos \frac{m^2t}{\tau}\right\}.$$

We find $$m^4 Z_m = l^4 \frac{d^4 Z_m}{dz^4}.$$

The solution of this equation takes the well-known form first given by Euler, (see our Art. 52*):

$$Z_m = C \sin\frac{mz}{l} + C_1 \cos\frac{mz}{l} + C_2 \sinh\frac{mz}{l} + C_3 \cosh\frac{mz}{l}.$$

To satisfy (ii) and the second of (iii) we must take

$$C_1 = C_3 = 0, \quad C_2 = -C\frac{\cos m}{\cosh m}.$$

Further $u = 0$ when $t = 0$, therefore we have finally u of the form

$$\left.\begin{aligned} u &= \Sigma \frac{\tau}{m^2} A_m Z_m \sin\frac{m^2t}{\tau} \\ \text{where} \quad Z_m &= \frac{\sin(mz/l)}{\cos m} - \frac{\sinh(mz/l)}{\cosh m} \end{aligned}\right\} \quad \text{(iv)},$$

and the first of conditions (iii) gives us the equation for m

$$m(\tan m - \tanh m) = 2P/Q \text{.....................(v)},$$

which may be termed the *characteristic* transcendental equation for m. Of this equation m, $-m$, $m\sqrt{-1}$, $-m\sqrt{-1}$ are all roots if m is a root, but they give rise to the same Z_m, so that we need only take the real and positive roots.

[347.] It remains to determine A_m. When $t=0$, let $\dot{u}=\psi(z)$, then

$$\Sigma A_m Z_m = \psi(z) \text{..............................(i)}.$$

Multiply both sides by $Z_{m'}$ and we have;

$$\Sigma A_m Z_m Z_{m'} = \psi(z)\, Z_{m'} \text{.........................(ii)}.$$

Put $z=l$, multiply by Q and add this to the integral of equation (ii) above with regard to $\frac{P}{2l}dz$ from $z=0$ to $2l$, and we find:

$$\Sigma A_m \left\{2\frac{P}{2l}\int_0^l Z_m Z_{m'}\,dz + QZ_m(l)\,Z_{m'}(l)\right\} = 2\frac{P}{2l}\int_0^l Z_{m'}\psi(z)\,dz + QZ_{m'}(l)\times\psi(l).$$

As Saint-Venant remarks this is really an integration of equation (ii) with regard to $d\mathfrak{q}$, where $\mathfrak{q}$ = total weight of beam and load $= P+Q$. Now Saint-Venant shews by straightforward integration that

$$\int_0^l Z_m Z_{m'}\,dz = -\frac{4l}{mm'}\frac{P}{Q} \text{........................(iii)},$$

or remembering the values of $Z_m(l)$, $Z_{m'}(l)$, we have, save when $m=m'$:

$$2\frac{P}{2l}\int_0^l Z_m Z_{m'}\,dz + QZ_m(l)Z_{m'}(l) = \int_0^{2l} Z_m Z_{m'}\,d\mathfrak{q} = 0.$$

Thus we find:

$$A_m = \frac{2\frac{P}{2l}\int_0^l Z_m\psi(z)dz + QZ_m(l)\psi(l)}{2\frac{P}{2l}\int_0^l Z^2{}_m dz + QZ^2{}_m(l)} \text{..............(iv)},$$

which may be written in the form:

$$A_m = \frac{\int_0^{2l} Z_m\psi(z)d\mathfrak{q}}{\int_0^{2l} Z^2{}_m d\mathfrak{q}} \text{........................(v)}.$$

[348.] Now arises a question as to the value we ought to give to $\psi(z)$. Saint-Venant puts $\psi(z)=0$ except $z=l$, when $\psi(l)=V$. Thus he obtains after some obvious reductions:

$$A_m = \frac{QVZ_m(l)}{\frac{P}{l}\int_0^l Z^2{}_m dz + QZ^2{}_m(l)}.$$

On evaluating this expression we find

$$A_m = \frac{4V}{m(\sec^2 m - \operatorname{sech}^2 m) + \dfrac{2P}{Qm}} \quad \text{..............(vi).}$$

Equations (vi) of this Article with (iv) and (v) of Art. 346 give the complete solution.

Is now this choice of initial velocities a proper one? Saint-Venant has defended it in the *Comptes rendus*, T. LXI. 1865, p. 43, against an objection raised to it. He says that if a small portion of the bar receive an initial velocity, Z_m will be nearly constant for this portion; accordingly equation (v) of the preceding Article gives us for the numerator of A_m, the expression $Z_m(l)\int\psi(z)d\mathfrak{q}$ where $\psi(z)$ is zero except over this small portion, where it has a value slightly less than V. But he remarks that the momentum possessed by this small portion and the weight Q ought to be exactly $\frac{Q}{g}V$, or $\int\psi(z)\frac{d\mathfrak{q}}{g} = \frac{Q}{g}V$.

[349.] We will now indicate the various problems which are dealt with analytically by Saint-Venant.

(*a*) In §§ 7–14 he treats as a general problem the cases when the bar is not prismatic (i.e. the rigidity $E\omega\kappa^2$ varies), when its ends are fixed in different fashions, when there are various bars or when one bar with a varying load forms the complete system. He shews that supposing the functions $Z_{\mathbf{m}}$ can be found which satisfy the equation:

$$\frac{d^2}{dz^2}\left\{E\omega\kappa^2\frac{d^2 Z_{\mathbf{m}}}{dz^2}\right\} = \mathbf{m}^4\frac{p}{g}Z_{\mathbf{m}},$$

then the integral $\int Z_{\mathbf{m}} Z_{\mathbf{m}'} d\mathfrak{q} = 0,$

where $\mathfrak{q}$ represents the total weight of the system and the integration extends from one end to the other of the system; $\mathbf{m}$ and $\mathbf{m}'$ are two unequal roots of the characteristic equation in $\mathbf{m}$ which arises from the terminal and load conditions (p. 506).

The coefficients of the time function $A_{\mathbf{m}}\mathbf{m}^{-2}\sin\mathbf{m}^2 t + B_{\mathbf{m}}\cos\mathbf{m}^2 t$ will be determined by equations similar to (v) of our Art. 347; $A_{\mathbf{m}}$ depending only on the initial velocities, $B_{\mathbf{m}}$ only on the initial displacements (p. 507).

The value of the denominator of these coefficients, i.e. $\int Z^2_{\mathbf{m}}d\mathfrak{q}$, can be obtained by the *differentiation* with regard to $\mathbf{m}$ of a certain function of $Z_{\mathbf{m}}$ and its fluxions with regard to z (p. 508). Compare Lord Rayleigh's *Theory of Sound*, Vol. I., pp. 209–10.

[350.] (*b*) The next special example given by Saint-Venant is that of a doubly built-in beam struck at the mid-point. He finds:

$$u = V\tau\Sigma \frac{2}{m^2} \cdot \frac{(1-\cos m\cosh m)(\cos m - \cosh m)(\sin m + \sinh m)}{(1-\cos m\cosh m)^2 + \frac{P}{Q}(\cos m - \cosh m)^2} Z_m \sin\frac{m^2 t}{\tau},$$

where $$Z_m = \frac{\sinh\frac{mz}{l} - \sin\frac{mz}{l}}{\cosh m - \cos m} - \frac{\cosh\frac{mz}{l} - \cos\frac{mz}{l}}{\sinh m + \sin m},$$

and the characteristic equation is:

$$\frac{m(1-\cos m\cosh m)}{\sin m\cosh m + \cos m\sinh m} = \frac{P}{Q}.$$

See pp. 511—3.

[351.] (*c*) When one end of the bar, for this particular case supposed of length l, is built-in and the other is struck we have:

$$u = V\tau'\Sigma \frac{2}{m^2} \frac{(\sin m\cosh m - \cos m\sinh m)(\sin m + \sinh m)(\cos m + \cosh m)}{(\sin m\cosh m - \cos m\sinh m)^2 + \frac{P}{Q}(\sin m + \sinh m)^2} \times Z_m \sin\frac{m^2 t}{\tau'},$$

where $$Z_m = \frac{\cosh\frac{mz}{l} - \cos\frac{mz}{l}}{\cosh m + \cos m} - \frac{\sinh\frac{mz}{l} - \sin\frac{mz}{l}}{\sinh m + \sin m},$$

and for this case: $$\tau'^2 = \frac{Pl^3}{gE\omega\kappa^2},$$

and the characteristic equation is:

$$m\frac{\sin m\cosh m - \cos m\sinh m}{1 + \cos m\cosh m} = \frac{P}{Q}.$$

See pp. 513—4.

[352.] (*d*) Suppose the bar of length $2l = a + b$ and the blow to be given at a distance a from one end, then if τ^2 be as in Art. 346:

$$u = V\tau\Sigma \frac{A_m Z_m}{m^2} \sin\frac{m^2 t}{\tau}, \text{ from } z = 0 \text{ to } a,$$

where $$Z_m = \frac{\sin\frac{mb}{l}\sin\frac{mz}{l}}{\sin m\cos m} - \frac{\sinh\frac{mb}{l}\sinh\frac{mz}{l}}{\sinh m\cosh m};$$

and, $$u' = V\tau\Sigma \frac{A_m Z'_m}{m^2} \sin\frac{m^2 t}{\tau}, \text{ from } z' = 0 \text{ to } b,$$

where $$Z'_m = \frac{\sin\frac{ma}{l}\sin\frac{mz'}{l}}{\sin m\cos m} - \frac{\sinh\frac{ma}{l}\sinh\frac{mz'}{l}}{\sinh m\cosh m}.$$

In both cases

$$A_m = \frac{4}{m}\frac{1}{\frac{dZ_m(a)}{dm}+\frac{2P}{m^2Q}},\ \text{(where obviously } Z_m(a)=Z'_m(b)),$$

and the characteristic equation is:

$$mZ_m(a) = \frac{2P}{Q}.$$

See pp. 514—7[1].

The results given in our Arts. 349—352 correspond with the *introduction of vibratory terms* to the solutions obtained by Cox's method in Art. 1437* and pp. 894—895, c (i), c (ii) and (*b*) respectively of our first volume. I have gone through Saint-Venant's analysis but not worked out independently his results.

[353.] We have next several cases in which the bar would not be immoveable if it were rigid, *i.e.* the bar is free or pivoted. Here the solution will have an algebraic part as well as a transcendental. This part can sometimes be obtained by retaining the root $m=0$, which has been divided out of the *characteristic* equation; but as a rule it is better to treat it separately as arising from the kinetic conditions of the problem and determine it by general dynamical principles such as the principle of momentum. I will briefly indicate Saint-Venant's treatment in the following example:

(*e*). A prismatic bar is struck transversely at its two terminals by bodies of weight q and Q moving with velocities v and V respectively. The length of the bar is l and its weight P; the origin is taken at the end at which q strikes the bar. As before let us take

$$q+P+Q=\mathfrak{q},\quad \frac{Pl^3}{gE\omega\kappa^2}=\tau'^2, \text{ as in Case } (c) \text{ Art. 351.}$$

We have then: $$\tau'^2\frac{d^2u}{dt^2}+l^4\frac{d^4u}{dz^4}=0 \text{.............................(i).}$$

For the free ends, $$\frac{d^2u}{dz^2}=0, \text{ when } z=0 \text{ or } l \text{..................(ii)};$$

$$\left.\begin{aligned}\tau'^2\frac{d^2u}{dt^2}+\frac{Pl^3}{q}\left(\frac{d^3u}{dz^3}\right)&=0,\text{ when } z=0\\ \tau'^2\frac{d^2u}{dt^2}-\frac{Pl^3}{Q}\left(\frac{d^3u}{dz^3}\right)&=0,\text{ when } z=l\end{aligned}\right\}\text{............(iii)};$$

$u=0$, for $t=0$; $du/dt=v$ when $z=0$, $=V$ when $z=l$, and equal zero for all other values of z at the epoch $t=0$.

[1] Saint-Venant has a for our l, b for our a and b_1 for our b, EI for our $E\omega\kappa^2$, with other slight differences. I have altered his notation to agree with that of our first volume.

Saint-Venant assumes a solution of the form:

$$u = \left(C + C'\frac{z}{l}\right) t + \Sigma Z_m A_m \frac{\tau'}{m^2} \sin \frac{m^2 t}{\tau'}.$$

By the principle of linear momentum we have:

$$qv + QV = q\left(\frac{du}{dt}\right)_{z=0} + Q\left(\frac{du}{dt}\right)_{z=l} + \int_0^l \frac{du}{dt}\frac{P}{l}\,dz \ldots\ldots \text{(iv)},$$

and by that of the moment of momentum about the point $z = 0$,

$$QVl = Ql\left(\frac{du}{dt}\right)_{z=l} + \int_0^l \frac{du}{dt}\, z\, \frac{P}{l}\,dz \ldots\ldots\ldots\ldots\ldots \text{(v)},$$

as equations connecting the velocities before and after the blow. Now Saint-Venant so chooses C and C' *that the algebraic part of u, namely $\left(C + C'\frac{z}{l}\right) t$, shall satisfy equations* (iv) *and* (v) *independently of the trigonometrical terms.* We easily find:

$$C \Big/ \left\{-1 + \frac{2qv}{QV}\left(1 + 3\frac{Q}{P}\right)\right\} = \frac{C'}{3} \Big/ \left\{1 + 2\frac{q}{P} - \frac{qv}{QV}\left(1 + 2\frac{Q}{P}\right)\right\}$$

$$= \frac{2QV}{P} \Big/ \left\{1 + \frac{4(Q+q)}{P} + 12\frac{qQ}{P^2}\right\} \ldots\ldots\ldots\ldots \text{(vi)}.$$

We can now determine the function Z_m from the equations (i) to (iii) as if the algebraic portion of u had no existence, for the latter disappears entirely from these equations. Saint-Venant finds:

$$Z_m = (\cosh m - \cos m)\left(\sinh\frac{mz}{l} + \sin\frac{mz}{l}\right) - (\sinh m - \sin m)\left(\cosh\frac{mz}{l} + \cos\frac{mz}{l}\right)$$

$$+ \frac{2mq}{P}\left(\sin m \sinh\frac{mz}{l} + \sinh m \sin\frac{mz}{l}\right) \ldots\ldots \text{(vii)},$$

with the characteristic equation:

$$1 - \cos m \cosh m + m\,\frac{Q+q}{P}\,(\sin m \cosh m - \cos m \sinh m)$$

$$+ \frac{2Qq}{P^2}\, m^2 \sin m \sinh m = 0 \ldots\ldots\ldots\ldots\ldots \text{(viii)}.$$

Initially $\qquad C + C'\frac{z}{l} + \Sigma A_m Z_m = \psi(z)$, say,

multiplying by $Z_m\, d\mathfrak{q}$ we have as the coefficients of C and C' respectively,

$$\left.\begin{aligned} \int Z_m\, d\mathfrak{q} &\equiv qZ_m(0) + QZ_m(l) + \frac{P}{l}\int_0^l Z_m\, dz \\ \int Z_m \frac{z}{l}\, d\mathfrak{q} &\equiv QZ_m(l) + \frac{P}{l}\int_0^l Z_m \frac{z}{l}\, dz \end{aligned}\right\} \ldots\ldots\ldots \text{(ix)}.$$

By straightforward integration we can shew that both these expressions $= \frac{2P}{m} \times$ {the function of m to the left of equation (viii)}, and therefore both $= 0$.

We have thus $\int \left(C + C' \frac{z}{l}\right) Z_m d\mathfrak{q} = 0$, or, *the algebraic part* has no influence on the determination of the value of A_m, which accordingly equals:

$$\frac{qvZ_m(0) + QVZ_m(l)}{qZ^2_m(0) + QZ^2_m(l) + \frac{P}{l}\int_0^l Z^2_m dz},$$

as before. Saint-Venant gives on p. 525 the lengthy expressions for the numerator and denominator of this quantity. Equation (vii) gives us easily the numerator and all but the expression $\int_0^l Z^2_m dz$ in the value of the denominator. The value of this integral I find to be:

$$\frac{1}{P}\Big[P^2\left\{\frac{3}{m}(\sin m\cosh m - \cos m\sinh m)(1-\cos m\cosh m) + (\sinh m - \sin m)^2\right\}$$
$$+ 2Pq\{6\sin m\sinh m(1-\cos m\cosh m) + m(\cosh m - \cos m)(\sinh m - \sin m)\}$$
$$+ q^2\{2m^2(\sinh^2 m - \sin^2 m) + 6m\sin m\sinh m(\sin m\cosh m - \sinh m\cos m)\}\Big].$$

There is one point, however, which we must notice, namely, that equations (iv) and (v) have only been proved for the *algebraic portions* of the solution, but they must hold generally. Substituting the full value of u, we find that these equations will still be satisfied, if:

$$q\Sigma AZ_m(0) + Q\Sigma AZ_m(l) + \Sigma\int_0^l AZ_m \frac{P}{l}dz = 0,$$

$$Ql\Sigma AZ_m(l) + \Sigma\int_0^l AZ_m \frac{Pz}{l}dz = 0.$$

But these equations are satisfied for each Z_m of the sum by reason of equations (ix).

I do not think this point is explicitly brought out by Saint-Venant, although in a long footnote pp. 521—4, he proves a more general proposition, namely:

On peut donc, dans les problèmes de mouvement des barres ou tiges élastiques libres ou pivotantes autour de points ou d'axes fixes, établir *séparément* la partie algébrique ou de solidification, et la partie transcendante ou vibratoire, de leur mouvement. Et même on peut généralement, ce qui est encore mieux, ne s'occuper que de celle-ci, qui seule intéresse le problème de la résistance de la matière, sans craindre que la non-prise en considération de celle-là *soit une cause d'erreur* (p. 524).

That is, the principles of kinetics will hold for the algebraic and transcendental parts of the solution separately as we have seen in the above example.

[354.] Saint-Venant on pp. 525—6 treats two special cases of the problem in Art. 353.

(i) If we put $q=0$ we get the case of a free bar struck transversely at one end. The solution given in the article referred to easily reduces to:

$$u = 2\frac{3\frac{z}{l}-1}{4+\frac{P}{Q}}Vt + 2V\tau'\Sigma\frac{(\sin m\cosh m - \cos m\sinh m)\,Z_m\sin\frac{m^2t}{\tau'}}{(\sin m\cosh m-\cos m\sinh m)^2+\frac{P}{Q}(\sinh m-\sin m)^2},$$

where Z_m

$$=(\cosh m-\cos m)\left(\sinh\frac{mz}{l}+\sin\frac{mz}{l}\right)-(\sinh m-\sin m)\left(\cosh\frac{mz}{l}+\cos\frac{mz}{l}\right);$$

the characteristic equation being:

$$m(\sin m\cosh m - \cos m\sinh m) + \frac{P}{Q}(1-\cos m\cosh m) = 0.$$

(ii) If we put $v=0$, and $q=\infty$, we have the case of a bar fixed or pivoted at one end and struck at the other.

We find

$$u = \frac{z/l}{1+P/(3Q)}Vt + 2V\tau'\Sigma\frac{1}{m^2}\frac{\frac{\sin(mz/l)}{\sin m}+\frac{\sinh(mz/l)}{\sinh m}}{1+\frac{P}{2Q}(\operatorname{cosec}^2 m - \operatorname{cosech}^2 m)}\sin\frac{m^2t}{\tau'},$$

the characteristic equation being:

$$\cot m - \coth m = 2m\,Q/P.$$

On pp. 526—30 are given various verifications of these results.

[355.] Saint-Venant on p. 530 (§ 24) passes to the consideration of *Impulsions graduelles ou tranquilles.* Under this term he includes problems involving the effect of the weights of both the striking body and the bar during the blow, or involving the constrained movement of a portion of the bar. For example, if a horizontal bar be struck vertically we have to solve the equation (i) of Art. 343, with $\mathfrak{g}$ put $=g$. I will briefly indicate Saint-Venant's method in the following problem: *A horizontal bar terminally supported, to the mid-point of which is attached a weight Q', receives a blow at the same point from a body of weight Q falling vertically with velocity V. To determine the transverse shift.*

[356.] Let $2l$ be the length, P the weight of the bar, $Pl^3/(2gE\omega\kappa^2)=\tau^2$, $P+Q+Q'=\mathfrak{q}$.

We have to solve the equation

$$\tau^2\left(\frac{d^2u}{dt^2}-g\right)+l^4\frac{d^4u}{dz^4}=0 \text{ .... (i)},$$

subject to the terminal conditions:

$$u=0,\quad \frac{d^2u}{dz^2}=0 \text{ when } z=0;\quad \frac{du}{dz}=0 \text{ when } z=l \text{........ (ii)},$$

$$\left.\begin{aligned} Q+Q'-\frac{Q+Q'}{g}\frac{d^2u}{dt^2} &= -2E\omega\kappa^2\frac{d^3u}{dz^3} \\ \text{or,}\qquad \tau^2\left(\frac{d^2u}{dt^2}-g\right) &= \frac{Pl^3}{Q+Q'}\frac{d^3u}{dz^3} \end{aligned}\right\} \text{when } z=l \text{.........(iii)}.$$

Saint-Venant takes $u=u_1+U$ where u_1 is independent of the time and chosen so that the gravitational terms disappear from equations (i) and (iii) i.e.:

$$-g\tau^2+l^4\frac{d^4u_1}{dz^4}=0;\quad -g\tau^2(Q+Q')=Pl^3\frac{d^3u_1}{dz^3}, \text{ when } z=l.$$

Thus we have $$\frac{d^3u_1}{dz^3}=\frac{g\tau^2}{l^4}(z-l)-g\tau^2\frac{Q+Q'}{Pl^3},$$

and integrating having regard to equations (ii), we find

$$u_1=\frac{Pl^3}{48E\omega\kappa^2}\left\{8\frac{z}{l}-4\left(\frac{z}{l}\right)^3+\left(\frac{z}{l}\right)^4\right\}+\frac{(Q+Q')\,l^3}{6E\omega\kappa^2}\left\{\frac{3}{2}\frac{z}{l}-\frac{1}{2}\left(\frac{z}{l}\right)^3\right\} \text{...(iv)}.$$

The equations for U can now be easily solved, we deduce:

$$U=\Sigma Z_m\left(\frac{\tau}{m^2}A_m\sin\frac{m^2t}{\tau}+B_m\cos\frac{m^2t}{\tau}\right) \text{ where } Z_m=\frac{\sin\frac{mz}{l}}{\cos m}-\frac{\sinh\frac{mz}{l}}{\cosh m},$$

and the characteristic equation is:

$$m(\tan m-\tanh m)=2P/(Q+Q') \text{ .... (v)}.$$

In order to determine B_m and A_m we have, if $\phi(z)$ and $\psi(z)$ be respectively the initial shift and velocity corresponding to U,

$$B_m=\frac{\int\phi(z)\,Z_m\,dz}{\int Z_m^2\,dz},\quad A_m=\frac{\int\psi(z)\,Z_m\,dz}{\int Z_m^2\,dz}.$$

Now, $U_{t=0}=$ the initial value of $u-u_1$,

$\dot U_{t=0}=$ the initial value of $\dot u-\dot u_1$.

Further the initial value of u is the deflection due to the bar's own weight P and the load Q' attached.

Hence we have: $$U_{t=0}=-\frac{Ql^3}{6E\omega\kappa^2}\left\{\frac{3}{2}\frac{z}{l}-\frac{1}{2}\left(\frac{z}{l}\right)^3\right\}.$$

Further $\dot u_1=0$, or $\dot U_{t=0}=V$ for the weight Q, whose abscissa is l, but for all other points $\dot U_{t=0}=0$.

Saint-Venant then proceeds to calculate A_m and B_m, and ultimately finds:

$$u = u_1 + \frac{Q}{Q+Q'} \Sigma \frac{4}{m^3} \left\{ \frac{\dfrac{\sin \dfrac{mz}{l}}{\cos m} - \dfrac{\sinh \dfrac{mz}{l}}{\cosh m}}{\sec^2 m - \operatorname{sech}^2 m + 2P/\{m^2 (Q+Q')\}} \right\}$$

$$\times \left\{ V\tau \sin \frac{m^2 t}{\tau} - \frac{g\tau^2}{m^2} \cos \frac{m^2 t}{\tau} \right\} \text{..........} \text{(vi)}.$$

Equations (iv), (v) and (vi) form the complete analytical solution of the problem. See pp. 531—5.

[357.] Saint-Venant now treats other problems of *gradual impulse*, or as I should prefer to term it *non-impulsive resilience*. For example:

(*a*) A vertical bar of weight P terminally fixed and having a weight Q attached to its mid-point, is acted upon at that point by a constant horizontal force q. See pp. 535—8.

(*b*) The same bar is acted upon at its mid-point by a horizontal force $q =$ some function $f(t)$ of the time. Here Saint-Venant for his method of treatment appeals to the memoir of Duhamel cited in our Art. 903*. The solution contains an integral of the form $\int_0^t \cos \frac{m^2}{\tau} (t-\epsilon) f'(\epsilon)\, d\epsilon$. See pp. 538—40.

(*c*) The same bar is subjected to a sudden small but afterwards invariable horizontal displacement α of its mid-point. See pp. 540—2.

(*d*) The same bar is subjected to a small horizontal displacement α of its mid-point which is a function of the time: $\alpha = F(t)$. See pp. 542—3.

On pp. 543—7 Saint-Venant indicates another method of treating problems in non-impulsive resilience. For this he appeals to Phillips' memoir of 1864: see our account of it later.

[358.] The next problem investigated is a more important one and is thus stated: *Balancier de machine à vapeur oscillant autour d'une situation horizontale; sa flexion, sa vibration et sa résistance quand il est soumis à l'action et à l'impulsion graduelle de forces périodiquement variables s'exerçant sur ses extrémités par des bielles restant sensiblement verticales.*

I will indicate the method adopted by Saint-Venant. He supposes the arms of the beam each equal l, and that the forces applied to each extremity may be represented by a periodic term of the form $2Q \cos \Omega t$ which practically acts perpendicular to the beam. He justifies this assumption in the following manner:

Quant à la forme à assigner aux expressions des efforts verticaux q et q_1 exercés par les bielles, observons que si, du côté gauche, l'on appelle $-Q_1$ l'effort opérateur qu'exerce, tangentiellement à sa circonférence, une roue montée sur l'arbre du volant, et d'un rayon égal à la longueur r_1 de la manivelle, effort qui est rendu sensiblement constant lorsque le volant a un moment d'inertie de grandeur suffisante, et si Ω est la vitesse angulaire de la manivelle, on doit prendre, le temps t étant supposé compté à partir de l'instant où celle-ci est horizontale,

$$q_1 = -2Q_1 \cos \Omega t.$$

En effet q_1 devra avoir son maximum négatif pour l'angle $\Omega t = 0$, son maximum positif pour $\Omega t = \pi$: il devra être nul aux *points morts*, où $\Omega t = \pi/2$ et $3\pi/2$; enfin comme l'espace parcouru *verticalement* par le bouton de la manivelle pendant le temps dt est $\Omega r_1\, dt \cos \Omega t$, le travail de la force q_1 pendant le parcours d'une demi-circonférence est $r_1 \int_{-\pi/2}^{+\pi/2} q_1 \cos \Omega t\, d(\Omega t)$; intégrale qui, si l'on y fait $q_1 = -2Q_1 \cos \Omega t$ est justement égale à $-Q_1 \pi r_1$, c'est-à-dire au travail de la force tangentielle constante $-Q_1$; en sorte que l'expression posée pour q_1 est bien ce qu'il faut pour que cette force verticale entretienne le mouvement du mécanisme en fournissant, à la fin de chaque période, le travail opérateur qui a été dépensé pendant sa durée (p. 549).

If we accept these values for the forces acting on the beam we can easily state the analytical conditions of the problem.

$$\left.\begin{array}{l}\text{For the right arm:}\quad \tau^2\, d^2u/dt^2 + l^4 d^4u/dz^4 = 0,\\ \text{with}\quad d^2u/dz^2 = 0 \text{ and } E\omega\kappa^2\, d^3u/dz^3 + q = 0, \text{ when } z = l\end{array}\right\} \quad\text{.............. (i).}$$

$$\left.\begin{array}{l}\text{For the left arm:}\quad \tau^2 d^2u_1/dt^2 + l^4 d^4u_1/dz_1^4 = 0,\\ \text{with}\quad d^2u_1/dz_1^2 = 0 \text{ and } E\omega\kappa^2\, d^3u_1/dz_1^3 + q_1 = 0, \text{ when } z_1 = l\end{array}\right\} \quad\text{.... (ii).}$$

$$\left.\begin{array}{l}\text{When}\quad z = z_1 = 0, \text{ we must have } u = u_1 = 0\\ du/dz = -du_1/dz_1 \text{ and } d^2u/dz^2 = d^2u_1/dz_1^2\end{array}\right\} \quad\text{.......... (iii).}$$

The initial conditions will be of the following kind:

When $t = 0$, $u = \phi(z)$, $du/dt = \psi(z)$, $u_1 = \phi_1(z_1)$, $du_1/dt = \psi_1(z_1)$...(iv).

We put also: $\quad q = 2Q \cos \Omega t, \quad q_1 = 2Q_1 \cos \Omega t,$

$$\Omega = n^2/\tau \text{ where } \tau^2 = Pl^3/(2gE\omega\kappa^2).$$

Now Saint-Venant takes $u = v + U$; $u_1 = v_1 + U_1$, and chooses v and v_1 so that q and q_1 shall disappear from the equations (i) and (ii), and shall separately satisfy all the conditions but (iv). Substituting u and u_1 in the equations (i) to (iii) we find they remain the same with the suppression of q and q_1, that is: d^3U/dz^3 and d^3U_1/dz_1^3 vanish for $z = l$ and $z_1 = l$ respectively.

The solutions for U and U_1 take the usual forms in Z_m functions as coefficients in a series of circular functions of m^2t/τ, the characteristic equation being now

$$1 + \cos m \cosh m = 0 \quad \text{(v)}.$$

This is the well-known equation of Bernoulli and Euler: see our Arts. 49*, 64* and the footnote Vol. I. p. 50.

It is obvious that v and v_1 will be single terms in circular functions of Ωt or $\frac{n^2}{\tau} t$ the phase of the forced vibration, while U and U_1 will contain series in terms of $\frac{m^2}{\tau} t$ where m satisfies equation (v), or gives a free vibration. We have then to determine the constants A_m and B_m so as to satisfy the relations (iv), or so that $U = \phi(z) - v$, $dU/dt = \psi(z) - dv/dt$, when $t = 0$, with similar values for the quantities with subscript unity. The solution is thus completed. (pp. 547—553.)

[359.] The reader will remark on examining Saint-Venant's results, that if n be nearly $= m$, or the fly wheel rotate with nearly a natural period of the rod vibration, the displacement due to that natural vibration will become excessive and the danger of the beam breaking will be great. This will occur when

$$\Omega = \frac{m^2}{\tau} = m^2 \sqrt{\frac{2gE\omega\kappa^2}{Pl^3}}.$$

Let $p =$ number of revolutions of the fly wheel per second, $= \Omega/2\pi$. Then there will be great danger when:

$$p = \frac{m^2}{2\pi} \sqrt{\frac{2gE\omega\kappa^2}{Pl^3}} \quad \text{(vi)}.$$

Saint-Venant in a footnote gives the following first 8 values of m^2:

3·557, 61·70, 199·8, 417, 712·9, 1088·3, 1542·1, 2075·1;

hence, since slackening speed would be dangerous, if p had a value lying between those obtained from (vi) by substituting any two values of m^2, we have the safe maximum number of rotations per second of the fly wheel given by

$$p < \frac{3{\cdot}557}{2\pi} \sqrt{\frac{2g\,E\omega\kappa^2}{Pl^3}}.$$

This seems to me an important condition[1]. I am not aware

[1] A similar condition ought also to be satisfied between the number of rotations of the fly wheel and the least free period of stretch vibrations in the connecting rod of an ordinary engine.

whether it has been previously noticed, or how far the dimensions of the beams of ordinary beam-engines ensure its fulfilment. We can throw it into a simpler form. Let f = the deflection of either extremity of the beam subject only to its own weight, then

$$f = \frac{Pl^3}{8E\omega\kappa^2},$$

or

$$p < \frac{3 \cdot 557}{4\pi}\sqrt{\frac{g}{f}} < \cdot 272\sqrt{\frac{g}{f}}.$$

[360.] Saint-Venant does not draw any numerical conclusions from his results, which seem to me to suggest several points of importance, but only remarks finally:

Nous n'insisterons pas sur la solution, dont nous croyons avoir posé les bases, de ce problème complexe et délicat, solution qui, une fois développée, fournira la connaissance des plus grandes dilatations à contenir dans de justes limites, en réglant les dimensions de cet organe de mécanisme, soumis à des forces toujours variables, le faisant fléchir et vibrer alternativement dans deux sens opposés (p. 553).

[361.] We now pass to that portion of Saint-Venant's work which is peculiarly characteristic of the man, namely to the practically important numerical calculation of the results given in the previous articles. This occupies pp. 553—576 (§§ 32—42). The appalling amount of work that lies behind the numbers given can only be appreciated by those who have attempted similar calculations. The graphical representation of the results, although the plates have been long engraved, has not yet been published (see footnote p. 557)[1]. The plaster model referred to in our Art. 105 will be found, however, of considerable service as offering a concise picture of the whole motion in a particular and most important case.

[362.] Saint-Venant treats in §§ 32—5 the problem of the doubly supported bar centrally struck: see our Arts. 104 and

[1] I much regret that it has been settled that these plates shall *not* be published, Saint-Venant at a date later than the footnote of 1883 having expressed an opinion that the curves ought to be plotted out for more frequent values of t/τ and z/l, as well as for a wider range of the ratio P/Q. It is to be hoped, having regard to the practical importance of the problem, that some one will be found willing to undertake the labour of the requisite numerical calculations.

346—8. He begins by tabulating the first seven values of m obtained from the characteristic equation:

$$m(\tan m - \tanh m) = 2P/Q,$$

when P/Q is very small, equals $\frac{1}{10}$, $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, or is very great (p. 554).

Then for the three cases $P/Q = \frac{1}{2}$, 1 and 2 he has calculated up to six terms in m the value of the amplitudes in $u/(V\tau)$ for each component harmonic at the points $z/l = \cdot 2$, $\cdot 4$, $\cdot 6$, $\cdot 8$, and 1. He has thus been able to trace the curves having the several terms of $u/(V\tau)$ for ordinate and t/τ for abscissa. Corresponding ordinates added together gave the total deflection for various values of z/l plotted to a time base. These curves were traced from $t/\tau = \cdot 05$ to $2\cdot 25$, except in the third case ($P/Q = 2$) when they were only taken to $t/\tau = 1\cdot 9$. Unfortunately we have not these curves to examine, but the following remarks of Saint-Venant sufficiently characterise the physical nature of the impulse:

Ces cinq courbes partant du même point ($u = 0$, $t = 0$) ne reviennent, au bout de ces temps, couper l'axe des abscisses $u/(V\tau) = 0$, qu'en des points légèrement différents les uns des autres, ce qui montre qu'à aucun instant la barre ne retourne exactement à son état primitif. Ces courbes, représentant la loi et la suite du mouvement de chacun des cinq points, sont fort sinueuses; cela vient de ce que le mouvement résulte de la superposition de vibrations ayant des durées et des amplitudes de moins en moins grandes, dont chacune a son *rebond* bien avant celui de l'oscillation principale provenant du premier terme du Σ ou de la valeur m_0 de m.

Toutes ces courbes serpentantes sont, pour $t = 0$, ou à l'origine, tangentes à l'axe des abscisses, avec lequel, même, elles se confondent dans de très petites étendues, parce que l'ébranlement ne se transmet pas instantanément du point milieu aux points d'appui.

Il y a exception, bien entendu, pour les courbes relatives à $z = l$. La tangente y fait un angle demi-droit avec l'axe; et cela devait être, car, à l'instant initial, les vitesses ne sont nulles qu'en exceptant le point milieu qui reçoit le choc, et où $du/dt = V$; ce qui donne bien 1 pour la tangente trigonométrique, quotient $d(u/V\tau)$ par $d(t/\tau)$, de l'angle fait avec l'axe des abscisses par le premier élément de la courbe représentative du mouvement du point milieu.

Ces courbes, pour des points proches des appuis, s'élèvent même au-dessus de l'axe $u = 0$ des abscisses, c'est-à-dire que, par une sorte de réaction ou de rebond qui suit de près un affaissement imperceptible, les u sont négatifs. (See pp. 557 and 889.)

The last remark should be compared with that of Stokes' in another case of resilience: see our Art. 1282*.

[363.] In § 33 Saint-Venant describes how he has traced the form taken by the rod at different intervals of time from $t = \cdot 1\tau$ up to $t = 2\cdot 25\tau$. From these curves he has deduced by graphical measurement the maximum curvatures and the times at which they occur. I reproduce some of his results in the accompanying table.

Resilience of a simple-supported beam, struck transversely.

When $P/Q=$	1/4	1/2	1	2	4
Maximum deflection	$1\cdot 091\, V\tau$	$\cdot 739\, V\tau$	$\cdot 477\, V\tau$	$\cdot 297\, V\tau$	$\cdot 167\, V\tau$
at about $t=$	$1\cdot 92\tau$	$1\cdot 34\tau$	$1\cdot 18\tau$	$\cdot 82\tau$	$\cdot 78\tau$
Maximum curvature	—	$2\cdot 60\, V\tau/l^2$	$1\cdot 75\, V\tau/l^2$	$1\cdot 30\, V\tau/l^2$	—
at about $t=$	—	$\begin{cases}1\cdot 25\tau\\1\cdot 65\tau\end{cases}$	$1\cdot 20\tau$	$\begin{cases}\cdot 75\tau\\1\cdot 1\tau\end{cases}$	—

It will be noted that the instant of maximum deflection and that of maximum curvature do not coincide. Saint-Venant remarks that at each instant of time the maximum curvature is not central, although the maximum of the maxima for the various times as above tabulated is central.

The maximum stretch at any instant $= h/R$, where h is the distance of the 'outer fibre' from the neutral axis and $1/R$ is the curvature; this must be less than T_0/E, where T_0 gives the fail-limit: see our Art. 173. Hence our condition for non-failure is

$$T_0/E > h\beta\, V\tau/l^2,$$

or

$$V < \frac{1}{\beta}\frac{l^2 T_0}{Eh\tau},$$

where $\tau^2 = \dfrac{Pl^3}{2gE\omega\kappa^2}$, and β must be put equal to $2\cdot 60$, $1\cdot 75$ or $1\cdot 30$ according as P/Q equal $\frac{1}{2}$, 1 or 2.

Saint-Venant throws this into a slightly different form. By substituting τ and squaring we find:

$$\frac{QV^2}{2g} < \epsilon\,.\,\frac{T_0^2}{E}\,2l\omega\,\frac{\kappa^2}{h^2}\,.$$

Here T_0^2/E is the modulus of resilience, $2l\omega$ is the volume of the beam, κ^2/h^2 is in general a number independent of the linear dimensions of the cross-section, i.e. the same for all similar beams, and $\frac{\epsilon}{3} = \frac{1}{6\beta^2}\frac{Q}{P}$, and so takes the values ·04928, ·05442, and ·04933 for the three cases respectively, so that for an approximation we might take $\epsilon = \cdot 15$ for these cases. This constancy of ϵ would give Young's Theorem which was established by neglecting the inertia of the bar ($\epsilon = \frac{1}{6}$), but, as Saint-Venant rightly observes, sufficient cases have not yet been calculated to allow a safe empirical formula to be proposed.

The reader should note, however, the contents of our Art. 371 (iv) as modifying the above results.

[364.] Some remarks of Saint-Venant on p. 627 bearing on the results of the previous article are so suggestive for directions of further physical research that we cite them here in the hope that some one may ultimately be induced to undertake the needful investigations:

Plusieurs questions, du reste, se présentent, dont l'analyse ne peut encore tirer, des faits actuellement connus, une solution suffisante.

1°. Doit-on (comme ont fait les auteurs qui ont traité les problèmes de résistance vive par première approximation) regarder la limite T_0/E des dilatations statiques ou permanentes non dangereuses des fibres, comme s'appliquant aux dilatations dynamiques ne durant qu'une fraction de seconde, et qu'un même choc ne produit qu'une seule fois dans toute leur grandeur; ou bien peut-on, sans péril, en adopter une moins élevée.

2°. Doit-on, dans le calcul (numérique ou graphique) de la plus grande courbure, ajouter, comme nous avons fait [Art. 363], à ce qui vient de la vibration principale et visible, donnée par le premier terme, en m_0, du Σ, ce que fournissent passagèrement les vibrations secondaires, tertiaires, etc., représentées par les autres termes, et dont la durée périodique est incomparablement plus petite ; ou bien peut-on négliger, comme sans danger, les surcroîts de dilatations de fibres qu'elles produisent par instants; ce qui reviendrait à s'en tenir aux valeurs[1] de $1/\rho$, en les affectant, tout au plus, de coefficients de sécurité ou de précaution, étrangers au calcul des vibrations accessoires?

3°. Y a-t-il, de la part des vibrations élastiques de peu de durée et d'amplitude, et vu le seul fait de leur fréquente répétition, une sorte particulière de danger, comme serait celui de détruire le *nerf* du fer forgé ou laminé, en le disposant, comme le feraient de fortes vibrations

[1] Saint-Venant here gives a reference to the equations he has given on p. 626, connecting *statical* curvature with statical deflection.

calorifiques ou une sorte de *fusion*, à revenir de l'état fibreux ou à particules entrelacées, à l'état cristallin ou grenu?

Des expériences, dont il est difficile de tracer le programme, mais où pourra jouer un rôle essentiel le mesurage de ces déformations persistantes regardées comme annonçant des commencements d'énervation et de désagrégation, seront nécessaires pour renseigner là-dessus la théorie qui devra, quels qu'en soient les résultats, se bien garder d'abdiquer son rôle et de renoncer aux considérations et patients calculs dont nous avons, à l'instar de nos maîtres, tâché de donner quelques specimens.

The experiments on repeated load to which we shall refer later in this volume have thrown light on some at least of Saint-Venant's problems.

[365.] Saint-Venant passes in § 35 to the problem of our Art. 355 with $Q' = 0$. He remarks that the maximum value of the second part of u (Equation vi) treated as *consisting only of the first term* will be reached when

$$\tan \frac{m^2 t}{\tau} = -\frac{m^2 V\tau}{g\tau^2}.$$

He thus deduces for the time-terms' bracket the value

$$\frac{1}{m^2}\sqrt{(g\tau^2)^2 + (m^2 V\tau)^2}.$$

Hence the total deflection f produced by the blow is given by:

f = maximum of u − initial deflection due to weight of beam,

$$= \frac{Ql^3}{6E\omega\kappa^2} + \frac{4}{m_0^4 + \dfrac{Qm_0^6}{2P}(\sec^2 m_0 - \operatorname{sech}^2 m_0)}\sqrt{\left(\frac{Pl^3}{2E\omega\kappa^2}\right)^2 + (m_0^2 V\tau)^2}.$$

Here m_0 is the first root of the characteristic Equation (v) of Art. 356, or since $Q' = 0$, of the Equation of Art. 362. Saint-Venant calculates on p. 562 the value of the coefficient of the radical and finds it has almost exactly for values of $P/Q = \frac{1}{4}, \frac{1}{2}, 1, 2, 4$ the same value, namely $Q/(3P)$, as when P/Q is extremely small.

Hence $$f = f_s + \sqrt{f_s^2 + f_D^2},$$

where f_s is the statical deflection and

$$f_D = \frac{Qm_0^2}{3P} V\tau \qquad \text{(i)}$$

is the dynamical deflection of Art. 363 to a first approximation.

If $V = 0$, we have non-impulsive resilience, and $f = 2f_s$, a theorem of Young's.

[366.] In the next sections (§§ 36—7), Saint-Venant shews that the solution obtained on the hypothesis of Cox, that the form of the beam is at each instant of the impact the same as it would be under the same *statical* central deflection, gives a close approximation to the maximum dynamic deflection. Now Cox has shewn that

$$f_D = \frac{V}{\sqrt{\frac{6gE\omega\kappa^2}{Ql^3}\left(1+\frac{17}{35}\frac{P}{Q}\right)}}$$

(see our Vol. I. p. 894, (*b*) with proper change of notation).

Equating this to $f_D = \frac{Qm_0^2}{3P} V_T$ (see Art. 365, Eqn. (i)), we have

$$m_0^4 = \frac{3P}{Q} \bigg/ \left(1+\frac{17}{35}\frac{P}{Q}\right),$$

which is Saint-Venant's second approximation to the value of m_0. It appears from his work that Cox's result for the *central maximum deflection* is accurate when we neglect m_0^8 (p. 568).

On p. 570 we have the maximum deflections calculated for the five typical cases:

	$P/Q=$	1/4	1/2	1	2	4
$\frac{f_D}{V_T}$	by Saint-Venant's series,	1·091	·739	·477	·297	·167
	by Cox's formula, or $f_D/V_T = \frac{1}{\sqrt{\frac{3P}{Q}\left(1+\frac{17}{35}\frac{P}{Q}\right)}}$	1·0904	·7375	·4737	·2908	·1683

We see that at any rate for these cases *Cox's formula gives the deflection with all the accuracy needful in practice.*

[367.] Saint-Venant next proceeds to a second approximation in other cases of resilience, i.e. he investigates the values of γ the *mass-coefficient of resilience*, see our Vol. I., p. 894 (*b*).

His § 40, 2° = our Vol. I., p. 895, c. (i).
„ § 40, 3° = „ „ p. 895, c. (ii).
„ § 40, 4° = „ „ p. 894, Eqn. (i).

For the case referred to in our Art. 355,

$$m_0^4 = \frac{3P}{Q+Q'} \bigg/ \left(1+\frac{17}{35}\frac{P}{Q+Q'}\right).$$

The case given in our Vol. I., p. 896, (iii), Saint-Venant does not appear to have considered.

In a footnote he remarks that the second approximation will be far from exact in cases like those of Arts. 353 and 354 where the bar is free or pivoted at one point only.

[368.] Saint-Venant next proceeds to obtain Cox's formula by an elementary method. In a long footnote he gives the history and a proof of the principle of virtual shifts as applied to impulsive forces (pp. 577—82). His method is more general and simpler than Cox's, and as it gives a general expression for the value of the mass-coefficient γ, we indicate it here: see his pp. 578—87:

Let V be the initial impact-velocity of the weight Q; let V_1 be the final impact velocity, or the velocity attained by Q when the beam begins to bend, let v_1 be the velocity of any point of the beam immediately after the impact, so that $v_1 = V_1$ at the mid-point. Take the shifts at the instant when the bending effect begins as the virtual shifts, then:

$$\left(\frac{Q}{g} V - \frac{Q}{g} V_1\right) V_1 dt - \int \frac{dP}{g} v_1 . v_1 \, dt = 0,$$

the integral extending along the length of the beam. Dividing by $Q V_1^2 \dfrac{dt}{g}$, we have

$$V/V_1 = 1 + \frac{P}{Q} \int \left(\frac{v_1}{V_1}\right)^2 \frac{dP}{P},$$

or
$$\left.\begin{aligned} V_1 &= \frac{V}{1 + \gamma P/Q} \\ \gamma &= \int \left(\frac{v_1}{V_1}\right)^2 \frac{dP}{P} \end{aligned}\right\} \quad \text{(i)}.$$
where

The determination of γ thus depends entirely on the relation we choose between v_1 and V_1. Cox's assumption is that: the relation between the statical shifts at the centre and any other point holds continuously during the motion. Thus if $u = U_1 \phi(z)$ be the relation,

$$\dot{u} = \dot{U}_1 \phi(z), \text{ or } v_1 = V_1 \phi(z).$$

This gives
$$\gamma = \int \{\phi(z)\}^2 \frac{dP}{P} \quad \text{(ii)}.$$

Now the total kinetic energy of the system after impact must be

$$= \frac{Q}{g} \frac{V_1^2}{2} + \int \frac{v_1^2}{2} \frac{dP}{g} = \frac{Q V_1^2}{2g} \left(1 + \gamma \frac{P}{Q}\right) = \frac{Q V^2}{2g} \Big/ \left(1 + \gamma \frac{P}{Q}\right).$$

This must be equal to the maximum strain-energy of the system, which is always of the form $af_D \times \frac{f_D}{2}$, a being a constant depending on the beam and f_D the maximum deflection. Thus we arrive at

$$f_D = V\sqrt{\frac{Q}{ag(1+\gamma P/Q)}} \quad \text{.....................(iii).}$$

This is Cox's formula: see our Arts. 1435*—7* and Vol. I., pp. 894—6.

If f_s be the statical deflection due to Q, $Q = af_s$ and

$$f_D = V\sqrt{\frac{f_s}{g(1+\gamma P/Q)}} \quad \text{.....................(iv).}$$

[369.] Saint-Venant adds to Cox's treatment the consideration of the approximate periodic time.

The body moved has the 'reduced total mass' $\frac{Q+\gamma P}{g}$, and the resistance to motion is au_0, where u_0 is the central shift at time t.

Hence we have $$\frac{Q+\gamma P}{g}\frac{d^2u_0}{dt^2} = -au_0 = -\frac{Q}{f_s}u_0,$$

or $$u_0 = A\sin(\beta t + B), \text{ where } \beta = \sqrt{\frac{g}{f_s}\frac{1}{1+\gamma P/Q}}.$$

But when $t = 0$, $u_0 = 0$ and $\dot{u}_0 = V_1$.

Thus finally

$$u_0 = V\sqrt{\frac{f_s}{g}\frac{1}{1+\gamma P/Q}}\sin\left(\sqrt{\frac{g}{f_s}\frac{1}{1+\gamma P/Q}}\cdot t\right) \quad \text{...........(v).}$$

[370.] (*a*) On pp. 587—589 the values of γ are obtained by Cox's method for the examples referred to in our Art. 367.

(*b*) On pp. 589—90 we have the case of a beam whose length exceeds the distance between the two points of support symmetrically placed. If P be the weight of beam in the span and P' of the total projecting portions we find

$$\gamma = \tfrac{17}{35} + \tfrac{3}{4}\left(\frac{P'}{P}\right)^2.$$

(*c*) On pp. 590—594 Saint-Venant treats the important case of resilience for the "solid of equal resistance," i.e. when the cross-sections are rectangles of equal breadths and of heights given by parabolic ordinates. He deals with this problem by two methods and finds in both cases that $\gamma = \frac{2}{4}\frac{3}{2}$. He remarks in a footnote that the end sections which are of course in practice not of zero height, must be calculated by the methods of the memoir on flexure: see our Art. 69. But the addition of this material only introduces into γ a term of the order $\left(\frac{\text{height of end-section}}{\text{height of mid-section}}\right)^7$ which is negligible.

(*d*) On pp. 595—597 Saint-Venant deduces the result of our Art. 365 by Cox's method.

[371.] Leaving on one side for a moment Saint-Venant's §§ 52—55 we observe the following points in the concluding pages of this long *Note*:

(i) pp. 620—623. An examination of the results of the *Iron Commissioners' Report* and Hodgkinson's experiments: see our Arts. 943* and 1409—10*. This amounts to little more than the remark that Hodgkinson's $\frac{1}{2}$ is almost equal to the theoretical value $\frac{17}{35}$ of γ, and the statement that the values of the modulus obtained by applying the resilience formulae to 67 experiments agree sufficiently well among themselves.

(ii) Saint-Venant remarks that $f_D = V\sqrt{\frac{f_s}{g}\frac{1}{1+\gamma P/Q}}$ can be applied to a variety of cases of impact, as those of carriage springs, etc.; the value of γ being known $\left\{\text{i.e. } \int\left(\frac{v_1}{V_1}\right)^2\frac{dP}{P}\right\}$, so soon as we have assumed v_1/V_1 to have the ratio of the corresponding statical deflections (p. 624). At the same time the method of vibrations involving the transcendental series ought to be used to control this result wherever it is possible (p. 625).

(iii) *The values obtained by Cox's method for the maximum curvature and so for the maximum stretch are not sufficiently exact*, and we must have recourse to the transcendental series or the numbers given in our Art. 363. Thus in the case of a simply supported beam centrally struck we should have by Cox's method $1/\rho = 3f_D/l^2$, but the values deduced from the Table in our Art. 363 give

$$3f_D/l^2 \times \begin{Bmatrix}1\cdot183\\1\cdot252\\1\cdot486\end{Bmatrix} \text{ according as } P/Q = \begin{Bmatrix}\frac{1}{2}\\1\\2\end{Bmatrix}.$$

Saint-Venant gives an empirical formula for these *three* cases on p. 627, but a better form (error $< \frac{1}{16}$) is given in the *Changements et Additions* p. 895, namely:

$$1/\rho = \sqrt{3}\frac{1}{\sqrt{E\omega\kappa^2 l}}\sqrt{\frac{QV^2}{2g}}.$$

This gives the same condition of resilience as the $\epsilon = \frac{1}{6}$ of our Art. 363.

It is noteworthy that in reality Young's Theorem is much more nearly fulfilled than would appear from the application of Cox's method: see our Vol. I., p. 895.

(iv) A second interesting point is raised in the *Changements et Additions* p. 896. Saint-Venant remarks that the formulae given are based upon the supposition that the disturbance due to the blow has had time to be reflected several times from the points of support before the moment of maximum flexure. They cease to be applicable when the bar is very long, and Q a very small weight with a very great velocity of impact:

En effet, préalablement à toute propagation, une flexion brusquement produite à l'endroit du choc peut engendrer des dilatations dangereuses, dépendant de la seule vitesse V et nullement du poids heurtant Q.

Let Ω = velocity of propagation of sound along the rod, or

$$\Omega^2 = \frac{E}{P/(2l\omega g)}.$$

We easily deduce $\dfrac{V\tau}{l^2} = \dfrac{V}{\Omega\kappa}$, where $\tau^2 = Pl^3/(2gE\omega\kappa^2)$.

The corresponding maximum values of the stretches are by Art. 363:

For	$P/Q = \frac{1}{2}$	$P/Q = 1$	$P/Q = 2$
$s_0 =$	$2{\cdot}60\frac{h}{\kappa}\frac{V}{\Omega}$	$1{\cdot}75\frac{h}{\kappa}\frac{V}{\Omega}$	$1{\cdot}30\frac{h}{\kappa}\frac{V}{\Omega}$

Now Boussinesq has shewn that the stretch produced in the element struck at the first instant of the blow has for magnitude, whatever be the relation between P and Q:

$$s_0 = \frac{h}{\kappa}\frac{V}{\Omega}.$$

Hence if we find that value for P/Q (say, n) for which the numerical coefficient of $\frac{h}{\kappa}\frac{V}{\Omega}$ is sensibly unity, we may say that the maximum stretch for that and all other larger values of P/Q is given by the expression $\frac{h}{\kappa}\frac{V}{\Omega}$, and takes place in the first instant of the impact.

See the Note in the *Comptes rendus* 1882, p. 1044, or Boussinesq, *Application des Potentiels à l'étude...du mouvement des solides élastiques*, p. 486.

From some slight calculations I have made I believe this result will be reached when P/Q lies between 2·5 and 3. If this be true, it very much limits the range within which there is any necessity to apply the transcendental series to ascertain the curvature and so the condition of failure. We may then, I think, say that after $P/Q = 2{\cdot}5$, the maximum-stretch is always given by the formula $\frac{V}{\Omega}\frac{h}{\kappa}$ or is independent of the mass of Q.

[372.] We must now return to pp. 597—619 of Saint-Venant's note which we have omitted above. They deal with *Willis' Problem* or the resilience of a horizontal beam subjected to a travelling load: see our Arts. 1417*—1422*. We shall include under our discussion the memoirs of Phillips[1] and Renaudot[2], because these writers have made mistakes in their analysis, which have been rectified by Saint-Venant. With Saint-Venant's additions and rectifications we shall thus be able to give the reader a more complete view of the advance made by the problem since the memoir of Stokes: see our Arts. 1276*—1291*.

[373.] We will first give the equations for the complete problem as propounded by Phillips. Let P be the weight, $2l$ the length, $E\omega\kappa^2$ the rigidity of the beam, u the shift to the right and u_1 to the left of the travelling load Q (distant $x = Vt$ from the right-hand terminal) of points distant z and z_1 from right and left-hand ends of the beam. We shall suppose the beam simply supported.

[1] *Calcul de la résistance des poutres droites telles que les ponts, etc. sous l'action d'une charge en mouvement. Annales des mines*, t. VII., pp. 467—506, 1855.

[2] *Etude de l'influence des charges en mouvement sur la résistance des ponts métalliques à poutres droites. Annales des ponts et chaussées*, t. I. 4e série, pp. 145—204, 1861.

Then for the beam we have

$$\left.\begin{aligned} E\omega\kappa^2 \frac{d^4u}{dz^4} - \frac{P}{2l} &= -\frac{P}{2lg}\frac{d^2u}{dt^2} = -\frac{P}{2l}\frac{V^2}{g}\frac{d^2u}{dx^2} \\ E\omega\kappa^2 \frac{d^4u_1}{dz_1^4} - \frac{P}{2l} &= -\frac{P}{2lg}\frac{d^2u_1}{dt^2} = -\frac{P}{2l}\frac{V^2}{g}\frac{d^2u_1}{dx^2} \end{aligned}\right\} \text{...........(i)},$$

$$\left.\begin{aligned} u = 0, \text{ and } \frac{d^2u}{dz^2} = 0 \text{ when } z = 0 \\ u_1 = 0, \text{ and } \frac{d^2u_1}{dz_1^2} = 0 \text{ when } z_1 = 0 \end{aligned}\right\} \text{..................(ii)};$$

and for the conditions at the load:

$$(u)_x - (u_1)_{2l-x} = 0,\ \left(\frac{du}{dz}\right)_x + \left(\frac{du_1}{dz_1}\right)_{2l-x} = 0,\ \left(\frac{d^2u}{dz^2}\right)_x - \left(\frac{d^2u_1}{dz_1^2}\right)_{2l-x} = 0 \text{ ... (iii)},$$

together with:

$$-E\omega\kappa^2\left\{\left(\frac{d^3u}{dz^3}\right)_x + \left(\frac{d^3u_1}{dz_1^3}\right)_{2l-x}\right\} - Q = -\frac{Q}{g}\frac{d^2y}{dt^2}, \text{ where } y = (u)_{z=x=Vt} \text{...(iv)}.$$

No general solution has yet been found for these equations. But omitting the condition of *initial zero velocities* it is possible to satisfy all the other Equations (i) to (iv) by algebraic expressions in z and x, when we neglect in successive approximations successive powers of a certain quantity which is small in all practical applications. Further, it is possible to add vibratory parts to the algebraic solution which satisfy very approximately the initial conditions (pp. 599 and 891).

[374.] Saint-Venant's method of solution differs from that of Stokes and includes the effect of the inertia of the beam. We will indicate its stages.

1*st Approximation*. Let us neglect the terms in V^2 in Equations (i) to (iv), or find only the statical shifts for the load Q at a point x. We have:

$$\left.\begin{aligned} u' &= Q\frac{2l-x}{12lE\omega\kappa^2}\left[(4lx - x^2)z - z^3\right] + P\frac{8l^3z - 4lz^3 + z^4}{48lE\omega\kappa^2} \\ u_1' &= Q\frac{x}{12lE\omega\kappa^2}\left[(4l^2 - x^2)z_1 - z_1^3\right] + P\frac{8l^3z_1 - 4lz_1^3 + z_1^4}{48lE\omega\kappa^2} \end{aligned}\right\} \text{.....(v)}.$$

2*nd Approximation*.

Now let $\dfrac{1}{\beta} = \dfrac{2QV^2l}{3gE\omega\kappa^2}$, or β is the same as Stokes' β of our Art. 1278* (where c is written for our present l), then in practice $1/\beta$ is always $< 1/12$ or even than $1/20$ and is the small quantity of our approximations. In the above equations we shall replace

$$\frac{PV^2}{2lgE\omega\kappa^2} \text{ by } \frac{1}{\beta}\frac{3P}{4Ql^2}, \text{ and } \frac{QV^2}{E\omega\kappa^2 g} \text{ by } \frac{1}{\beta}\frac{3}{2l}.$$

Let us assume

$$u = u' + \frac{1}{\beta} U, \quad u_1 = u_1' + \frac{1}{\beta} U_1,$$

substitute and neglect $\left(\frac{1}{\beta}\right)^2$. We find from Equations (i) by dividing out by $1/\beta$:

$$\left.\begin{aligned} \frac{d^4U}{dz^4} &= \frac{3P}{8l^3E\omega\kappa^2}(2l-x)z = \frac{3P}{8l^3E\omega\kappa^2}x_1 z, \text{ if } x_1 = 2l - x, \\ \frac{d^4U_1}{dz_1^4} &= \frac{3P}{8l^3E\omega\kappa^2}xz_1 \end{aligned}\right\} \ldots \text{(vi)}.$$

Equations (ii) now become:

$$\left.\begin{aligned} U = 0, \text{ and } \frac{d^2U}{dz^2} = 0 \text{ when } z = 0, \\ U_1 = 0, \text{ and } \frac{d^2U_1}{dz_1^2} = 0 \text{ when } z_1 = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots \text{(vii)}.$$

Integrating (vi) we have

$$\left.\begin{aligned} U &= \frac{3Px_1}{8l^3E\omega\kappa^2}\left\{\frac{z^5}{5\,!} + \frac{Cz^3}{3\,!} + Dz\right\} \\ U_1 &= \frac{3Px}{8l^3E\omega\kappa^2}\left\{\frac{z_1^5}{5\,!} + C_1\frac{z_1^3}{3\,!} + D_1 z_1\right\} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots \text{(viii)}.$$

These satisfy equations (vii).
It remains to determine C, D, C_1, D_1, by Equations (iii) and (iv).
But they become:

$$\left.\begin{aligned} (U)_{z=x} = (U_1)_{z_1=x_1}, \quad \left(\frac{dU}{dz}\right)_{z=x} + \left(\frac{dU_1}{dz_1}\right)_{z_1=x_1} = 0, \\ \left(\frac{d^2U}{dz^2}\right)_{z=x} = \left(\frac{d^2U_1}{dz_1^2}\right)_{z_1=x_1} \end{aligned}\right\} \ldots\ldots\ldots \text{(ix)}.$$

Further, (iv) may be written

$$\left(\frac{d^3U}{dz^3}\right)_{z=x} + \left(\frac{d^3U_1}{dz_1^3}\right)_{z_1=x_1} = \frac{Q\beta}{E\omega\kappa^2 g}\frac{d^2y}{dt^2} = \frac{3}{2l}\frac{d^2y}{dx^2}.$$

Now $y = u_1$ when $z = x_1$, or after a short reduction

$$y = \frac{Qx^2x_1^2}{6lE\omega\kappa^2} + \frac{Pxx_1(4l^2 + xx_1)}{48lE\omega\kappa^2}.$$

Since $x_1 = 2l - x$, we have: $\frac{d(xx_1)}{dx} = 2(l - x) = x_1 - x$, and thus find

$$\frac{3}{2l}\frac{d^2y}{dx^2} = \frac{1}{E\omega\kappa^2}\left\{2Q - \frac{3xx_1}{l^2}\left(Q + \frac{P}{8}\right)\right\}.$$

Hence

$$\left.\begin{aligned}\left(\frac{d^3U}{dz^3}\right)_x + \left(\frac{d^3U_1}{dz_1^{\,3}}\right)_{x_1} &= \frac{1}{E\omega\kappa^2}\left\{2Q - \frac{3xx_1}{l^2}\left(Q + \frac{P}{8}\right)\right\}\\ &= \frac{Q}{E\omega\kappa^2 l^2}(2l^2 - 6lx + 3x^2) + \frac{3P}{8E\omega\kappa^2 l^2}(-2lx + x^2)\end{aligned}\right\}\text{...(x).}$$

This result does not agree with Saint-Venant's on p. 606 (Equation (t_9)) but it will do with that in the Errata, p. 900, if the coefficients of the brackets of the latter are inverted.

From the third Equation of (ix) and from (x) we easily find with the help of (viii) the following equations to determine C, C_1,

$$\frac{x^2}{6} + C = \frac{x_1^{\,2}}{6} + C_1\,;\quad x_1C + xC_1 = -2lxx_1 + \frac{8Ql}{3P}(2l^2 - 3xx_1)\ldots\text{(xi).}$$

This differs in the sign of the bracket in the second equation from Saint-Venant's Equation (x_9) on p. 606.

Solving (xi) we have

$$\left.\begin{aligned}C &= -\frac{x}{6}(x + 5x_1) + \frac{4Q}{3P}(2l^2 - 3xx_1)\\ C_1 &= -\frac{x_1}{6}(x_1 + 5x) + \frac{4Q}{3P}(2l^2 - 3xx_1)\end{aligned}\right\}\ldots\ldots\ldots\ldots\text{(xii).}$$

[375.] We can easily test these results. The bending-moment

$$M = -E\omega\kappa^2\frac{d^2u}{dz^2} = -E\omega\kappa^2\frac{d^2u'}{dz^2} - \frac{E\omega\kappa^2}{\beta}\frac{d^2U}{dz^2}$$

$$= \frac{Qx_1z}{2l} + \frac{P(2l - z)z}{4l} - \frac{3Px_1}{8l^3\beta}\left\{\frac{z^3}{6} - \frac{x}{6}(x + 5x_1)z + \frac{4Q}{3P}(2l^2 - 3xx_1)z\right\}\ldots\text{(xiii}^{\text{a}}\text{).}$$

Put $z = l$, and $x = x_1 = l$, and we find

$$M_l = \frac{Ql}{2}\left(1 + \frac{1}{\beta}\right) + \frac{Pl}{4}\left(1 + \frac{5}{4}\frac{1}{\beta}\right)\ldots\ldots\ldots\ldots\ldots\text{(xiii}^{\text{b}}\text{).}$$

This is Saint-Venant's result (z_9) on p. 607.

It gives the bending moment at the centre when the train is passing that point. If we put $P = 0$ or neglect the weight of the beam, we have Stokes' result. Phillips finds by overlooking several terms and by means of a longer analysis $3/(4\beta)$ in the second bracket.

[376.] We are now in a position to find D and D_1 and so determine the deflection at any point. From Equations (ix) I have calculated the following value for D:

$$D = \frac{1}{360}\{x_1x(58x^2 + 8x_1^{\,2} + 92xx_1) + 7x^4\} + \frac{2Q}{9P}(3xx_1 - 2l^2)x(x + 2x_1)\ldots\text{(xiii).}$$

Equations (xii) and (xiii) determine C, D, and so U from (viii).

Adding $\frac{1}{\beta} U$ to u' of (v) we obtain the complete solution to this degree of approximation. We may write down the complete value thus obtained, u_1 being obviously given by interchanging x_1 with x and z_1 with z:

$$u = \frac{Qx_1}{12lE\omega\kappa^2}\{x(2x_1+x)z - z^3\} + \frac{P}{48lE\omega\kappa^2}(8l^3z - 4lz^3 + z^4)$$

$$+ \frac{1}{\beta}\frac{3Px_1}{8l^3E\omega\kappa^2}\left[\frac{z^5}{120} + \frac{z^3}{6}\left\{-\frac{x}{6}(x+5x_1) + \frac{4Q}{3P}(2l^2 - 3xx_1)\right\}\right.$$

$$\left. + \frac{z}{360}\{x_1x(58x^2+8x_1^2+92xx_1)+7x^4\} + \frac{2z}{9}\frac{Q}{P}(3xx_1-2l^2)x(x+2x_1)\right] \ldots\ldots(\text{xiv}^a).$$

This embraces both Saint-Venant's forms (ω) and (ω'), p. 615 g, and I have tested them, and find they agree with this result.

If $x = x_1 = z = l$ we find:

$$u_l = \frac{Ql^3}{6E\omega\kappa^2} + \frac{5Pl^3}{48E\omega\kappa^2} + \frac{1}{\beta}\left\{\frac{Ql^3}{6E\omega\kappa^2} + \frac{9}{80}\frac{Pl^3}{E\omega\kappa^2}\right\}$$

$$= \frac{Ql^3}{6E\omega\kappa^2}\left(1+\frac{1}{\beta}\right) + \frac{5Pl^3}{48E\omega\kappa^2}\left(1+\frac{27}{25}\frac{1}{\beta}\right) \ldots\ldots\ldots\ldots(\text{xiv}^b).$$

If we put $P = 0$, we obtain Stokes' result: see our Art. 1287*. It will be observed that these expressions for the bending-moment and the deflection have been reached *without any assumption as to the value of the ratio Q/P.*

[377.] We may make some remarks on the above results. Phillips first gave the complete equations for the problem and included the effect of the inertia of the beam (i.e. the terms in P). He obtained erroneous coefficients, however, for the terms in $1/\beta$. The correct values were first obtained by Saint-Venant, and his process is much shorter than Phillips'. In § 54 (pp. 609—612) Saint-Venant gives an elementary proof of the value of the bending moment in our equation (xiii^a). He does not make use of the general differential equations, but calculates and sums the parts of the bending moment due to statical loading, to the 'centrifugal force' of the travelling load $\left(\frac{Q}{g}\frac{V^2}{\rho}\right)$, and to the mass accelerations $\left(\frac{P}{2lg}dz\frac{d^2u}{dt^2}\right)$ of each element dz of the beam. The parts due to the last two influences are of the first order in $1/\beta$ and so we use in them the statical values for $1/\rho$, the curvature, and u.

We may ask whether the expressions in Equations (xiii^b) and (xiv^b) give the maxima values of M and u.

In the value of M the part affected by Q has its maximum when $z =$ its greatest value x; further, the principal portion of the same part $\left\{\frac{Q}{2}\frac{x(2l-x)}{l}\right\}$ has its maximum when $x = l$. Again the principal portion of the part in P, namely $\frac{P}{4} z \frac{2l-z}{l}$, has its maximum for $z = l$. The other parts of the expression for M are always much less and thus will give only an influence of the second order on the maximum values of z and x, i.e. their influence will not be sensible on the value of the maximum moment (p. 607). It is also easy to see that the maximum deflection is the mid-point deflection at the instant of transit of the load over the mid-point.

Throughout his discussion of the problem Saint-Venant does justice to Stokes' memoir; it will be observed that he frequently adopts Stokes' methods, but the extension of the results to any ratio of Q/P is in itself no small advance.

[378.] We shall now shew how the results obtained in (xiv[a]) must be modified in order that the condition for initial zero-velocity in the parts of the beam may be satisfied. This involves the introduction of periodic terms. Stokes had introduced such a periodic term on the assumption that Q/P was small (see our Art. 1289*). Phillips had endeavoured to measure the magnitude of the periodic terms which would enable us to dispose of the finite initial velocities which the above solution pre-supposes; he found that these terms were much smaller than the principal algebraic terms (Saint-Venant, pp. 613—614), but this does not prove that we may neglect them as compared with the terms in $1/\beta$. Saint-Venant adopting Stokes' approximate method, but without his assumption of the smallness of Q/P, introduces a periodic term which allows approximately (to the order $1/\beta$) for the zero initial velocities of the beam.

[379.] It will be remembered that Stokes' method consists in replacing each force acting on the beam by a uniformly distributed force which produces the same mean deflection as would be produced by the actual force taken alone (see our Art. 1288*). By this method he arrives at the following equation:

$$\frac{15E\omega\kappa^2}{Pl^3}v + \frac{155}{126}\frac{V^2}{g}\frac{d^2v}{dx^2} = \frac{Q}{P}X\left\{1 - \frac{V^2}{g}\frac{d^2(vX)}{dx^2}\right\} \text{.........(i)},$$

where $$\left.\begin{matrix}X\\Z\end{matrix}\right\} = 5\,\frac{8l^3\begin{Bmatrix}x\\z\end{Bmatrix} - 4l\begin{Bmatrix}x^3\\z^3\end{Bmatrix} + \begin{Bmatrix}x^4\\z^4\end{Bmatrix}}{16l^4},$$

and the deflection at z is given by

$$u = Z\left(\nu + \frac{Pl^3}{15E\omega\kappa^2}\right) \text{.........................(ii)},$$

thus determining what is represented by ν.

The equation (i) shews that ν is of the same order as Q/P, and Stokes solves it on the supposition that Q/P is so small that quantities of the order $(Q/P)\times\nu$ may be neglected, i.e. he omits the last term of the bracket on the right-hand side. Saint-Venant, however, seeks a value of ν by approximations in which powers of $1/\beta$ are neglected, in other words, *he makes no assumptions as to the value of the ratio Q/P except that P/Q is not to be extremely large.* In most practical cases Q and P will not be very far from equality, and the exception is accordingly legitimate. If we take

$$r^2/l^2 = \frac{31}{252}\frac{P}{Q}\frac{1}{\beta}, \quad \text{where } \frac{1}{\beta} = \frac{2QlV^2}{3gE\omega\kappa^2},$$

we have the small quantities in terms of which Saint-Venant solves the equation (i).

It will be found that $r/2l = 1/q$, where q is the constant of Stokes' investigation: see our Art. 1290*.

$$\left\{\text{We may note that } S_1 \text{ of Stokes} = \frac{Pl^3}{6E\omega\kappa^2},\right.$$

$$\left.\text{and } f_s' \text{ of Saint-Venant} = \frac{5Pl^3}{48E\omega\kappa^2}\right\}.$$

[380.] The solution found by Saint-Venant is given by:

$$\nu = \frac{Ql^3}{15E\omega\kappa^2}\left(X - r^2\frac{d^2X}{dx^2} + \frac{1}{\beta}X\frac{6lx - 3x^2 - 2l^2}{l^2}\right)$$
$$- \frac{Ql^3}{6E\omega\kappa^2}\left\{1 - \frac{2}{\beta} + 3\left(\frac{r}{l}\right)^2\right\}\frac{r}{l}\sin\frac{x}{r} \text{ .... (iii)}.$$

See his p. 615 e.

Substituting for ν in equation (ii) of Art. 379 we have u. For the central-deflection as the load passes we find

$$\left.\begin{aligned} u_l = \frac{125}{128}\frac{Ql^3}{6E\omega\kappa^2}\left(1 + \frac{1}{\beta}\right) + \frac{5Pl^3}{48E\omega\kappa^2}\left(1 + \frac{155}{336}\frac{1}{\beta}\right) \\ - \frac{25l^3}{96E\omega\kappa^2}\left[Q\left(1 - \frac{2}{\beta}\right) + \frac{31}{84}\frac{P}{\beta}\right]\frac{r}{l}\sin\frac{l}{r} \end{aligned}\right\} \text{...(iv)}.$$

The algebraic terms as might be supposed owing to the method of approximation, are not exactly the same as in (xiv^b). The factor $\frac{125}{128}$ instead of 1 is not, however, important, while the factor $\frac{155}{336}$ instead of

$\frac{27}{25}$ occurs only in terms involving $1/\beta$. Saint-Venant concludes that the algebraic terms given by the first method are the correct ones, and that we may add to them the expression

$$-\frac{25l^3}{96E\omega\kappa^2}\left[Q\left(1-\frac{2}{\beta}\right)+\frac{31}{84}\frac{P}{\beta}\right]\frac{r}{l}\sin\frac{l}{r},$$

in order to approximately account for the periodic terms. This result and (iv) differ from those of Saint-Venant (α') p. 615 h and (δ') p. 615 i but seem to me to give the correct value of u_1.

The corresponding part to be subtracted from the bending-moment at the centre as the load passes it is

$$-\frac{5}{8}l\left[Q\left(1-\frac{2}{\beta}\right)+\frac{31}{84}\frac{P}{\beta}\right]\frac{r}{l}\sin\frac{l}{r}.$$

This again differs from Saint-Venant's results (β') p. 615h and (ϵ') p. 615j. By a misprint which has escaped correction he has the fraction $\frac{35}{24}$ where I have $\frac{31}{84}$.

[381.] The last extension of the problem which we shall consider here is that of Renaudot, who does not deal with the case of an isolated oad (as a locomotive) but with that of a continuous load (as a train of trucks or carriages) crossing the bridge. Let p be the weight per foot-run of the girder, p' that of the travelling load the head of which is distant $x = Vt$ from the right hand terminal. In this case equations (i) of our Art. 373, are replaced, on the supposition that the train is longer than bridge, by

$$E\omega\kappa^2\frac{d^4u}{dz^4}-(p+p')=-\frac{p}{g}\frac{d^2u}{dt^2}-\frac{p'}{g}\frac{d^2w}{dt^2}\;\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

$$E\omega\kappa^2\frac{d^4u_1}{dz_1^{\,4}}-p=-\frac{p}{g}\frac{d^2u_1}{dt^2}=-\frac{pV^2}{g}\frac{d^2u_1}{dx^2}\;\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

Here w is the shift of the element $(p'/g)\,dz$ of the train on the bridge, and z is to be put in w equal to Vt less the constant distance between the given element and the head of the train. Thus while the z in d^2u/dt^2 is not a function of t, that in d^2w/dt^2 is to be treated as a function of t, or since $x = Vt$ we may write:

$$\frac{d^2w}{dt^2}=V^2\left\{\frac{d^2u}{dx^2}+2\frac{d^2u}{dx\,dz}+\frac{d^2u}{dz^2}\right\}.$$

Thus the first equation becomes:

$$E\omega\kappa^2\frac{d^4u}{dz^4}-(p+p')=-\frac{pV^2}{g}\frac{d^2u}{dx^2}-\frac{p'V^2}{g}\left\{\frac{d^2u}{dx^2}+2\frac{d^2u}{dx\,dz}+\frac{d^2u}{dz^2}\right\}\;\ldots\text{(iii)}.$$

Starting from equations (ii) and (iii) with the necessary terminal conditions for each portion of the girder, we may proceed as in our Arts. 373—376 to determine first the statical and then the first dynamical approximation. The maximum bending-moment will be greatest when the load just covers the whole girder. It is then given by

$$M_l = \frac{p+p'}{2} l^2 \left(1 + \frac{5}{8}\frac{1}{\beta'}\right), \text{ where } \frac{1}{\beta'} = \frac{p' l^2 V^2}{gE\omega\kappa^2}.$$

Similarly we may deduce the bending-moment when the train is headed by a locomotive of weight Q followed by a train of weight p' per foot-run. In order to obtain the position of the maximum bending-moment, it will, as in Art. 377, be sufficient to find the values of z and x which give the maximum moment for the statical approximation. These are

$$x = 2l - Q/p', \quad z = l\left\{1 + \frac{Q}{2pl} \cdot \frac{Q}{2(p+p')l}\right\},$$

and they must be substituted in the second approximation involving the terms in $1/\beta$ and $1/\beta'$ (see pp. 616—618).

Renaudot neglects the term $2\frac{d^2u}{dx\,dz}$ in equation (iii) as of small importance. He arrives at a wrong value, i.e. $\left(1 + \frac{2}{3\beta'}\right)$, for the bracket in the value of the bending-moment.

[382.] Saint-Venant remarks that Phillips has also treated the case of a travelling load crossing a beam *doubly built-in.* His solution is, however, erroneous, as has been pointed out both by Bresse and Saint-Venant, nor would it be of much value to correct his results, for built-in ends (*encastrements*) never produce their full effect, and such alternating motions as occur with travelling loads in bridges soon deprive such ends of nearly all their effect (see p. 619 and our Arts. 733* and 188).

There is also a reference on p. 619 to Bresse's exact solution for the case of a bridge across which a very long train is continuously moving with velocity V, so that the bridge takes up a permanent form. In this case equation (iii) of Art. 381 becomes

$$E\omega\kappa^2 \frac{d^4u}{dz^4} - (p+p') = -\frac{pV^2}{g}\frac{d^2u}{dz^2},$$

and we can find an exact solution. It gives for the maximum bending-moment

$$M_l = \frac{p+p'}{2} l^2 . 2\beta'' \{\sec\sqrt{1/\beta''} - 1\}, \text{ where } \frac{1}{\beta''} = \frac{pl^2V^2}{gE\omega\kappa^2},$$

or, $$M_1 = \frac{p+p'}{2} l^2 \left(1 + \frac{5}{12}\frac{1}{\beta''}\right) \text{ approximately.}$$

This result is less than that of our Art. 381, or we see that the dangerous instant is that in which the train just covers the whole bridge. It is then producing impulsive changes in the elastic line of the bridge, and not a steady form of the elastic line as in the case of a very long train imagined to be continuously crossing.

[383.] We now reach Saint-Venant's last contribution to the annotated Clebsch, namely, the *Note finale du* § 73, pp. 689—752. It is entitled: *Théorie de la flexion et des autres petites déformations des plaques élastiques planes minces, tirée directement des équations différentielles générales de l'équilibre d'élasticité des solides.*

The *Note* consists of four essentially distinct parts: (i) a deduction of the general elastic body-shift equations for thin plates; (ii) a full discussion of the contour conditions, and the controversy with regard to them; (iii) the solutions for statical equilibrium of thin circular plates; and (iv) a reproduction with extensions of Navier's results obtained in the memoir of 1820, and hitherto only published in extract: see our Arts. 258*—64*. I propose to deal somewhat at length with this *Note* as it forms distinctly the best treatment hitherto given for thin plates. Saint-Venant adopts Boussinesq's method (see the memoirs of 1871 and 1879 in the Chapter devoted to that elastician) but with certain important modifications. He describes Clebsch's investigation, notwithstanding that it starts with unnecessary simplifications, as "obscure, indirect and very complex." I think the terms are fully warranted.

[384.] Let the mid-plane of the plate be taken as that of $z = 0$ and let its faces be $z = \pm \epsilon$. We shall endeavour to deduce from the three body-stress equations, a single equation involving only the stresses $\widehat{xx}$, $\widehat{yy}$, $\widehat{xy}$ and given quantities. Let the body-stress equations be

$$\left.\begin{array}{l} \dfrac{d\widehat{xx}}{dx} + \dfrac{d\widehat{yx}}{dy} + \dfrac{d\widehat{zx}}{dz} + X = 0 \\[2ex] \dfrac{d\widehat{xy}}{dx} + \dfrac{d\widehat{yy}}{dy} + \dfrac{d\widehat{zy}}{dz} + Y = 0 \\[2ex] \dfrac{d\widehat{xz}}{dx} + \dfrac{d\widehat{yz}}{dy} + \dfrac{d\widehat{zz}}{dz} + Z = 0 \end{array}\right\} \dots\dots\dots\dots\dots\dots\text{(i)}.$$

Adding the third of these equations to the differentials of the first two with regard respectively to x and y, such differentials before addition being multiplied by z, we find

$$z\left\{\frac{d^2\widehat{xx}}{dx^2} + 2\frac{d^2\widehat{xy}}{dxdy} + \frac{d^2\widehat{yy}}{dy^2}\right\} + \frac{d}{dz}\left\{\frac{d}{dx}(z\,.\,\widehat{zx}) + \frac{d}{dy}(z\,.\,\widehat{yz}) + \widehat{zz}\right\} + z\left(\frac{dX}{dx} + \frac{dY}{dy}\right) + Z = 0.$$

Integrating this from $z = +\epsilon$ to $z = -\epsilon$,

$$\int_{-\epsilon}^{+\epsilon} \frac{d^2\widehat{xx}}{dx^2} zdz + 2\int_{-\epsilon}^{+\epsilon} \frac{d^2\widehat{xy}}{dx\,dy} zdz + \int_{-\epsilon}^{+\epsilon} \frac{d^2\widehat{yy}}{dy^2} zdz + \phi(xy) = 0 \text{......(ii)},$$

where

$$\phi(xy) = Z' + (\widehat{zz})_{+\epsilon} - (\widehat{zz})_{-\epsilon} + \frac{dX''}{dx} + \frac{dY''}{dy} + \frac{d}{dx}[\epsilon\{(\widehat{zx})_{+\epsilon} + (\widehat{zx})_{-\epsilon}\}]$$
$$+ \frac{d}{dy}[\epsilon\{(\widehat{zy})_{+\epsilon} + (\widehat{zy})_{-\epsilon}\}] \text{...(iii)},$$

and $$Z' = \int_{-\epsilon}^{+\epsilon} Zdz, \quad X'' = \int_{-\epsilon}^{+\epsilon} zXdz, \quad Y'' = \int_{-\epsilon}^{+\epsilon} zYdz;$$

the subscripts denote as usual that the stresses are to be given their values at the surfaces $z = \pm\epsilon$.

All the terms in the expression $\phi(xy)$ are thus known quantities.

[385.] The question of what further assumptions we shall make now arises. Those usually made are the following:

1°. $\widehat{zz} = 0$. (This is made even by Boussinesq and Lévy, the most recent writers on the subject.)

2°. $s_x = z/\rho$, $s_y = z/\rho'$, where ρ and ρ' are the two curvatures of the plate at *its mid-plane* for the point x, y. It follows that:

$$\left.\begin{aligned} s_x &= -z\frac{d^2w_0}{dx^2} \\ s_y &= -z\frac{d^2w_0}{dy^2} \end{aligned}\right\} \text{.......................... (iv)},$$

where w_0 is the normal shift of the point x, y of the mid-plane.

Using the stress-strain relations for three planes of elastic symmetry (see our Art. 117 (a)), we easily find from 1° and 2°:

$$\left.\begin{aligned} \widehat{xx} &= (a - e'^2/c)\, s_x + (f' - d'e'/c)\, s_y \\ \widehat{yy} &= (f' - d'e'/c)\, s_x + (b - d'^2/c)\, s_y \end{aligned}\right\}, \text{ and } \frac{d^2\widehat{xy}}{dx\,dy} = -2zf\frac{d^4w_0}{dx^2dy^2} \text{...(iv}^b\text{)}.$$

Substituting in (ii) and integrating we have the equation:

$$(a - e'^2/c)\frac{d^4w_0}{dx^4} + 2(2f + f' - d'e'/c)\frac{d^4w_0}{dx^2dy^2} + (b - d'^2/c)\frac{d^4w_0}{dy^4} = \frac{3}{2\epsilon^3}\phi(xy) \text{...(v)}.$$

This becomes in the case of elastic isotropy parallel to the mid-plane:

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)^2 w_0 = \frac{3}{2H\epsilon^3}\phi(xy) \text{ ... .(vi)},$$

where $H = a - e'^2/c$, the plate-modulus of our Art. 323.

This is the equation obtained by Lagrange, Poisson and Cauchy: see our Arts. 284*, 484* and 640*.

[386.] On pp. 696—700 (§§ 4—5) Saint-Venant considers what are the arguments in favour of the assumptions 1° and 2° of the previous Article. He remarks that owing to the thinness of the plate, the normal or z variations of both the stresses and the strains must be large as compared with the longitudinal variations. Hence as a first approximation, we have the fluxions with regard to x and y of both stress and strain components more and more nearly zero as the plate is taken thinner and thinner. It is sufficient however to assume that those of $\widehat{xx}$, $\widehat{xy}$ and $\widehat{yy}$ are zero or small. The body stress equations then give:

$$\frac{d\widehat{zx}}{dz} + X = 0, \quad \frac{d\widehat{zy}}{dz} + Y = 0.$$

Thus the stresses $\widehat{zx}$, $\widehat{zy}$ on integration will be of the order ϵ, or as Saint-Venant puts it:

Si on les intègre par rapport à la petite coordonnée z on voit que les composantes $\widehat{zx}$, $\widehat{zy}$ n'ont de valeurs, à l'intérieur d'un tronçon ou élément de plaque, que celles qu'elles peuvent avoir sur une des deux bases, plus ce qui vient des forces A, B, agissant sur sa masse. Ces forces locales n'ont qu'une influence insignifiante qui n'est presque rien en comparaison de ce qui vient à la fois de toutes les forces agissant sur le reste de la plaque ainsi que sur ses bords par les réactions des appuis ou autrement, et dont les effets accumulés se transmettent au tronçon à travers ses quatre faces latérales, ce qui s'applique surtout aux composantes agissant horizontalement (pp. 697—8).

The third body stress equation, however, shows that $\widehat{zz}$ is very small as compared with $\widehat{zx}$, $\widehat{zy}$ because these quantities occur with *lateral variation*, hence $\widehat{zz}$ is doubly small as compared with $\widehat{xx}$, $\widehat{xy}$ and $\widehat{yy}$. Thus we may take $\widehat{zz} = 0$ as all writers have hitherto done.

[387.] This argument is not, perhaps, quite convincing. It would seem at first sight better to assume *$\widehat{zz}$ to be very approximately a function of x, y only.* The expressions then for $\widehat{xx}$, $\widehat{xy}$, $\widehat{yy}$ would contain together with the terms linear in z, terms not involving z, but functions of x, y only. These terms disappear when we substitute them in equation (ii) and integrate between $z = +\epsilon$ and $-\epsilon$. But here a new difficulty arises; suppose the surface of the plate $z = +\epsilon$ subjected to a load $\widehat{zz} = \chi(x, y)$. This will make no change in the first three terms of equation (ii) of Art. 384 although we cannot suppose $\widehat{zz} = 0$, but it will lead to a difficulty with regard to the expression $\phi(xy)$.

This expression contains terms of the form $(\widehat{zz})_{+\epsilon}$ and $(\widehat{zz})_{-\epsilon}$; the former $= \chi(x, y)$ and the latter is zero. Hence it follows that $\widehat{zz}$ must vary with z from $+\epsilon$ to $-\epsilon$. Saint-Venant (p. 699) says we must take $(\widehat{zz})_{+\epsilon} = \chi(x, y)$ and put $(\widehat{zz})_{-\epsilon} = 0$, but this seems to me to destroy the basis of his approximation. Possibly, following the hint he gives on p. 700, the true method is to consider that, when the dimensions of a body are very small in any sense, then a *surface-load* in the same sense will give the same strains perpendicular to that sense as the *integral of a body-force* also in that sense. Thus the flexure-equations for a beam are deduced on the assumption that there is *no* lateral stress, yet we do not hesitate to use them for beams subject to continuous lateral load[1]. I conclude then that it is best to put $\widehat{zz}$ always *zero* (and not a definite value as Saint-Venant does on p. 699) and assume, when plates have a surface distribution of load, that the result of such load so far as the shifts of the points of the mid-plane are concerned can be represented by a body force, whose integral between the faces is equivalent per unit area to the surface load.

[388.] In § 5 Saint-Venant shews that from the assumptions, or approximate values:

$$\frac{d\,(\sigma_{zx},\ \sigma_{zy})}{dx,\ dy}=0, \qquad \frac{d^2 s_z}{dx^2,\ dxdy,\ dy^2}=0 \ldots\ldots\ldots\ldots\ldots(\alpha),$$

(which are less restrictive than $\sigma_{zx}=\sigma_{zy}=0$, and $\widehat{zz}=0$) we can deduce results embracing those of our Art. 385, 1° and 2°.

Writing the first set of expressions at length we easily find that:

$$\frac{ds_x}{dz}=-\frac{d^2w}{dx^2}, \qquad \frac{ds_y}{dz}=-\frac{d^2w}{dy^2}, \qquad \frac{d\sigma_{xy}}{dz}=-2\frac{d^2w}{dxdy} \ldots\ldots\ldots(\beta).$$

Whence we see by differentiating with regard to z that:

$$\frac{d^2s_x}{dz^2}=-\frac{d^2s_z}{dx^2}=0, \qquad \frac{d^2s_y}{dz^2}=-\frac{d^2s_z}{dy^2}=0, \qquad \frac{d^2\sigma_{xy}}{dz^2}=-2\frac{d^2s_z}{dxdy}=0.$$

Thus the second z fluxions of s_x, s_y, σ_{xy} are zero, or we may write w_0 for w in (β); it follows that we must have:

$$s_x=s^0_x-z\frac{d^2w_0}{dx^2}, \qquad s_y=s^0_y-z\frac{d^2w_0}{dy^2}, \qquad \sigma_{xy}=\sigma^0_{xy}-2z\frac{d^2w_0}{dxdy}\ldots(\gamma),$$

where the zero affixed refers the quantity to the mid-plane.

Saint-Venant remarks of the equations (γ):

Elles montrent, comme conséquence cinématique des égalités posées (α), que les dilatations de petites droites matérielles horizontales de direction donnée varient *linéairement* le long de toutes les lignes primitivement verticales, des divers points desquelles ces petites horizontales auraient été tirées.

Il convient de remarquer en passant que cela n'entraîne nullement, comme conséquence, que ces verticales resteront exactement droites et normales au feuillet moyen devenu courbe, car leurs petits intervalles horizontaux peuvent très bien croître linéairement avec z quoiqu'elles soient devenues courbes, si celles qui sont voisines affectent des inflexions pareilles, ainsi qu'il arrive pour les *sections* voisines, dans les tiges éprouvant *la flexion* dite *inégale* (p. 699: see our Art. 325).

It will be noted that this treatment brings out the real difficulties and assumptions of the problem, better than those which start by

[1] Since writing the above I have obtained the *full* solution for a simply supported beam continuously loaded on its upper surface, I find $\widehat{zz}$ is of the *same order* as $\widehat{zy}$, where x is the direction of the axis of the beam, and z the direction of the load.

assuming the strain-energy to be a function of the curvatures and so deduce by Lagrange's, or other, method the fundamental equation of the plate: see Thomson and Tait, § 639, or Lord Rayleigh's *Sound*, Vol. I., § 214.

I may remark that the equations (ii) and (iii) obtained in Art. 384 still hold if 2ϵ the thickness of the plate changes gradually with x and y.

[389.] Returning to the body-stress equations of Art. 384, let us integrate the first two between the limits $\pm\epsilon$ of z. We note first, however, since:

$$\widehat{xx} = \left(a - \frac{e'^2}{c}\right) s_x + \left(f' - \frac{d'e'}{c}\right) s_y$$
$$= \left(a - \frac{e'^2}{c}\right)\left(s^0_x - z\frac{d^2 w_0}{dx^2}\right) + \left(f' - \frac{d'e'}{c}\right)\left(s^0_y - z\frac{d^2 w_0}{dy^2}\right)$$

by equations (γ) of Art. 388, that

$$\left.\begin{array}{l} \displaystyle\int_{-\epsilon}^{+\epsilon} \widehat{xx}\,dz = 2\epsilon\left\{\left(a - \frac{e'^2}{c}\right) s^0_x + \left(f' - \frac{d'e'}{c}\right) s^0_y\right\} = 2\epsilon\,\widehat{xx}_0, \\ \text{similarly} \quad \displaystyle\int_{-\epsilon}^{+\epsilon} \widehat{xy}\,dz = 2\epsilon\,\widehat{xy}_0, \text{ and } \int_{-\epsilon}^{+\epsilon} \widehat{yy}\,dz = 2\epsilon\,\widehat{yy}_0 \end{array}\right\} \ldots\text{(i)},$$

where the affixed $_0$ denotes a mid-plane value.

Hence from the integration of the body-stress equations we obtain:

$$\left.\begin{array}{l} 2\epsilon\left(\dfrac{d\widehat{xx}_0}{dx} + \dfrac{d\widehat{xy}_0}{dy}\right) + (\widehat{zx})_{+\epsilon} - (\widehat{zx})_{-\epsilon} + X' = 0, \\ 2\epsilon\left(\dfrac{d\widehat{xy}_0}{dx} + \dfrac{d\widehat{yy}_0}{dy}\right) + (\widehat{zy})_{+\epsilon} - (\widehat{zy})_{-\epsilon} + Y' = 0, \end{array}\right\}$$

where X' and Y' are the integrals of X and Y across the plate.

Substitute the values of the mid-plane stresses in terms of the mid-plane shifts u_0, v_0 and we have:

$$\left.\begin{array}{l} 2\epsilon\left\{(a - e'^2/c)\dfrac{d^2u_0}{dx^2} + (f + f' - d'e'/c)\dfrac{d^2v_0}{dxdy} + f\dfrac{d^2u_0}{dy^2}\right\} + (\widehat{zx})_{+\epsilon} - (\widehat{zx})_{-\epsilon} + X' = 0 \\ 2\epsilon\left\{(b - d'^2/c)\dfrac{d^2v_0}{dy^2} + (f + f' - d'e'/c)\dfrac{d^2u_0}{dxdy} + f\dfrac{d^2v_0}{dx^2}\right\} + (\widehat{zy})_{+\epsilon} - (\widehat{zy})_{-\epsilon} + Y' = 0 \end{array}\right\} \ldots\text{(ii)}.$$

These equations reduce in the case of isotropy parallel to the plate to the simpler forms (p. 702):

$$\left.\begin{array}{l} 2\epsilon\left\{H\dfrac{d}{dx}\left(\dfrac{du_0}{dx} + \dfrac{dv_0}{dy}\right) + f\dfrac{d}{dy}\left(\dfrac{du_0}{dy} - \dfrac{dv_0}{dx}\right)\right\} + (\widehat{zx})_{+\epsilon} - (\widehat{zx})_{-\epsilon} + X' = 0 \\ 2\epsilon\left\{H\dfrac{d}{dy}\left(\dfrac{du_0}{dx} + \dfrac{dv_0}{dy}\right) - f\dfrac{d}{dx}\left(\dfrac{du_0}{dy} - \dfrac{dv_0}{dx}\right)\right\} + (\widehat{zy})_{+\epsilon} - (\widehat{zy})_{-\epsilon} + Y' = 0 \end{array}\right\} \ldots\text{(iii)},$$

where H is the plate-modulus of Art. 323.

It will be noticed that these equations for the shifts u_0, v_0 are independent of that for w_0, or the transverse and longitudinal strain exercise no influence on each other. This has already been remarked by Cauchy and Poisson: see our Arts. 483* and 640*.

[390.] In §§ 8—10 Saint-Venant considers the effect of great stresses parallel to the mid-plane on the normal shift w_0. Thus he obtains what may be called the terms due to the action of the plate as a transverse membrane. He finds that in the function $\phi(xy)$ of equations (v) and (vi) of Art. 385 we must include the expression:

$$2\epsilon\left(\frac{d^2w_0}{dx^2}\widehat{xx}_0 + 2\frac{d^2w_0}{dxdy}\widehat{xy}_0 + \frac{d^2w_0}{dy^2}\widehat{yy}_0\right) - \frac{dw_0}{dx}\{(\widehat{zx})_{+\epsilon} - (\widehat{zx})_{-\epsilon} + X'\}$$
$$- \frac{dw_0}{dy}\{(\widehat{zy})_{+\epsilon} - (\widehat{zy})_{-\epsilon} + Y'\}.$$

From the sum of this expression and $Z' + (\widehat{zz})_{+\epsilon} - (\widehat{zz})_{-\epsilon}$ equated to zero we deduce the equation for the transverse equilibrium of a membrane. In its present form it has been obtained on the supposition that 2ϵ is constant; the alterations for 2ϵ variable are indicated by Saint-Venant in a footnote, p. 704.

[391.] In § 13 Saint-Venant commences his treatment of the contour conditions. Let α be the angle between the normal to the mid-plane contour at any point and the axis of x, let P, Q, R be the components of the applied load parallel to the axes, and ds, dn elements of the arc and normal of the mid-plane contour.

We find at once:

$$P = \widehat{xx}\cos\alpha + \widehat{yx}\sin\alpha, \quad Q = \widehat{xy}\cos\alpha + \widehat{yy}\sin\alpha, \quad R = \widehat{xz}\cos\alpha + \widehat{yz}\sin\alpha.$$

Hence by equations (i) of Art. 389, we have:

$$2\epsilon\,(\widehat{xx}_0\cos\alpha + \widehat{yx}_0\sin\alpha) = \int_{-\epsilon}^{+\epsilon} Pdz = P',$$

$$2\epsilon\,(xy_0\cos\alpha + \widehat{yy}_0\sin\alpha) = \int_{-\epsilon}^{+\epsilon} Qdz = Q'.$$

Substitute for the mid-plane stresses in terms of the shifts and we have:

$$2\epsilon\left\{\left[(a - e'^2/c)\frac{du_0}{dx} + (f' - d'e'/c)\frac{dv_0}{dy}\right]\cos\alpha + f\left(\frac{du_0}{dy} + \frac{dv_0}{dx}\right)\sin\alpha\right\} = P',$$

$$2\epsilon\left\{\left[(f' - d'e'/c)\frac{du_0}{dx} + (b - d'^2/c)\frac{dv_0}{dy}\right]\sin\alpha + f\left(\frac{du_0}{dy} + \frac{dv_0}{dx}\right)\cos\alpha\right\} = Q'.$$

These are the sufficient and necessary contour conditions for longitudinal strain. When there is elastic isotropy parallel to the mid-plane they reduce to

$$2\epsilon\left\{\left[H\left(\frac{du_0}{dx} + \frac{dv_0}{dy}\right) - 2f\,\frac{dv_0/dy}{du_0/dx}\right]\frac{\cos\alpha}{\sin\alpha} + f\left(\frac{du_0}{dy} + \frac{dv_0}{dx}\right)\frac{\sin\alpha}{\cos\alpha}\right\} = \frac{P'}{Q'}.$$

[392.] Saint-Venant next turns to the more controverted conditions involving the normal shift w_0. He proceeds to calculate M_s, M_n and R', the first two symbols representing the moments round tangent and normal respectively to the mid-plane contour for the load applied to the strip $2\epsilon \times ds$, and R' being the total shearing load on the same strip.

Now $$M_s = \int_{-\epsilon}^{+\epsilon} (P \cos\alpha + Q \sin\alpha)\, zdz = P'' \cos\alpha + Q'' \sin\alpha$$
$$= -\frac{2\epsilon^3}{3}\{\widehat{xx}'' \cos^2\alpha + 2\widehat{xy}'' \sin\alpha\cos\alpha + \widehat{yy}'' \sin^2\alpha\},$$

where, r and p being any two directions,

$$\widehat{rp}'' \times \frac{2\epsilon^3}{3} = -\int_{-\epsilon}^{+\epsilon} \widehat{rp}zdz,$$

and as before, a single dash on the loads denotes an integration with regard to z from $+\epsilon$ to $-\epsilon$, and a double dash an integration after multiplication by z, between the same limits. Substituting from equations (i) of Art. 389 for the stresses we find:

$$M_s = -\frac{2\epsilon^3}{3}\left\{\left[(a - e'^2/c)\frac{d^2w_0}{dx^2} + (f' - d'e'/c)\frac{d^2w_0}{dy^2}\right]\cos^2\alpha + 4f\frac{d^2w_0}{dxdy}\sin\alpha\cos\alpha\right.$$
$$\left. + \left[(b - d'^2/c)\frac{d^2w_0}{dy^2} + (f' - d'e'/c)\frac{d^2w_0}{dx^2}\right]\sin^2\alpha\right\}.$$

Or, for elastic isotropy parallel to the mid-plane:

$$M_s = -\frac{2\epsilon^3}{3}\left[(H - 2f)\left(\frac{d^2w_0}{dx^2} + \frac{d^2w_0}{dy^2}\right) + 2f\frac{d^2w_0}{dn^2}\right] \text{............(i).}$$

This first condition is not the subject of discussion but has been generally accepted.

[393.] In a similar manner we find:

$$M_n = Q'' \cos\alpha - P'' \sin\alpha,$$
$$= \frac{2\epsilon^3}{3}\{\sin\alpha\cos\alpha\,(\widehat{xx}'' - \widehat{yy}'') - (\cos^2\alpha - \sin^2\alpha)\,\widehat{xy}''\},$$

where

$$\widehat{xx}'' - \widehat{yy}'' = \left(a - f' + \frac{d' - e'}{c}e'\right)\frac{d^2w_0}{dx^2} - \left(b - f' - \frac{d' - e'}{c}d'\right)\frac{d^2w_0}{dy^2},$$

or, $$= (a - f')\left(\frac{d^2w_0}{dx^2} - \frac{d^2w_0}{dy^2}\right)$$ with elastic isotropy parallel to the mid-plane.

And again, $$\widehat{xy}'' = 2f\frac{d^2w_0}{dxdy}.$$

Further: $$R' = \int_{-\epsilon}^{+\epsilon} Rdz = \cos\alpha \int_{-\epsilon}^{+\epsilon} \widehat{zx}dz + \sin\alpha \int_{-\epsilon}^{+\epsilon} \widehat{yz}dz$$
$$= \cos\alpha\left[\epsilon\{(\widehat{zx})_{+\epsilon} + (\widehat{zx})_{-\epsilon}\} + X'' - \frac{2\epsilon^3}{3}\left(\frac{d\widehat{xx}''}{dx} + \frac{d\widehat{yx}''}{dy}\right)\right]$$
$$+ \sin\alpha\left[\epsilon\{(\widehat{zy})_{+\epsilon} + (\widehat{zy})_{-\epsilon}\} + Y'' - \frac{2\epsilon^3}{3}\left(\frac{d\widehat{xy}''}{dx} + \frac{d\widehat{yy}''}{dy}\right)\right],$$

where $$\frac{d\widehat{xx}''}{dx} + \frac{d\widehat{xy}''}{dy} = (a - e'^2/c)\frac{d^3w_0}{dx^3} + (2f + f' - d'e'/c)\frac{d^3w_0}{dxdy^2},$$

$$\frac{d\widehat{xy}''}{dx} + \frac{d\widehat{yy}''}{dy} = (b - d'^2/c)\frac{d^3w_0}{dy^3} + (2f + f' - d'e'/c)\frac{d^3w_0}{dx^2dy},$$

reducing respectively to the differentials with regard to x and y of $H\left(\frac{d^2w_0}{dx^2} + \frac{d^2w_0}{dy^2}\right)$, when there is elastic isotropy parallel to the mid-plane.

These results can be easily deduced by integrating $\widehat{zx}$ and $\widehat{yz}$ from expressions of the form:

$$\widehat{xz} = \frac{d(z.\widehat{xz})}{dz} - z\frac{d\widehat{xz}}{dz}$$

$$= \frac{d(z.\widehat{xz})}{dz} + z\left(\frac{d\widehat{xx}}{dx} + \frac{d\widehat{xy}}{dy} + X\right).$$

I have reproduced the values of M_s, M_n and R', because they are the most complete hitherto given and will be useful for reference hereafter.

[394.] Saint-Venant adopts Thomson and Tait's 'reconciliation' (see our Art. 488*) and replaces the couples M_n by an additional shear $\frac{dM_n}{ds}$ added to R'. In other words he equates the contour load to the couple M_s and the shear $R' + \frac{dM_n}{ds}$.

He attributes this method of reconciling Kirchhoff and Poisson to Boussinesq (p. 715). There are two points which arise in this reconciliation which deserve to be noted. The first objection to the replacement of the couples M_n by a distributed shear is that referred to in our Art. 488*, namely that the Kirchhoff contour conditions could not be used for the case of a *discontinuous* distribution of shearing force and normal couple. Saint-Venant replies to this:

S'il y avait des forces extérieures isolées, appliquées en certains points de ce cylindre et faisant couples autour de ses normales, elles seraient capables d'y imprimer à la plaque, entre ces points, des torsions finies. Aucun auteur n'a supposé l'existence de pareilles forces, qui sont capables de produire des altérations permanentes de la contexture de la matière de la plaque si elles agissent avec une certaine intensité sur des portions excessivement petites de sa surface. Tous supposent que les forces se répartissent sur des surfaces d'étendue finie; et nous ne considèrerons même, ainsi qu'ils l'ont tous fait, que des forces agissant sur le cylindre contournant d'une manière continue et graduelle (p. 714).

The second objection is that due to M. Lévy (see our Art. 397); he holds that when the couple M_n is due to vertical forces we can replace it by a shear distribution perpendicular to the plate, but that when it is due to horizontal forces this is not allowable. This point has been discussed at length by Boussinesq (see the Chapter we have devoted to that elastician), and Saint-Venant sums up his arguments in the following words:

Si ces couples sont formés par des forces horizontales tangentes au cylindre, agissant en sens opposés, les unes au-dessus, les autres au-dessous de la périphérie moyenne, et si la plaque est supposée avoir une épaisseur comparable aux deux autres dimensions, ces couples pourront conspirer pour produire certains effets d'ensemble dont nous ne nous occupons pas, tels qu'une *inflexion* imprimée à toutes les arêtes, et accompagnée de cette torsion générale autour d'un axe vertical dont il a été traité dans les chapitres relatifs aux tiges. Mais si la plaque est extrêmement mince, ces sortes de déformations sont négligeables. Les couples de forces horizontales dont il s'agit s'exerçant d'une manière continue sur les arêtes successives, ne produiront que ces torsions *locales* dont nous nous occupons ici; et leurs effets seront sensiblement les mêmes que ceux de couples de forces verticales *de même moment*, qu'on leur substituerait en faisant tourner ceux-là de 90 degrés, substitutions qui se font, comme on sait, dans la statique élémentaire des corps solides (p. 714).

We may I think conclude that:

1°. The shift-equation ((vi) of Art. 385) for thin plates is only an approximation and depends upon the assumptions that $\widehat{zz} = 0$ and that s_x, s_y, σ_{xy} contain *only the first power* of z, as in Eqns. (γ) of our Art. 388. These assumptions are, however, probable and the approximation is close when the thickness of the plate is extremely small.

2°. To the same degree of approximation the *two* boundary conditions of Kirchhoff are true for very thin plates.

3°. When the plate has a thickness small but not indefinitely small compared with its other dimensions, the equation of Lagrange can under certain conditions still hold, but it is not then legitimate to replace the normal couple by a distribution of shearing-load.

This latter conclusion is *opposed* to Saint-Venant's opinion on p. 720. He shews that if the following conditions hold:

$$\frac{d^2\widehat{zz}}{dx^2,\ dxdy,\ dy^2} = \frac{d^2(\sigma_{zx},\ \sigma_{zy})}{dx^2,\ dxdy,\ dy^2} = \frac{d^3 s_z}{dx^3,\ d^2xdy,\ d^2ydx,\ dy^3} = 0,$$

we can still deduce Lagrange's equation, but these conditions allow of a definite but small thickness for the plate. He then states that Kirchhoff's contour conditions remain true. Now it seems to me that we can no longer replace normal couples by vertical shearing-loads, for this will cause a difference in the strain of the plate to a distance into its material of the same order as its thickness, and this distance is no longer vanishingly small as compared with the other dimensions of the plate.

[395.] Saint-Venant now proceeds to an interesting summary of other writers' treatment of the problem of thin plates. He notes that Poisson and Cauchy assume that the stresses can be expanded in powers of z giving convergent series. From this assumption Saint-Venant deduces equations (γ) of our Art. 388. He remarks of this assumption that it has never found supporters—*elle n'est pas suffisamment fondée, et peut se trouver souvent en défaut.* I must notice, however, that Saint-Venant's own assumptions of our Art. 385 really lead to the expression of the stresses $\widehat{xx}$, $\widehat{xy}$ and $\widehat{yy}$ as *linear functions* of z, (see equations (iv^b) of Art. 385 and (γ) of Art. 388,) while from the first two body stress-equations we obtain by integration for $\widehat{zx}$ and $\widehat{yz}$ *quadratic functions* of z together with terms $-\int_0^z Xdz$ and $-\int_0^z Ydz$ which will in the great majority of cases be linear in z. Thus Poisson's and Cauchy's assumption is only a too general statement of the results reached by Saint-Venant himself.

[396.] Saint-Venant appears to think that the terms $Z' + \frac{dX''}{dx} + \frac{dY''}{dy}$ which occur in the function $\phi(xy)$ of his result (see equation (iii) of our Art. 384) do not agree with the similar terms obtained by Poisson (see equation (9) of our Art. 484*) and Cauchy (see Equation (70) of our Art. 640*). With proper transformation of symbols these are:

$$2\epsilon\left\{\left[Z + \frac{\epsilon^2}{6}\frac{d^2Z}{dz^2}\right]_{z=0} + \frac{\epsilon^2}{3}\left[\frac{d}{dz}\left(\frac{dX}{dx} + \frac{dY}{dy}\right)\right]_{z=0}\right\}.$$

Now Poisson and Cauchy assume forms such as:

$$X = X_0 + z\left(\frac{dX}{dz}\right)_0 + \frac{z^2}{2}\left(\frac{d^2X}{dz^2}\right)_0 + \text{etc.},$$

$$\therefore X'' = \int_{-\epsilon}^{+\epsilon} Xzdz = \frac{2\epsilon^3}{3}\left(\frac{dX}{dz}\right)_0 + \text{terms involving fifth and higher}$$

powers of ϵ.

Similarly:

$$Z' = \int_{-\epsilon}^{+\epsilon} Z dz = 2\epsilon Z_0 + \frac{2\epsilon^3}{6}\left(\frac{d^2Z}{dz^2}\right)_0 + \text{terms involving fifth and higher}$$

powers of ϵ,

$$\text{or} \quad Z' + \frac{dX''}{dx} + \frac{dY''}{dy} = 2\epsilon \left\{ \left[Z + \frac{\epsilon^2}{6}\frac{d^2Z}{dz^2} \right]_{z=0} + \frac{\epsilon^2}{3}\left[\frac{d}{dz}\left(\frac{dX}{dx} + \frac{dY}{dy} \right) \right]_{z=0} \right\}$$

+ terms involving fifth and higher powers of ϵ, which may be neglected.

Thus their assumption does not lead to an error in this point as Saint-Venant suggests. It seems to me also that Poisson and Cauchy's hypothesis is more valid than that of Clebsch and other writers who simply put $\widehat{zz} = \widehat{zx} = \widehat{zy} = 0$.

[397.] At this point Saint-Venant notices Maurice Lévy's memoir of 1877 (see *Journal de Liouville*, 1877, pp. 219—306, or our discussion of the memoir later). Lévy investigates what are the possible solutions for a prism with two free faces, when the shifts u, v, w are supposed capable of being expressed in a series of ascending powers of z and the forces acting on the lateral sides of the prism have a given resultant load. It follows that the stresses will now be given in ascending powers of z, and that there is no limitation as to thickness of the plate (or height of the prism). Lévy finds (1) that the powers of z in u, v, w and in $\widehat{xx}$, $\widehat{xy}$ and $\widehat{yy}$ cannot surpass the third, (2) that the stress $\widehat{zz} = 0$ throughout the prism, and (3) that the stresses $\widehat{xz}$, $\widehat{yz}$ contain only second powers of z, which appear through the factor $(\epsilon^2 - z^2)$.

It will be seen that these results of Lévy give the proper limitation to Cauchy and Poisson's hypothesis, and shew clearly its relation to Saint-Venant's assumptions. Saint-Venant on pp. 726—733 deals with another part of Lévy's memoir; namely, the term he has introduced into the values of the shifts u and v in order that the *three* surface conditions of Poisson may be satisfied for thin plates. This term is periodic in z, but Saint-Venant following Boussinesq rejects it as producing effects only of the same order as those we are neglecting in our approximation. We shall return to this point later when treating of Lévy's memoir and his controversy with Boussinesq.

[398.] Saint-Venant now turns to the concrete application of the thin plate formulae. He first deals with circular plates and obtains the following results:

Let there be a uniform surface load p per unit of area.

(i) *When the contour simply rests on a ring of its own radius a.*

Shift of mid-plane at radius r:

$$w_0 = \frac{3pa^4}{128H\epsilon^3}\left[\left(\frac{r}{a}\right)^4 - 1 - \frac{4H-2f}{H-f}\left\{\left(\frac{r}{a}\right)^2 - 1\right\}\right].$$

Central Shift $$f_0 = \frac{3H-f}{H-f}\,\frac{3pa^4}{128H\epsilon^3}.$$

(ii) *When the contour is built-in.*

$$w'_0 = \frac{3pa^4}{128H\epsilon^3}\left\{1 - 2\left(\frac{r}{a}\right)^2 + \left(\frac{r}{a}\right)^4\right\},$$

$$f'_0 = \frac{3pa^4}{128H\epsilon^3}.$$

Further we find that:

When the plate is simply supported: the line of inflection, given by $\frac{d^2w_0}{dr^2} = 0$, is determined by $r = a\sqrt{\frac{2H-f}{3(H-f)}}$, or $= \cdot 931 \,.\, a$ if $H/f = 8/3$, as in the case of uniconstant isotropy.

When it is built-in: the line of inflexion is determined by $r = \cdot 5773 \,.\, a$, and the line of zero-moment (i.e. where $M_s = 0$) by $r = a\sqrt{\frac{H-f}{2H-f}}$, or $= \cdot 6202 \,.\, a$ in the case of uniconstant isotropy.

The maximum stretches in the two cases, given by the greatest values of $s_r = -z\frac{d^2w_0}{dr^2}$, are respectively:

$$s_0 = \frac{2H-f}{H-f}\cdot\frac{3pa^2}{32H\epsilon^2}\left(= \frac{117pa^2}{1280\mu\epsilon^2}, \text{ for uniconstant isotropy}\right),$$

$$s'_0 = \frac{3pa^2}{16H\epsilon^2}\left(= \frac{9pa^2}{128\mu\epsilon^2}, \text{ for uniconstant isotropy}\right).$$

The conditions for the fail-limit are thus easily written down. Compare with these results those of Poisson in our Arts. 497*—504*.

[399.] The last pages of Saint-Venant's *Note* are occupied with a reproduction and extension of Navier's memoir on rectangular plates (pp. 740—52).

Let us take the origin at one of the angles of the plate (sides a, b, and $a < b$) the sides being the axes.

We have here to solve the equation (vi) of our Art. 385, namely:

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)^2 w_0 = \frac{3}{2H\epsilon^3}\phi(xy),$$

subject in the case of a simply supported edge to the conditions

$$\left.\begin{aligned} &H\frac{d^2w_0}{dy^2}+(H-2f)\frac{d^2w_0}{dx^2}=0, \text{ when } x=0 \text{ or } a, \text{ for values of } y \text{ from } 0 \text{ to } b,\\ &H\frac{d^2w_0}{dx^2}+(H-2f)\frac{d^2w_0}{dy^2}=0, \text{ when } y=0 \text{ or } b, \text{ for values of } x \text{ from } 0 \text{ to } a,\\ &\qquad\qquad w_0=0 \text{ for all points of the contour.}\end{aligned}\right\}$$

The solution is easily found to be (p. 743):

$$w_0=\frac{3}{2\pi^4 H\epsilon^3}\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} A_{mn}\frac{\sin\dfrac{m\pi x}{a}\sin\dfrac{n\pi y}{b}}{\left(\dfrac{m^2}{a^2}+\dfrac{n^2}{b^2}\right)^2},$$

where $$A_{mn}=\frac{4}{ab}\int_0^a d\alpha\int_0^b d\beta \sin\frac{m\pi\alpha}{a}\sin\frac{n\pi\beta}{b}\phi(\alpha,\beta).$$

This result is applied to the calculation of the following special examples:

Case (a). A uniformly distributed load p per unit of surface area.

Here:

$$w_0=\frac{24p}{\pi^6 H\epsilon^3}\sum_1^\infty\sum_1^\infty\frac{1}{m'n'}\frac{\sin\dfrac{m'\pi x}{a}\sin\dfrac{n'\pi y}{b}}{\left(\dfrac{m'^2}{a^2}+\dfrac{n'^2}{b^2}\right)^2},$$

the summations being for odd numbers m', n' only.

The maximum or central deflection f_0 is very nearly given by the first term of this series with $x=a/2$, $y=b/2$, or we find

$$f_0=\frac{24\,(pab)\,a^3b^3}{\pi^6 H\epsilon^3\,(a^2+b^2)^2}.$$

The second term will, for $a=b$, be only 1/75 of this.

The maximum stretch s_0 is very nearly given by

$$s_0=\frac{24ab^3}{\pi^4 H\epsilon^2\,(a^2+b^2)^2}pab.$$

Case (b). An isolated central load $= P$. Here:

$$w'_0=\frac{6P}{\pi^4 Hab\epsilon^3}\sum_1^\infty\sum_1^\infty\frac{(-1)^{\frac{m'-1}{2}}(-1)^{\frac{n'-1}{2}}}{\left(\dfrac{m'^2}{a^2}+\dfrac{n'^2}{b^2}\right)^2}\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b},$$

where m' and n' are odd numbers only.

Here, $$f'_0=\frac{6Pa^3b^3}{\pi^4 H\epsilon^3\,(a^2+b^2)^2},$$

nearly, but not so approximately as in the like result for f_0 in *Case* (*a*), and

$$s'_0 = \frac{6ab^3}{\pi^2 H c^2 (a^2+b^2)^2} P.$$

Hence f'_0/f_0 and s_0'/s_0 for the same total loads $=\pi^2/4 = 2{\cdot}5$ nearly: see our Art. 263*.

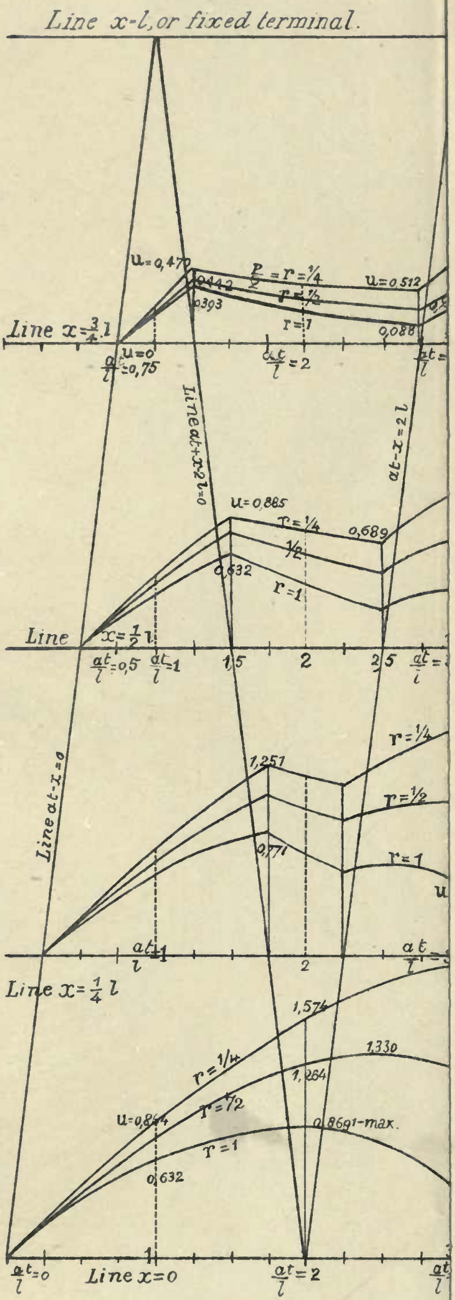
[400.] We have given a large amount of space to this monumental work because it contains much that is of value to both physicist and technologist, and we would gladly bring home to both the important services which Saint-Venant has rendered to the science of elasticity. His annotated *Clebsch* will long form the standard book of reference on our subject, but it is possible that the results we have here collected will reach some to whom it may not be accessible.

[401.] *Détermination et Représentation graphique des lois du choc longitudinal.* This memoir was presented to the Academy on July 16, 23, 30, and August 6, 1883. It appeared in the *Comptes rendus*, T. XCVII., 1883, pp. 127, 214, 281 and 353. An off-print of it with a note by Boussinesq (*Comptes rendus*, T. XCVII., 1883, p. 154) was afterwards put together and repaged. This off-print was distributed also as an appendix to the annotated *Clebsch*. Our references will be to the *sections* which are the same in the *Comptes rendus* and in the offprint.

The memoir is due to Saint-Venant in conjunction with M. Flamant the co-translator of the *Clebsch* and professor at the École des Ponts et Chaussées, Paris.

[402.] After a short account as in the *Clebsch* (see our Art. 341) of the evolution of the problem the authors refer to the analytical solution given by Boussinesq (see the same article and Boussinesq's *Application des potentiels à l'étude de l'équilibre... des solides élastiques*, p. 508 *et seq.*) and reproduced by them on pp. 480 k—480 gg of the *Clebsch*.

D'après cette Note (du § 60 de Clebsch), le choc longitudinal s'accomplit suivant des lois ayant des expressions analytiques différentes, se succédant l'une l'autre à des intervalles déterminés. Par exemple, les dérivées des déplacements des divers points de la barre varient, d'un instant à l'autre, tantôt avec gradation continue, tantôt par *bonds* considérables donnant aux mouvements une empreinte périodique de l'acquisition brusque de vitesse qui a été faite au premier instant du choc par l'élément heurté.



To face p. 277, Art. 404.

Il nous a donc paru utile de présenter ici aux regards, par une suite d'épures ou de diagrammes, une peinture de ces singulières et remarquables lois, afin de les éclairer et d'en faire bien comprendre la nature et les intéressantes conséquences. (§ 1.)

What then our authors accomplish is the *graphical* representation of the results of Boussinesq's analytical solution which was obtained in terms of discontinuous functions. It is one of the many instances in which Saint-Venant has helped to make of practical value the results of most intricate analysis. He was ever conscious that till theoretical formulae are reduced to simple numbers, the task of the mathematician is very far indeed from completion. Only the final diagram or numerical table can fitly crown the analytical labours of the mathematical physicist.

By means of such graphical representation we see at a glance the chief laws of the phenomena investigated, and are able to determine which approximate formulae we may fairly accept, which we must replace by others better adapted to represent the exact facts of the case.

[403.] I reproduce the more important diagrams of the memoir as their practical value for engineers and technologists seems to me very considerable.

The following notation is adopted[1]:

l = length, ω = cross-section, ρ = density, P = weight, E = stretch-modulus, u = shift (at distance x from impelled end) of the bar, $a = \sqrt{E/\rho}$, or the velocity of sound. One end of the bar is fixed, and we may suppose it placed horizontally and struck horizontally by a mass of weight Q with velocity V. If the bar be vertical the effect produced by its weight must also be taken into account.

At the instant at which the blow ends, $du/dx = s_x = 0$ for $x = 0$, (see our Art. 204) and the following numerical values are obtained:

$Q/P < 1\cdot7283$............ the blow ends between the times $t = 2l/a$ and $4l/a$,
$Q/P > 1\cdot7283 < 4\cdot1511$ .... $= 4l/a$ and $6l/a$,
$Q/P > 4\cdot1511 < 7\cdot35$ .... $= 6l/a$ and $8l/a$.

[404.] The first diagram which I reproduce gives the shifts u for zero, quarter, half and three-quarter span for times $at/l = 0$ to $at/l = 7\cdot5$. Along the horizontal axes at/l is measured, along the

[1] In the memoir the authors use a for our l, σ for our ω, and ω for our a.

vertical axis $u = au/(Vl)$. Three curves are drawn for $r \equiv P/Q = 1$, $\frac{1}{2}$ and $\frac{1}{4}$ respectively, and having for scale 20 mm. for the unit of both at/l and u.

The shifts for the duration of the impulse are denoted by a heavy line ending in a small circle which marks the end of the impulse; the shifts after the impulse are marked by dotted lines, till they begin to repeat themselves when the lines become again heavy.

Whenever $at - x$ or $at + x - 2l$ is a multiple of $2l$ we note sudden changes in the slope (or the *shift-velocity*) of these curves; the points where these changes occur are termed by the authors *points de brisures*.

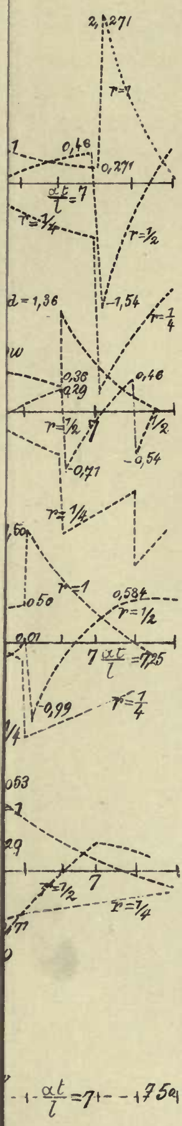
Les pieds des ordonnées de ces points de brisures sur les lignes horizontales d'abcisses marquées $x = l/4$, $x = l/2$, $x = 3l/4$ se trouvent, ainsi, aux rencontres de ces trois horizontales avec les obliques joignant en deux sens opposés les points $at/l = 0, 2, 4$...de l'horizontale $x = 0$ du bas, avec ceux $at/l = 1, 3, 5$...d'une horizontale $x = l$ tracée au haut. Celles de ces obliques qui montent de gauche à droite ont, en effet, pour équations $at - x = 0, 2l, 4l$...et celles qui descendent ont $at + x - 2l = 0, 2l, 4l$....Ces lignes obliques *figurent, en x et t, la marche de l'onde d'ébranlement*, tant directe que réfléchie aux extrémités de la barre, ou ce que parcourrait la tête de cette onde, si la barre vibrante était emportée perpendiculairement à sa longueur avec une vitesse a/l. Cela montre bien que les *bonds* et les *brisures* sont déterminés *par le passage de cette onde;* et cela donne une raison sensible du binôme et du trinôme $at - x$ et $at + x - 2l$ que M. Boussinesq a fait figurer dans ses formules de déplacements, etc. (§ 8).

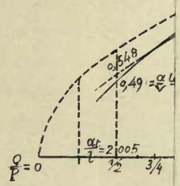
We see that in all cases the maximum shift is at the end which receives the impulse.

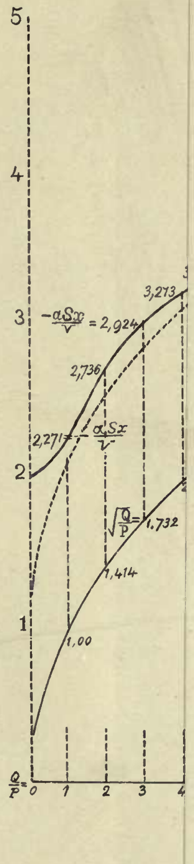
[405.] The second diagram (Fig. 4 of the memoir) gives graphically the law of squeeze at the terminals and at $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$ span for the same values of P/Q. Here the abscissae are at/l, and the ordinates $d = (-s_x)\, a/V$, or the squeezes reduced in the ratio of a to V. The scale of the abscissae is 20 mm. for $at/l = 1$ and of the ordinates 10 mm. for $d = 1$.

We note that in all cases the maximum squeeze is at the fixed end.

[406.] The third and fourth diagrams (figures 5 and 6 of the memoir) give:


2,271
r=1
0,48
0,271
r=1/4
r=1/2
d=1,36
-1,54
r=1/4
0,36
0,29
0,46
r=1/2
1/2
-0,71
-0,54
r=1/4
r=1
0,50
0,584
r=1/2
0,01
-0,99
r=1/4
053
r=1/2
r=1/4


0,548
0,49
1/2
3/4


5
4
3
2
1
3,273
2,924
2,736
2,271
1,732
1,414
1,00
0
1
2
3
4

(1) *The maximum shifts at the end struck* (u_m).

Here the abscissa gives Q/P from 0 to 6, and the ordinate $u_m a/(Vl)$, the scale of the unit of both being equal to 20 mm. The heavy line is given by the true theory; the broken line is the parabola given by the first approximation $\frac{a}{V}\frac{u_m}{l} = \sqrt{\frac{Q}{P}}$; the pointed line is the curve given by the second approximation $\frac{a}{V}\frac{u_m}{l} = \sqrt{\frac{Q}{P}\frac{1}{1+\frac{1}{3}\frac{P}{Q}}}$: compare our Arts. 943*, 368, and the *Historique Abrégé*, *Leçons de Navier*, footnote p. ccxxiii.

We see at once that the *Hodgkinson-Saint-Venant approximation gives the terminal shifts with very considerable accuracy, and may be adopted with safety for all values of* $Q/P > \frac{1}{2}$.

In the course of the calculations the following numerical results not indicated on the diagram are obtained:

The maximum shift u_m is reached if

$$\begin{array}{lll} Q/P < \;\; 5{\cdot}686 & \text{between} & t = 2l/a \text{ and } 4l/a, \\ Q/P > \;\; 5{\cdot}686 < 13{\cdot}816 & \;\;\;\;\text{,,} & t = 4l/a \text{ and } 6l/a, \\ Q/P > 13{\cdot}816 < 25{\cdot}16 & \;\;\;\;\text{,,} & t = 6l/a \text{ and } 8l/a. \end{array}$$

(2) *The maximum squeezes* ($-s_x$) *at the fixed end.*

The three curves have for abscissae the values of Q/P from 0 to 25·10; the upper heavy curve has for ordinates the exact values of $(-s_x)\,a/V$ where s_x is given its maximum value, i.e. at the fixed end. The lower heavy curve is the parabola obtained by taking for ordinates

$$d = \frac{a}{V}(-s_x) = \sqrt{\frac{Q}{P}};$$

and the dotted curve by taking for ordinates

$$d = \frac{a}{V}(-s_x) = \sqrt{\frac{Q}{P}} + 1.$$

For the abscissa-scale 5 mm. is taken for $Q/P = 1$, and for the ordinate scale 20 mm. for $d = 1$.

It will be seen that the true values differ immensely, from the values given by the old formula $d = \sqrt{Q/P}$. Thus that formula never suffices for finding the maximum squeeze[1].

[1] It may be obtained as follows: Suppose the shift uniformly distributed and its maximum mean value $= u_m$. Then work done $= \frac{1}{2}E\omega l\left(\frac{u_m}{l}\right)^2 = \frac{1}{2}\frac{Q}{g}\frac{V^2}{2}$ when the maximum is reached. Hence

$$(-s_x)_m = \frac{u_m}{l} = \frac{V}{a}\sqrt{\frac{Q}{P}}, \quad \text{or} \quad \frac{a}{V}\left(\frac{u_m}{l}\right) = \sqrt{\frac{Q}{P}}.$$

If, however, we take

$$d = \sqrt{\frac{Q}{P}} + 1,$$

we get the dotted curve of our fourth diagram, which from $Q/P > 5$ approaches closely to the true curve. Saint-Venant gives the following practical rules:

(*a*) For values of $Q/P > 24$ take $d = \sqrt{\frac{Q}{P}} + 1$,

(*b*) .... > 5 and < 24...... $d = \sqrt{\frac{Q}{P}} + 1{\cdot}10$.

(*c*) ....Q/P between 0 and 5...... $d = 2\ (1 + e^{-2P/Q})$,

this latter being the exact formula.

[407.] There are one or two other points in § 12 which we may note:

(1) The authors refer to the condition for *cohésion permanente* which is to be obtained from the maximum squeeze given by the results of Art. 406 (2).

Si les chocs ont eu pour tendance de raccourir, et *s'il ne devait en résulter que des compressions*, ces mêmes quantités numériques $-(du/dx)_m$ seraient à égaler à un nombre plus grand T'_0/E, T'_0 étant la limite, toujours très au-dessus de T_0, des forces comprimantes non dangereuses. Mais, comme nous avons vu que, dans la première période *de la détente libre* qui suit le choc, il se produit des dilatations égales aux compressions ayant précédé, le danger de désagrégation de la matière survit à la jonction, et la prudence conseille de traiter les compressions sur le même pied que les dilatations, ou d'égaler leurs valeurs numériques (see our Art. 175) à la même limite T_0/E que si c'étaient des dilatations.

This does not seem to me the necessarily true condition for the safety of the structure. The real limit of T_0, here, *must be that for a repeated positive and negative strain*, and if we are to give credence to Wöhler's experiments this is not the T_0 for a fail-point in pure tractional experiments. According to Wöhler the former is much less than the latter: see our account of his researches later.

(2) In a footnote (§ 12) the authors remark that the negative traction must be such that it does not cause the bar to buckle. They add that no bar will buckle unless the load is $> \pi^2 E\omega\kappa^2/l^2$, so that they treat the bar as a doubly pivoted strut (see *Corrigenda* to our Vol. I., Art. 959*). It seems just as probable that the bar would have one end built-in, in which case we might take double of the above load. The footnote then continues:

L'on peut admettre analogiquement, et même, ce semble, comme un *a fortiori*, que cette barre d'un poids P, sollicitée par le choc comprimant d'un corps Q tendant à y développer en un seul endroit, et comme maximum, une pression longitudinale égale à $E\omega(-s_m)$, s_m étant la valeur ci-après (see Art. 406 (*a*)) ne fléchira pas si l'on a

$$E\omega \frac{V}{a}\left(\sqrt{\frac{Q}{P}}+1\right) < \frac{\pi^2 E\omega\kappa^2}{l^2}.$$

This result[1], I think, extremely doubtful, in particular when we take into account the want of accord between the Eulerian theory of struts and experience: see our Arts. 957*—961*, 1255*—1262*. The authors remark of the above condition that it is *presque toujours remplie*, but I should be uneasy with regard to any structure where the above quantities had any approach to equality.

(3) At the end of § 12 it is shewn by a process involving the determination of mean values that the expression given in our Art. 406 (*a*) is really a close approximation to the true result. This is also proved in Boussinesq's note attached to the memoir: see also the work of his referred to in our Art. 402, p. 544.

[408.] *Remarques relatives à la Note de M. Berthot sur les actions entre les molécules des corps: Comptes rendus*, 1884, T. XCIX., pp. 5—7.

Berthot in a memoir of 1884 (*Comptes rendus*, T. XCVIII., p. 1570) had suggested the following law of intermolecular force

$$F(r) = Kmm' \frac{r - r_0}{r^3},$$

where m, m' are the masses of two molecules at distance r and K, r_0 are constants. It is obvious that the force changes from attraction to repulsion at $r = r_0$.

Saint-Venant remarks that in 1878 in a footnote to a memoir, *Sur la constitution des atomes* (p. 37: see our Art. 275), he had referred to a law of somewhat like form.

In both cases the force tends to follow the gravitational law when r is much greater than r_0. Saint-Venant refers to the forms given by Poisson and Poncelet for representing intermolecular force (see our Arts. 439* and 977*), but he holds that although such laws are suggestive, it is very unlikely that in the present state of science we shall hit upon the correct one. He

[1] The memoir has $\frac{\sqrt{Q}}{P}$ for $\sqrt{\frac{Q}{P}}$. I may note also the following errata:

In equations (11), (12) and (13) the exponentials following the *curled* brackets should be placed inside them.

In equation (46) for first P/Q read Q/P.

observes that the discovery of its absolute form indeed is unnecessary for the establishment of the formulae of elasticity, hydraulics and electricity.

[409.] *Sur la flexion des prismes. Comptes rendus*, T. CII. 1886, pp. 658—664 and pp. 719—722. This memoir by Resal professes to point out an error in Saint-Venant's memoir on the flexion of prisms of 1856: see our Art. 69. The writer notes that Saint-Venant fixes the direction of a rectilinear element of the first face and not the direction of the first element of the prismatic axis. He then proceeds to assert that Saint-Venant has not taken account of the relation $\int \widehat{xz} d\omega + P = 0$: see our Art. 81. He endeavours to shew that this has led Saint-Venant to erroneous results in the case of the elliptic cross-section, but he himself falls into an error in his algebra, and so gives the colour of an error to Saint-Venant's work.

Boussinesq in a note in the same volume of the *Comptes rendus*, pp. 797—8, entitled: *Observations relatives à une Note récente de M. Resal sur la flexion des prismes*, points out Resal's algebraic error, and remarks that the difference between the terminal conditions of Saint-Venant and those proposed by Resal only produces a small rotation of the coordinate axes, and introduces no change into the expressions for the strains or stresses.

Resal in a few words (p. 799) acknowledges his error.

[410.] Saint-Venant died on January 6, 1886. The President of the Academy on announcing his death at the meeting of the Academy on the 11th used the following words:

> La vieillesse de notre éminent confrère a été une vieillesse bénie. Il est mort plein de jours, sans infirmités, occupé jusqu'à sa dernière heure des problèmes qui lui étaient chers, appuyé pour le grand passage sur les espérances qui avaient soutenu Pascal et Newton (*Comptes rendus*, T. CII., 1886, p. 73).

Short notices of his life appeared in the *Comptes rendus*, T. CII., pp. 141—7, 1886 by Phillips, and in *Nature*, Feb. 4, 1886 by the Editor of the present volume. A full and excellent account by Boussinesq and Flamant of his life and work, together with a complete bibliography of his contributions to science, was published in the *Annales des Ponts et Chaussées* for November, 1886. In presenting this paper to the Academy, Boussinesq said:

Nous avons tâché d'y rappeler, avec tous les détails que comportait l'étendue matérielle de texte dont nous pouvions disposer, l'existence si bien remplie et les travaux les plus marquants du profond ingénieur-géomètre, notre maître à tous deux, qui a été une des gloires de l'Académie à notre époque et un modèle pour les travailleurs de tous les temps. (*Comptes rendus*, T. CIV., 1887, p. 215.)

A more popular account of Saint-Venant's life based chiefly on the notices in the *Annales* and *Nature* will be found in the *Tablettes biographiques; Dixième Année*, 1888.

[411.] *Summary.* In estimating the value of Saint-Venant's contributions to our subject, we have first of all to note that he is essentially the founder of practical, or better, technical elasticity. In his whole treatment of the flexure, torsion and impact of beams he kept steadily in view the needs of practical engineers, and by means of numerical calculations and graphical representations he presented his results in a form, wherein they could be grasped by minds less accustomed to mathematical analysis. At the same time he was no small master of analytical methods himself, and he undertook in addition purely numerical calculations before which the majority would stand aghast. His memoirs on the distributions of elasticity round a point and of homogeneity in a body opened up new directions for physical investigation, while his numerous discussions on the nature of molecular action have greatly assisted towards clearer conceptions of the points at issue. The hypotheses of modified molecular action and of polar molecular action may either or both be true, or false; but we see now clearly that it is to the investigation of these hypotheses and not to the experiments of Oersted etc. nor to the viscous fluid and ether jelly arguments of the first supporters of multi-constancy to which we must turn if we want to investigate the question of rari-constancy[1]. Saint-Venant's foundation, on the basis of Tresca's investigations, of the new branch of theoretical science, which he has termed *plastico-dynamics*, has not only direct value, but shews clearly the fallacy of those who would identify plastic solids and viscous fluids. The fundamental equations in the two cases differ in character; a difference which may be expressed in the words—the plastic solid

[1] This is well brought out by the comparison of Voigt's recent memoir (*Göttinger Abhandlungen*, 1887) with those of the early supporters of multiconstancy.

requires a certain magnitude of stress (shear), the viscous fluid a certain magnitude of time for any stress whatever, to permanently displace their parts.

Not the least merit of Saint-Venant's work is the able band of disciples he collected around him. His influence we shall find strongly felt when investigating the work of Boussinesq, Lévy, Mathieu, Resal and Flamant. He formed the connecting link between the founders of elasticity and its modern school in France.

The vigorous spirit, the striking mental freshness, the perfect fairness of his thought enabled him to penetrate to the basis of things; the depth of his affection, his kindly foresight and consideration, his rare personal devotion attached to him all who came in his way and stimulated them to renewed investigation (Flamant and Boussinesq: *Notice sur la vie et les travaux de B. de St. V.*, p. 27).

INDEX[1].

The numbers refer to the articles of the book and not *to the pages unless preceded by p. C. et A.* = *Corrigenda and Addenda to Volume I. attached to this Part. ftn.* = *footnote.*

[1] This index will be incorporated in that for the entire second volume on its completion.

CORRIGENDA AND ADDENDA
TO VOLUME I.

CORRIGENDA.

Art. 922.

I have used an expression in this article with regard to Weyrauch's contribution to the problem of rari-constancy which is undoubtedly liable to misinterpretation. It might be supposed from what I have written that Weyrauch had obtained rari-constant equations on the assumption that the intermolecular action although central was any function *whatever*, e.g. a function of 'aspect' or involving 'modified action terms'. What he really does (*Theorie elastischer Körper*, 1884, p. 132) is to take a central action R between two elements of masses m and m', at distance r of the form:

$$R = mm' \{F(r) - i\} \text{.........................(i)},$$

where, in his own words:

"$mm'i$ ganz allgemein eine Function derjenigen Grössen bedeutet, welche neben der Entfernung r auf R Einfluss nehmen."

This of course is something different from taking R of the form:

$$R = mm' F(r, i) \text{.......................(ii)}.$$

Further, if i_0 represents the value of i before strain or at time t_0, and i the value at time t, Weyrauch assumes (p. 134) that $i - i_0$ for the material in the neighbourhood of the element m may be treated as constant and brought outside the sign of summation for elementary actions. This would be impossible, if $i - i_0$ were due to 'modified action,' because the modifying elements (or molecules) would be themselves in the immediate neighbourhood of m, and the modifying action would probably be a function of their distances which are themselves commensurable with the linear dimensions of the "neighbourhood of the element m."

By taking R of the form (i) and not (ii) Weyrauch much limits the generality of his results, and by choosing $i - i_0$ a constant for the neighbourhood of an element, he practically reduces his $(i - i_0)$ to little more than the temperature-effect. But even this may serve to indicate that wider laws of intermolecular action than that in which it is central and a function of the distance only may be found to lead to rari-constant equations.

Art. 959.

The formulae for the buckling load on struts were taken from notes of mine in which $2l$ and *not* l was the length of the strut. This, however, does not apply to the point of maximum traction or other results of this same Article. We have with this correction the following results for a strut of length l:

Buckling force for doubly built-in strut

$$= E\omega \frac{\dfrac{4\pi^2\kappa^2}{l^2}}{1 + \dfrac{4\pi^2\kappa^2}{l^2}}.$$

Buckling force for built-in pivoted strut

$$= E\omega \frac{\dfrac{\pi^2\kappa^2}{l^2}\, 2{\cdot}047}{1 + \dfrac{\pi^2\kappa^2}{l^2}\, 2{\cdot}047}.$$

Buckling force for doubly pivoted strut

$$= E\omega \frac{\dfrac{\pi^2\kappa^2}{l^2}}{1 + \dfrac{\pi^2\kappa^2}{l^2}}.$$

I much regret that this error should have escaped my attention, and trust all possessors of the first volume will make the above changes in the text.

Arts. 795—6.

I have reproduced an error of Neumann's which I ought to have seen and corrected. The wrong signs are given to all the quantities M, N, P in Art. 796. If these are corrected a negative sign must be inserted in the second table of Art. 795 before all the $1/F$'s. The value of $1/E_r$ in Art. 799 is then accurate.

Index, p. 899, Column (ii) and Arts. 813—16.

The title *Bresse* has been inserted between *Bevan* and *Binet*, when it ought to follow *Braun* on p. 900, Column (i). There should also be a reference under *Bresse* to Arts. 813—16. I find that the lithographed course of lectures there referred to is due to this scientist, to whom we thus probably owe the first theory of the 'core.'

ADDENDA.

Arts. 352, 353, 354—5, 745—6.

A paper by A. Müttrich on Chladni's figures for square-plates appeared in 1837 in the *Geschichte des altstädtischen Gymnasiums. Dreizehntes Stück*, Königsberg. It is entitled: *Beitrag zur Lehre von den Schwingungen der Flächen*, and contains 8 pages and a plate of figures. Pp. 1—5 suggest practical methods of supporting the plates, of setting them vibrating, and of keeping their surfaces dry and clean. Pp. 6—8 give Müttrich's conclusions and the grounds on which he bases them. Two of them are opposed to Strehlke's views of 1825 as given in our Vol. I., Art. 354, namely Müttrich holds:

(i.) Straight lines are possible forms for the nodal lines of plates with free edges.

(ii.) Nodal lines can intersect one another.

The experimental proof of these results lies in the demonstration of a *gradual* transition from one system of nodal lines to another, when intermediate stages are necessarily intersecting straight lines.

Müttrich's third conclusion is that the nodal lines themselves are in a state of vibration and that only their nodal points are true nodes for the plate. It seems to me possible that this oscillation of the nodal lines results from longitudinal vibrations in the plate which again are due to its sensible thickness, or to the mode of support and excitation.

Art. 937.

A copy of Ardant's work which was printed as a separate publication by "order of the minister of war" has reached me since the printing of Vol. I. The title is: *Études théoriques et expérimentales sur l'établissement des charpentes à grande portée*, Metz, 1840. It contains *Avertissement* pp. i—v; the report referred to in our Vol. I., Art. 937, pp. vi—xvii; the text of the work pp. 1—94; *Appendice* pp. 95—122, and concludes with five pages (123—127) of contents and twenty-nine plates of figures. It is obvious that the work is one of considerable size, and as it possesses some importance, I give here a résumé of its contents.

[i.] Chapter I. (pp. 1—11), briefly describes the origin and history of wooden trusses designed to cross considerable spans, more especially roof-trusses. These range from the 4th century roof of the Basilica of Saint-Paul's, through the frame 'à la Palladio,' the arched truss of Philibert de l'Orme, and the Gothic roof to the English truss with iron tie-bars, and to the arched forms common in France in 1840. Ardant gives at the end of the chapter a summary of the conclusions he has formed upon the comparative merits of arched timber trusses and trusses built up of straight pieces of timber. He believes the former to be very inferior to the latter in both economy and strength; while the latter can be easily made to present as pleasing an artistic effect. He holds the adoption of the former to have arisen partly from the mistaken notion that a semi-circular arch produced little or no thrust on the abutments, partly from an unreasoning extension of the theory of stone arches to wood and iron:

> Dans la première de ces constructions, on utilise la pesanteur, la rigidité et l'inflexibilité relatives des pierres; dans les secondes, c'est l'élasticité et la cohésion des parties qui sont les qualités essentielles (p. 10).

Chapter II. gives an account of the fifteen arches and frames (with spans so large as 12·12 metres and rise so large as 5·41 metres), upon which experiments were made, as well as the apparatus with which they were made.

[ii.] Chapters III., IV. and V. cite the theoretical results of the Appendix for the thrust in terms of the load in the cases of circular arches and of a simple roof-truss of straight timbers. The thrust for the latter is not materially greater than that for the former. Hence no gain is obtained by combining the two which appears to have been frequently done in practice:

On tirera de cette comparaison une conclusion assez opposée à l'opinion de la plupart des constructeurs, savoir;

Que dans les cas ordinaires de la pratique, un cintre demi-circulaire exerce autant de poussée que la ferme droite sans tirant, à laquelle on le réunit pour composer une charpente en arc; et que, par conséquent, on pourrait, en augmentant l'équarrissage de cette ferme, supprimer le cintre sans qu'il en résultât sur les appuis, une action horizontale plus considérable (p. 25).

These chapters then compare the experimental measure of the thrust with that given by theory. The comparison gives an accordance fairly within the limits of experimental error. Unfortunately Ardant did not make a sufficiently wide range of observations for the results to be quite conclusive. He cites an experiment of Emy which led the latter to believe that circular arches had no thrust. He then considers experiments made by Reibell at Lorient. These appear to be the only other important experiments which had been made on large circular wooden arches. An account of them was published in the *Annales maritimes et coloniales* 22e année, 2e série, T. XI., p. 1009. Reibell did not get rid of the friction at the terminals of the arch, but allowing for this Ardant finds the corrected values of the thrusts agree well with his formulae (pp. 32—33). From this double set of experiments he draws the following conclusions:

(*a*) The thrust of a semi-circular arch due to an isolated central load never exceeds $\frac{1}{5}$ of the load.

(*b*) Whatever be the manner in which a continuous load is distributed along the arch, the thrust for a semi-circular arch never exceeds $\frac{1}{4}$ to $\frac{1}{3}$ of the total load.

(*e*) That flatter arches produce thrusts which are to those which arise in the case of a semi-circular arch in the ratio of the half span to the rise.

(*f*) That the thrust is independent of the particular mode of construction of the arch, when its figure, dimensions and the load-distribution are the same.

Chapter V. shews that the thrust-formula obtained in the Appendix for the truss with straight timbers and without a tie, is confirmed by experiment.

[iii.] Chapter VI. begins with some general discussion on elasticity, the elastic constants and the coefficients of rupture. Ardant then cites a formula of the following kind for the deflection, f, of a circular arch at the summit, the terminals being both pivoted:

$$f = K\frac{PY^2X}{E\omega\kappa^2},$$

where $2X$ is the span, Y the rise, E the stretch-modulus, $\omega\kappa^2$ the moment of inertia of the cross-section, P the total load and K a constant depending on the distribution of the load etc. Here the arch is supposed to be of continuous homogeneous material and of uniform cross-section. Ardant now applies this formula to the deflections he has found by experiment for his arches built up of curved pieces or planks pinned or bound together. The results given in Chapter VII. he holds to satisfy this formula, provided E be given values depending on the nature of the structure, from $\frac{3}{5}$ to $\frac{3}{50}$ of its value for a continuous arch or beam of the same material. The experiments even on the same arch seem to me to give such divergent values for E, that I think this method of exhibiting the deflection can only be looked upon as an expression of experimental results for practical purposes. With certain assumptions Ardant also obtains an expression for the deflection of a roof truss without tie, built up of straight beams (pp. 48—49). I do not consider this expression to be theoretically or experimentally justified. Ardant proceeds at the end of Chapter VII. (pp. 61—68) to determine the resistance to rupture of his arches. Here he applies to rupture a formula deduced from the theory of continuous arches on the hypothesis that linear elasticity holds up to rupture. At best the theory could only apply to the *fail-point* (i.e. failure of linear elasticity) of continuous arches. A like treatment of rupture leads to absurd results in the case of the flexure of beams, so it can hardly be expected to give better results in the case of arches: see our Vol. I. Art. 1491 and Vol. II. Art. 178. Thus, as we might naturally expect, his "coefficient of rupture" varies from arch to arch, and its ratio in each case to the "coefficient of rupture" for a continuous arch is equally variable. The results, however, of his experiments resumed on (pp. 67—8) are suggestive for the practical design of such arches and roof-trusses as he has experimented on.

[iv.] In Chapter VIII., it is sufficient to notice here Ardant's conclusion that the truss built up of straight beams is for the same amount of material stronger than the built-up wooden arch:

Il semble d'après cela que si les charpentes en arc conservent quelque avantage sur les fermes droites, c'est uniquement celui d'avoir une forme plus gracieuse, et que sous les rapports importants de la solidité et de l'économie, les premières sont très-inférieures aux autres (p. 75).

Chapter IX. gives methods of calculating suitable cross-sections for the various parts of arches of the types on which Ardant has experimented. It also gives some attention (pp. 77—80) to the thickness and height of the masonry which will stand the thrust of a given roof-truss. It concludes with two numerical examples of the application of the formulae of the appendix to the calculation of the dimensions of metal arches.

[v.] We now reach the *Appendice*, which is entitled: *Théorie de la flexion des corps prismatiques dont l'axe moyen est une droite ou une courbe plane* (pp. 95—122). This contains the first theory of circular arches which attains to anything like completeness (see our Vol. I. Arts. 100, 278, 914) and it anticipates Bresse's later work on this subject: see our Vol. I. Arts. 1457—8, and Vol. II. Chapter XI. for an account of the book referred to in these Articles. We note a few points with regard to this Appendix.

(*a*) Pp. 95—100 give the ordinary Bernoulli-Eulerian theory of flexure. On p. 98 Ardant speaks of the product of the stretch-modulus and moment of inertia of the cross-section (namely $E\omega\kappa^2$ in our notation) as improperly termed the *moment d'élasticité*. It is the *moment de roideur* of Euler (Ek^2 in his notation: see our Vol. I. Art. 65) or the 'moment of stiffness.' This 'moment of stiffness,' $E\omega\kappa^2$, occurs so frequently that we have ventured to term it the 'rigidity' of a beam. It follows from this definition that the product of the rigidity and curvature is equal to the bending-moment. Thus for the same value of the bending-moment the curvatures of a series of beams vary inversely as their rigidities.

(*b*) Pp. 100—103 deal with rupture on the old lines, i.e. as if linear elasticity lasted up to rupture. The results obtained are thus only of value when we treat the 'coefficient of rupture' R which occurs in them as the 'fail-limit.' Accordingly the Tables on p. 103 for rupture-stresses are meaningless when applied to the previous flexure formulae. On pp. 99 and 101 we have the rigidity and fail-moment (here called *moment de rupture*) calculated for 'skew-loading' or for the case when the load-plane does not pass through a principal axis of inertia of each cross-section: see our Vol. I. Arts. 811, 1581, Vol. II. Arts. 14, 171. To judge by Ardant's reference to Persy's lithographed *Cours*, the latter possibly did more for the theory of skew-loading than I judged from an examination of only one edition of that *Cours*: see Vol. I. Art. 811. The value given by Ardant on p. 101 for the fail-moment of a beam of *rectangular* cross-section under skew-loading is incorrect, it applies only to the case of *square* cross-section. The true value is given in our Vol. II. Art. 14.

(*c*) Pp. 104—115 are occupied with a consideration of the elastic line under various systems of loading in the case of straight beams, besides a discussion of combined strain. The results obtained are afterwards applied to various types of simple roof or bridge trusses, in which the members are supposed mortised and not merely pinned at the joints. Ardant's treatment of these trusses seems to me from the theoretical standpoint extremely doubtful, and I should hesitate before applying his results even to the practical calculation of dimensions. The remark in § 34, p. 107, on the sign to be given to a certain quantity is, I think, erroneous. The fail-point of a beam is not necessarily where the *stress* is greatest as Ardant like Weisbach (see Vol. I. Art. 1378) holds. It will be at the point of *maximum stretch*, and this will be at the side of the cross-section in tension or compression according as the load-point is outside or inside the whorl of the cross-section: see Vol. I. p. 879.

(*d*) Pp. 115—121 contain the theory of flexure of circular ribs or arches. Ardant's work here was up to his date the most complete treatment of the subject, and his Table on p. 45 for thrust and deflection based upon this theory may even now be of practical service. He obtains the thrust and deflection for circular ribs with an isolated load, or with uniform loading distributed along either the span or rib, when the terminals of the rib are *pivoted*. He finds also for a complete semi-circle, that the points of maximum horizontal shift are about 63° from the vertical. He throws all his results into very simple approximate forms, which he holds accurate enough for practice. I refrain from quoting these theoretical results, because they have been worked out with greater generality and accuracy by Bresse in a work with which I shall deal fully in Chapter XI. At the same time Ardant's researches must be remembered as an important historical link between those of Navier and Bresse. That the latter had studied them may be seen from our Vol. I. Art. 1459.

What I have noted in Ardant's memoir, will probably be sufficient to mark its importance. Experiments on such large wooden arches and frames have I believe not been repeated and it seems improbable that they ever will be. The results obtained will therefore remain of value, so far as roof-structures of the types with which Ardant dealt, are concerned. In addition to the experimental data of the memoir I may mark Ardant's conclusion, that the same theoretical formulae hold for an arch of continuous material and one built-up of bent pieces of wood or planks bolted or bound together provided we reduce the stretch-modulus in a certain proportion. Finally I have already noted the historical value of the memoir as a step in the theory of circular arches or ribs.

Art. 974.

Poncelet. *Cours de mécanique industrielle, fait aux artistes et ouvriers messins, pendant les hivers de* 1827 *à* 1828, *et de* 1828 *à* 1829, *Première partie. Préliminaires et applications.* Metz, 1829. I have procured a copy of this work since the publication of Vol. I. It contains xvi pages of prefatory matter, 240 pages of text, and 8 pages of contents at the end. The first preliminary 145 sections agree with those in the third edition by Kretz (1870). In the *Applications* the Metz edition agrees fairly with Kretz's up to section 197; after this it deals with the resistance and motion of fluids, thus containing nothing concerning the resistance of solids to which the *Deuxième Partie* of the 3rd edition is devoted. The few paragraphs on the *Élasticité des corps*, pp. 17—20, are thus all it contributes to our subject: see our Vol. I. Art. 975. The chief interest of the work is the place it takes in the origin of modern technical instruction.

Art. 1249.

A further memoir by Brix which had escaped my attention may be referred to here: *Ueber die Tragfähigkeit aus Eisenbahnschienen zusammengesetzter horizontaler Träger.* This is an offprint from the *Verhandlungen des Vereins zur Beförderung des Gewerbfleisses in Preussen*, Berlin, 1848, 16 pages and a plate.

Owing to some peculiar local conditions at a Berlin mill it was necessary to build bridges, of which the girder-depth had to be very small, over the mill-races. For this purpose pairs of railway rails with flat bases ('*sogenannte Vignolsche*') were placed base to base and used as girders. The bases were riveted together at short intervals. Experiments were made on the flexure and ultimate strength of two such girders; in the one the bases were riveted close together, in the other there were placed at the rivets small intervening blocks of cast-iron. The first part of the paper (pp. 1—6) is occupied with an account of the experiments made upon these two girders, for the details of which,—too individual to be of much general use—I must refer to the paper itself. The rupture, by shearing of the rivets, only seems to shew that the area of the riveting was very insufficient, as the load required to produce failure in a bar under flexure by longitudinal shearing is immensely greater than that required to produce failure by stretch in the 'fibres', the order of the ratio of these loads being practically that of the length to the diameter of the bar.

The second part of the paper,—that specially due to Brix—deals with the theory of the flexure of a beam (a) with both terminals supported, (b) with one terminal supported and one built-in, (*c*) with both terminals built-in,—the load in all cases being partially uniform and continuous and partially isolated and central. The treatment of these problems by the Bernoulli-Eulerian theory presents no difficulties, but it has long been known that the *absolute strength* of beams under flexure calculated by this theory is very far from according with experiment (see our Vol. II. Art. 178). Hence there does not seem much value in the numerical results given on pp. 12—16 and based on the preceding experiments. Two points in Brix's work may be noticed. He *assumes* the maximum curvature (which gives the maximum stretch and so the fail-point) to be either at the built-in end or the centre of the beam in case (*b*), but this is by no means obvious, it requires an investigation similar to that given by Grashof in Arts. 58—9 of his *Theorie der Elasticität,* 1878. Secondly, he shews, I believe for the first time, that the fail-point for a *uniformly* loaded beam, either doubly-built-in or built-in and supported, is at the built-in end; in the former case the bending-moment at the centre is only half its value at the built-in ends.

Arts. 1180 and 1402, ftn.

Just as this page goes to press a copy of Seebeck's paper in the *Programm* of the Dresden Technical School (1846) reaches me. It contains a good deal of valuable matter, and I shall take the opportunity of referring to it with other papers of Seebeck's in the course of Vol. II.

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