Strength of Materials

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PREFACE

In preparing this book the author has had in mind primarily the needs of his own students in strength of materials. He hopes, however, that it will meet a real want in other colleges and technical schools also.

This book has been written with the aim of making intelligible the fundamental principles of the strength of materials without the formal use of the calculus. The works which do not use the ordinary calculus treatment usually omit some important parts such as the deflection of beams, strength of columns, horizontal shear, combined stresses, impact loads, etc. This book is designed to give a fairly complete course in the subject for students who have not had the calculus, or when graphical presentations are preferred. However, a separate chapter giving the derivation of the elastic curve of beams by the calculus method has been included for those who desire such treatment.

Effort has been made to present the derivation of the formulas in a clear and concise manner, in such a way as to enable the student to obtain an adequate comprehension of the principles involved. While the aim is to emphasize the elementary principles and to develop independent reasoning in the student, the ground covered is that usually given in a college course for engineering students. Many illustrative examples and problems are given for the purpose of making clear the application of the theory. Answers to some of the problems are given in order that the student may occasionally check his numerical work. The order of the arrangement is one that has given good satisfaction.

In the deduction of the shear formula it is brought out at the first that the shearing stress is not uniformly distributed over the sectional area of the beam and that the maximum stress is greater than that obtained by dividing the vertical shear by that area. A chapter on graphic integration is included and the graphical method of determining the deflection of beams is utilized. The graphical method appeals to the eye as well as to the reason, and thus supplies an additional avenue of conception. It also shows to advantage the meaning of the constants of integration. The graphical method is also much more readily applicable to beams carrying non-uniform distributed loads, and to beams for which the moment of inertia of the cross section is not constant. When one set of curves is drawn for a given beam carrying a given system of loading, those curves may be used for all similar beams with similar loading. In the chapter on the calculus method an attempt is made to give the physical conception of the constants of integration rather than to treat them simply as mathematical symbols.

As the nature of the behavior of columns under load is very uncertain, the treatment given to columns is largely empirical. Emphasis is laid on the straight line formula, although the Euler and the Rankine formulas are also given.

The author wishes to acknowledge his indebtedness to the following professors and instructors in the College of Engineering of the University of Illinois: to Dr. N. Clifford Ricker for the interest shown in the preparation of this work, and the use of tables and data prepared by him; to Professor A. N. Talbot and Professor H. F. Moore for many suggestions as to the form, arrange-

ment, and subject matter, and much assistance in the preparation of the book; to Mr. G. P. Boomsliter for his criticizing and checking the examples and problems; to Mr. C. R. Clarke and Mr. C. E. Noerenberg for their criticisms and help in preparing the manuscript for the publishers.

Although the work has been carefully checked, errors may exist, and for any intimation of these I shall be obliged.

H. E. MURDOCK.

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CHAPTER I

MATERIALS OF CONSTRUCTION

1. INTRODUCTION. **Strength of Materials** treats of the action of the parts or members of structures or machines in resisting loads and other forces which come upon them. By the use of the principles of mechanics and the properties of materials, it determines the internal forces or stresses which are developed in the simpler forms of construction, as beams and columns, when they are subjected to loads. The properties of the engineering materials are obtained through experimental tests. Many of the formulas derived in strength of materials are based on both theoretical analysis and experimental data, and the subject, therefore, is of a semi-empirical nature.

In architectural and engineering construction, stability, strength, durability, and economy are essential elements. The proper proportioning, spacing, and connection of the parts are important. Too little material in a member would make the structure unsafe, and too much would mean a waste. In general, one member should not be designed in such a way that it will be weaker than others in the structure. Proper design, then, takes into account the properties and qualities of materials and the mechanics of their action in a structure in such a way as to insure safety and economy.

2. Mechanical and Physical Properties. The materials of construction possess characteristic properties known as mechanical and physical properties. These properties measure the fitness and ability of the material to sustain external loads or forces under given conditions. Different materials possess these properties in different degrees, and, of course, different grades of the same material differ in their properties. Some of these characteristic properties can be expressed quantitatively between fairly well defined limits which are determined by test, while others may be specified in terms of ability to withstand certain tests and fulfill certain requirements. The mechanical properties include strength, elasticity, stiffness, and resilience. Other physical properties frequently referred to are toughness, ductility, malleability, hardness, fusibility, and weldableness.

When a load is applied to a piece or member of a structure the material undergoes a change in size and shape. If on the removal of the load the original size and shape are resumed the material is said to be elastic. **Elasticity,** then, is the property of a material by which it will regain its original size and shape on the removal of an applied load. A material which will not recover its original dimensions after deformation is termed **plastic.** If it will only partially recover its original dimensions after deformation it is said to be partially elastic and partially plastic. Most constructional materials are nearly or quite perfectly elastic up to a certain limit of deformation, beyond which they are partly elastic and partly plastic.

The ability to resist change in shape and size when a load is applied is termed **stiffness.** In elastic materials the amount of change in size and shape is generally proportional to the amount of the load applied.

Materials will differ in their tensile, compressive, and shearing strengths. The **strength** of a material is ordinarily determined under the application of a static load applied in a slowly increasing amount. The effect of permanent loads, of suddenly applied loads, and of impact loads, and of the repetition of a load many times, requires separate consideration.

A material possesses the property of **ductility** if the length can be increased and the cross section decreased considerably before rupture occurs. **Toughness** is that property by which a material will not rupture until it has deformed considerably under loads at or near its maximum strength. This deformation may be produced by stretching, bending, twisting, etc. A tough material gives warning of failure. It will resist impact and will permit rougher treatment in the manipulations which attend fabrication and use. A **brittle** material will rupture without developing much deformation and without giving warning. Brittle materials are unfitted to resist shock or sudden application of load.

3. MASONRY. Masonry is mostly used to carry compression loads, such as come on foundations, walls, piers, chimneys, etc.

(a) **Stone Masonry.** The kinds of stone that are best adapted to building and construction purposes are those that can be worked satisfactorily, can be obtained in suitable size, have great compressive strength, and are durable. Sandstone, limestone, marble, granite, trap, and slate are those in most common use. Stone masonry is laid up in mortar, and the quality and character specified will depend upon the purpose and need of the structure. The weight of stone masonry is about 160 pounds per cubic foot.

(b) **Brick Masonry.** Many grades of brick are used. This great variety affords the designer opportunity for selecting the kind specially adapted to his purpose. Special kilns are required for burning bricks to fulfill special requirements, such as paving brick, fire brick, pressed brick, etc. The range in the quality of brick is indicated in part by the compressive strength which varies from 400 pounds per square inch to 15,000 pounds per square inch. The strength of brick masonry depends largely upon the kind of mortar used in the joints and upon the workmanship, but it is much smaller than that of the individual brick, ranging from one-sixth to one-third as much. The weight of brick masonry is about 125 pounds per cubic foot.

(c) **Concrete, Plain and Reinforced.** In recent years concrete has come into common use for building and structural purposes. The convenience with which it can be made into the required form, its durability, and its fireproofing qualities make it a desirable material. For foundations and for places where only compression comes on the structure the plain concrete is more generally used, but where tension exists steel is embedded in the concrete to take the tension.

4. **TIMBER.** Timber has been used extensively for building purposes. There are many varieties and qualities on the market, affording good opportunity for the selection of the timber most suitable for the desired purpose. The cost of timber is gradually increasing, and some species have disappeared and others are disappearing from the market for structural purposes. The strength depends upon the species, the condition of growth, the seasoning, the defects in the timber, etc.

5. **CAST IRON.** Cast iron is a brittle metal. Its cheapness, the ease with which it is cast into special forms and machined into exact shapes, and its high

compressive strength make it valuable for a great many purposes; but its low tensile strength, compared with that of other metals, and its brittleness make it an undesirable material for resisting shock or tension. Cast iron is made by smelting ore in a blast furnace. In its crude form it is called pig iron. The strength of cast iron and its other properties vary widely and depend upon the amount and condition of the carbon and other ingredients which it contains.

6. WROUGHT IRON. Wrought iron is made from pig iron in a reverberatory furnace by what is called the puddling process. The puddled balls obtained in the process are run through a squeezer and much of the cinder expelled. The material is rolled into muck bars which are cut, piled together, heated, and finally rolled into the shapes desired. The strength and other qualities depend upon the quality of the pig iron used, and upon the details of manufacture. Because wrought iron can be easily worked and welded it is adaptable to many uses, but its use has diminished and steel has taken its place until very little is now manufactured.

7. STEEL. The term **steel** covers a wide range of material, — soft steel, mild steel, medium steel, hard steel, tool steel, etc., all being expressions used in connection with the various steels. In structural steel the element carbon is the one generally used to control strength and hardness, though other elements like phosphorus, sulphur, and manganese exercise important influence upon other properties.

The best structural steel is made by the open-hearth process. In this process pig iron, together with scrap steel and some iron ore, are melted in an open-hearth furnace, the carbon, silicon, and other elements are burned out, and a recarburizer is added to give the proper carbon content and to remove the iron oxide and increase the manganese, the final product being molded into ingots. Acid open hearth steel is produced in a furnace which has a siliceous lining; no reagent is added to remove the phosphorus, and hence the phosphorus content of the product is the same as that of the charge. Basic open hearth steel is produced in a furnace having a dolomitic lining (giving a basic chemical reaction), and lime is added to remove the phosphorus.

In the Bessemer process, melted pig iron is placed in a Bessemer converter and, by the action of air which is blown through the charge, most of the carbon, silicon, and manganese are burned out, and a recarburizer is later added to give the proper carbon content and to remove the iron oxide and to increase the manganese, the final product being molded into ingots. As used in the United States the Bessemer converters have siliceous linings, and no phosphorus is removed. Relatively little Bessemer steel is now used for structural purposes.

In the crucible process, crude wrought iron is fused with a carbon flux in a sealed air-tight vessel. The crucible process is in use for making hard steel, like tool steel, spring steel, etc.

The carbon content of steel varies from less than one-tenth per cent for the softest steels to more than one and one-half per cent for the hardest carbon steels. Metals like nickel, tungsten, vanadium, etc., are also added to give special amounts of strength or hardness and produce grades of steel which have special adaptability for various purposes.

8. OTHER MATERIALS. Many other materials used by the engineer and architect are specially adapted to the purpose for which they are intended. Rope is made of fibrous materials such as hemp, manilla, cotton, etc., and of wire. Belting is made of leather, canvas and rubber, and of metallic links. Several alloys having copper as a basic element are made, such as phosphor bronze, brass, etc. Several kinds of artificial stone are manufactured, for most of which sand and hydraulic cement are used as the basic constituents. Metals such as lead and aluminum are also used for various purposes.

Table 1 gives average values of the weights of various materials used in constructional work, but variation from the tabulated values is to be expected.

TABLE 1

WEIGHTS OF VARIOUS MATERIALS USED IN CONSTRUCTION

Material.	Weight, lb. per cu. ft.	Material.	Weight, lb. per cu. ft.
Timber.......	25 to 45	Sandstone............	150
Cast iron.....	450	Granite..............	170
Wrought iron .	480	Marble...............	170
Steel.........	490	Slate................	175
Brass.........	515	Terra cotta, facing....	110
Copper, Bronze	550	Terra cotta, fireproofing..........	50
Aluminum....	160	Book tile............	60
Brick.........	100 to 150	Concrete.............	150
Limestone....	165		

EXAMPLE

What is the weight of a solid stone masonry pier with uniformly sloping sides and rectangular section, 4 feet by 8 feet at the top and 8 feet by 16 feet at the bottom and 20 feet high?

This example is most easily worked by using the prismoidal formula to obtain the volume. This is $V = \frac{h}{6}(A + 4B + C)$ in which V is the volume, h is the altitude, A and C are the areas of the two bases, and B is the sectional area at the middle point.

For the given example,

$$A = 4 \times 8 = 32 \text{ square feet.}$$
$$B = 6 \times 12 = 72 \text{ square feet.}$$
$$C = 8 \times 16 = 128 \text{ square feet.}$$

Therefore the weight of the pier is

$$W = 160 \times \tfrac{20}{6} (32 + 4 \times 72 + 128) = 160 \times 1493\tfrac{1}{3} = 239{,}000 \text{ lb.}$$

PROBLEMS

1. What is the weight of a wrought iron rod of 1 square inch sectional area 1 yard long? *Ans.* 10 lb.

2. What is the weight of a hollow log $3\frac{1}{2}$ feet in external diameter, 2 feet in internal diameter, and 16 feet long?

3. What is the weight per lineal foot of a concrete dam 4 feet high, 1 foot thick at the top, and 2 feet thick at the base?

4. What is the weight of a solid granite obelisk 40 feet high, 1 foot square at the top, and 3 feet square at the base? *Ans.* 29,500 lb.

5. What is the weight of a square brick chimney 30 feet high, the inside dimensions being 2 feet at the top and $2\frac{1}{2}$ feet at the bottom, and the thickness of the walls uniformly 8 inches?

6. A certain white oak log 12 feet long and 2 feet in diameter weighed 1885 pounds, what was the weight of a cubic foot of that white oak?

CHAPTER II

DIRECT STRESSES

9. **DEFINITIONS.** **Force** is an action of one body upon another which tends to change its shape and to produce a change of motion in the body. In this book the use of the term will generally be restricted to forces which are externally applied to the member.

Stress is an internal action which is set up between the adjacent particles of a body when forces or loads are applied to the body. It is developed whenever the body undergoes a change in shape. Stress may be considered an internal force.

A **unit-stress** is obtained by dividing the total stress by the area over which it acts if the stress is uniformly distributed. In the case of this uniform distribution the unit-stress is the amount of stress per unit of area of the sectional area. If the stress is not uniformly distributed, the unit-stress, or the intensity of stress, at a point of the sectional area is equal to the amount of stress that would be developed upon a unit of area if the stress were uniform over the area and if its intensity were the same as that at the point.

Deformation is a change in a dimension of a specimen.

Shortening is a decrease in the length of a specimen.

Elongation is an increase in the length of a specimen.

Detrusion is a lateral deformation in which the particles apparently slip past each other. It is caused by a shearing force.

An **axial load** is one whose line of action coincides with the axis of the member. The axial load may be the resultant of several loads.

An **axial stress** is one developed by an axial load.

If a plane is passed perpendicular to the axis of a bar, its intersection with the bar is called the **cross section,** or the **section,** and its area the **sectional area.**

10. **TENSION.** When a load tends to pull the particles of a material directly apart in the direction of the load the material is under **tension** and the load is a **tension load.** The internal stresses developed are **tensile stresses.** The resulting deformation is an elongation. As long as rupture does not occur, the forces acting on all, or on a part of the specimen, are in equilibrium.

By the principles of theoretical mechanics it is shown that the conditions of equilibrium are that there shall be no resultant force and no resultant moment. These conditions are expressed in three fundamental equations

$$\Sigma F_x = 0, \quad (1)$$
$$\Sigma F_y = 0, \quad (2)$$
$$\Sigma M = 0. \quad (3)$$

These conditions of equilibrium are essential for determining the internal stresses produced by external forces.

For a homogeneous specimen in direct tension, under

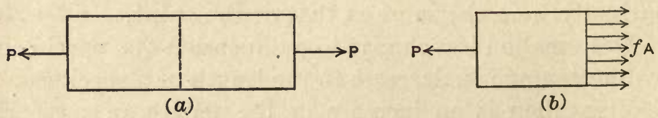


FIG. 1.

an axial load the stress is uniform over the entire sectional area A. Let Fig. 1(*a*) represent the member carrying the tension load P. Imagine the member cut as indicated. Fig. 1(*b*) shows the left portion of the member with the forces and stresses acting upon it indicated.* The total resisting stress is fA, where f is the tensile unit-stress developed. The resisting stress is treated as an external force in the free-body diagram, then by taking as the X-axis the axis of the piece or member, the summation of the X-forces gives

$$\Sigma F_x = fA - P = 0,$$

$$\therefore \quad f = \frac{P}{A}.$$

If the load becomes great enough to cause rupture the maximum unit-stress developed at any time before rupture is called the **ultimate tensile strength.** In some materials, such as wrought iron and soft steel, the load will increase to a maximum value, then decrease before rupture occurs. The ultimate strength differs for different specimens of the same material, and for purposes of design the value should be determined for each material used in the structure.

The unit-stress at the point of rupture is called the **rupturing strength.** The rupturing strength is of no practical value. For brittle materials the rupturing strength and the ultimate strength are equal.

When a specimen is broken by a tension load, its final length will be greater than its original length. The ratio of the increase in length to the original length is called the **ultimate elongation.** For ductile materials the length of the specimen has an influence upon this ratio. So for purposes of uniformity the ultimate deformation is usually obtained for specimens of standard size, either two or eight inches in gauge length. The average values of the ultimate tensile strength and of the ultimate elongation for specimens of eight-inch gauge length are given in Table 2.

* Fig. 1(*b*) is called a *free-body diagram.*

TABLE 2

ULTIMATE TENSILE STRENGTH AND ULTIMATE ELONGATION OF MATERIALS

Material.	Ultimate tensile strength, lb. per sq. in.	Ultimate elongation, per cent.
Timber	6,000 to 10,000	1.5
Cast iron	20,000	.3
Wrought iron	50,000	30.0
Structural steel	60,000	25.0 to 30.0
Steel wire	60,000 to 250,000	10.0 to 25.0

11. COMPRESSION. When a force acting on a member tends to push the particles closer together in the direction of the force the member is in **compression.** The stresses arising are **compressive stresses.** For compression there is a shortening. If the load is axial and is applied in such a manner that the stress developed is uniformly distributed over a section of the member, the compressive unit-stress developed is $f = \frac{P}{A}$. The average values of the ultimate compressive strength are given in Table 3. The table does not include values of the ultimate compressive strengths of malleable materials. Their values, however, should not be considered greater than the ultimate tensile strengths.

TABLE 3

ULTIMATE COMPRESSIVE STRENGTH OF MATERIALS

Material.	Ultimate compressive strength, lb. per sq. in.
Timber	7,000
Cast iron	90,000
Brick	6,000
Brick masonry	1,500
Rich concrete	2,500
Stone	10,000

12. **SHEAR.** When external forces tend to cause two adjacent sections of a member to slip past each other the member is in **shear.** Stresses resisting such forces are **shearing stresses.** When the two shearing forces are near together the shear is considered as a simple

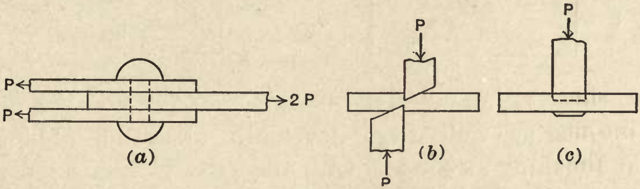


FIG. 2.

stress. Fig. 2 shows cases of direct shear. If a force P tends to shear a specimen along an area A the average shearing unit-stress is $s = \frac{P}{A}$. Table 4 gives average values of the ultimate strength in shear.

TABLE 4

ULTIMATE SHEARING STRENGTH OF MATERIALS

Material.	Ultimate shearing strength, lb. per sq. in.
Timber:	
Along grain	400
Across grain	3,000
Cast iron	20,000
Wrought iron	40,000
Structural steel	50,000
Rivet steel	45,000

13. **OBLIQUE SHEAR.** Shearing stresses are developed in structural members which are subjected to direct tension or compression. Let Fig. 3 represent a specimen under the compression force P, and imagine it cut along the plane AB. The two plane surfaces made

by the cut would slip past each other under the action of the force. This tendency of the sections to slip past each other, which gives rise to shearing stresses, always exists in members under load. To deduce the value of

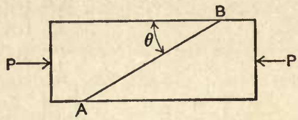


FIG. 3.

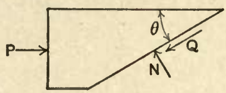


FIG. 4.

the shearing unit-stress developed along an oblique plane making an angle θ with the axis, let Fig. 4 represent the free-body diagram of one end of the specimen. The resisting stresses acting on the fibers of the plane may be resolved into their components Q and N parallel and normal to the plane respectively. Taking the X-axis along the plane there results

$$\Sigma F_x = P\cos\theta - Q = 0,$$
$$\therefore\ Q = P\cos\theta.$$

The component Q parallel to the plane is the force that keeps this end from slipping past the other one, and therefore Q is the resultant of the shearing stresses, which act parallel to the plane. If A is the area of the cross section of the specimen, $\frac{A}{\sin\theta}$ is the area of the section cut. The shearing unit-stress then is

$$s = P\cos\theta \div \frac{A}{\sin\theta} = \frac{P}{A}\sin\theta\cos\theta.$$

This is the value of the shearing unit-stress along any oblique plane. To find the value of θ for the maximum shearing stress developed in a specimen the relation $\sin^2\theta + \cos^2\theta = 1$ exists. It is shown by algebra that when the sum of two variables is constant the product of those variables is a maximum when they are equal; *

* See "Higher Algebra," by John F. Downey, page 252.

therefore for the maximum shearing unit-stress $\sin \theta = \cos \theta$ which is true when $\theta = 45°$ then

$$s_m = \frac{P}{A} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{P}{2A}.$$

14. Stress-deformation Diagrams. Whenever a load is applied to a specimen of any material there is a corresponding deformation. A graphical representation showing the values of the unit-stress developed in

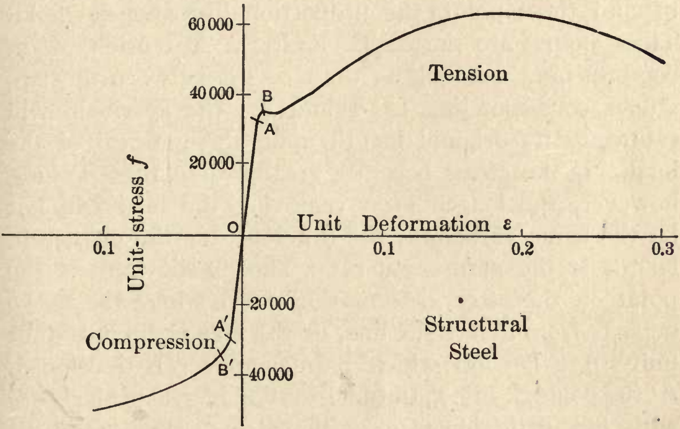


FIG. 5.

the specimen along one axis and the corresponding values of its unit deformation along the other axis is called a **stress-deformation diagram.** Fig. 5 is such a diagram for a specimen of soft steel. The unit-stress $f = \frac{P}{A}$ is plotted along the vertical axis and the unit deformation $\epsilon = \frac{e}{l}$ is plotted along the horizontal axis. In these equations P is the total load on the specimen, A is the cross-sectional area, e is the total deformation at the load P, and l is the original length of the specimen. In the diagram shown, the stresses measured upward from O are tension and those measured downward are compression, and the deformations measured to the right are elongations and those measured to the left shortenings.

15. ELASTIC LIMIT AND YIELD POINT. For stresses between the two points A and A', Fig. 5, the deformation is proportional to the stress, while for stresses beyond these points the proportionality does not hold. These points are the elastic limits, A in tension, A' in compression. As long as the stress is between the two values corresponding to A and A', the specimen will return to its original length upon the removal of the load. If the stress becomes greater than these values, however, the length after removing the load will not be the same as before; the difference or the change in length is the **permanent set.** The **elastic limit** is the point on the stress-deformation curve where the curve departs from a straight line, or the **elastic limit** is that unit-stress beyond which permanent set is developed. At the point B, Fig. 5, the deformation increases markedly with but little increase in the stress. That point is the **yield point.** The **yield point** then may be defined as the unit-stress at which there is a marked increase in the deformation with but little or no increase in the stress. Table 5 gives average values of the elastic limits of wrought iron and steel as commonly determined in the laboratory. The values for tension and compression are about the same. Values for timber and cast iron have not been included on account of the uncertainty in their determination, but when used they may be taken to be about one-third to two-thirds of the ultimate strength.

TABLE 5

ELASTIC LIMIT OF WROUGHT IRON AND STEEL

Material.	Elastic limit, lb. per sq. in.
Wrought iron..........	25,000
Structural steel........	35,000
Hard steel.............	50,000

16. The Modulus of Elasticity. For values of the stress less than the elastic limit the rate at which the unit deformation increases with the increase in the unit-stress is constant,* i.e., the unit deformation is proportional to the unit-stress (Fig. 6). This is commonly called **Hooke's Law.** Then for stresses below the elastic limit the unit-stress divided by the unit deformation gives a constant. This ratio is the **modulus of elasticity** or the **coefficient of elasticity. Young's modulus** is the modulus of elasticity for direct tension or compression.

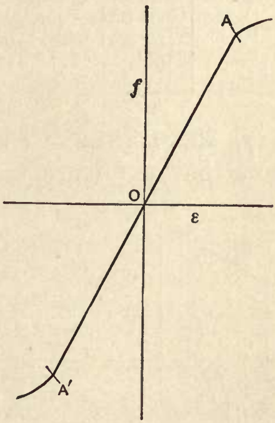


FIG. 6.

If E is the modulus of elasticity, f the unit-stress below the elastic limit, and ϵ the corresponding unit deformation the value of the modulus of elasticity is

$$E = \frac{f}{\epsilon} = \frac{\frac{P}{A}}{\frac{e}{l}} = \frac{Pl}{Ae}.$$

* Experiments indicate that the increase of deformation is not absolutely proportional to the increase in stress, but for practical purposes they may be taken as varying directly with each other.

In the formula for E, ϵ is an abstract number; consequently the unit for E is the same as that for f, pounds per square inch, tons per square foot, etc. In Table 6 are given average values of the modulus of elasticity in tension and compression for some materials.

TABLE 6

MODULUS OF ELASTICITY

Material.	Modulus of elasticity, lb. per sq. in.
Timber	1,500,000
Cast iron	15,000,000
Wrought iron	25,000,000
Steel	30,000,000

17. RESILIENCE. The stress-deformation diagrams show that a force acts through a distance and thus does work on the specimen. When the load is released the specimen gives up energy stored in it. This energy a specimen under stress is capable of giving up in returning to its original dimensions is called **resilience.**

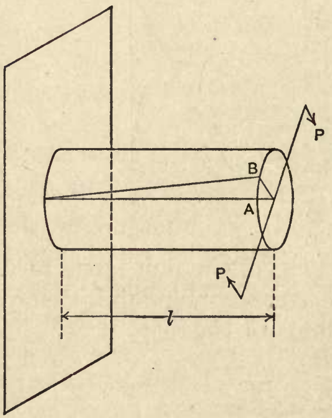


FIG. 7.

18. THE SHEARING MODULUS OF ELASTICITY. Under shearing forces a specimen will undergo a detrusion. The unit detrusion is obtained by dividing the total detrusion by the length over which it occurs. The ratio of the unit-stress developed in the specimen to the unit detrusion is called the **shearing modulus of elasticity.** It is also called the **modulus of transverse elasticity,** and the **modulus of rigidity.** When a specimen is subject to shearing stresses as in Fig. 7 the unit-stress can be calculated and the detrusion measured. If E_s is the shearing modulus of elasticity, ϵ_s the unit detrusion, s the shearing unit-stress,

$$E_s = \frac{s}{\epsilon_s}.$$

In Table 7 are given average values of the shearing modulus of elasticity.

TABLE 7

SHEARING MODULUS OF ELASTICITY

Material.	Shearing modulus of elasticity, lb. per sq. in.
Timber, across grain....	400,000
Cast iron...............	6,000,000
Wrought iron...........	10,000,000
Steel....................	12,000,000

19. POISSON'S RATIO. As the length of a specimen is increased by a tension load the lateral dimensions decrease. For a compression load the lateral dimensions increase. For stresses below the elastic limit the ratio of the lateral unit deformation to the longitudinal unit deformation is called **Poisson's ratio.** This ratio is considerably less than 1 and for most metals ranges between $\frac{1}{3}$ and $\frac{1}{4}$.

20. REDUCTION OF AREA. When a specimen is ruptured by a tension force the final sectional area is less than the original area. The ratio of the amount the section at rupture is decreased to the original area is called the **reduction of area.** Thus, if A_1 is the original area and A_2 is the final area at rupture, the reduction of area is $\frac{A_1 - A_2}{A_1}$.

21. USES OF THE MODULUS OF ELASTICITY. The maximum unit-stress for which the formula $E = \frac{Pl}{Ae}$ may be used in calculations is the elastic limit. For stresses below the elastic limit E may be calculated from data observed in the laboratory; and the change in length of a specimen may be calculated by the formula $e = \frac{Pl}{AE}$. The modulus of elasticity has an important application also in determining the deflection of beams and the strength of columns.

22. STRESSES USED IN DESIGN. In making a design safety and economy must be considered. Experiments indicate that at stresses slightly beyond the elastic limit there is a marked change in the structure of the material, and therefore the working stresses should not be carried beyond that value. At least there should not be noticeable permanent set. **Working stresses** are the allowable stresses used for designing; they should always be well below the elastic limit. The method of fixing upon values for allowable working stresses is by making a set of experiments in which the elastic limit and ultimate strength of a number of specimens are determined, and then by taking a certain per cent either of the elastic limit or of the ultimate strength as the working stress. A knowledge of the behavior of the material under stress is essential for a proper determination of working stresses.

TABLE 8

SAFE WORKING STRESSES IN POUNDS PER SQUARE INCH FOR STEADY LOADS

Material.	Tension.	Shear.	Compression.		Bending (fiber).
			Perpendicular to grain.	Parallel to grain.	
Timber:					
Cedar, white........	800	100	180	1,100	1,000
Cypress............	600	100	180	1,100	1,000
Elm................	1,000	240	300	1,200	1,200
Fir, Washington	1,200	100	300	1,600	1,200
Gum...............	1,000	200	340	1,300	1,100
Hemlock...........	800	80	180	1,000	800
Larch..............	800	120	240	1,200	1,300
Maple, sugar (hard).	1,000	200	800	1,800	1,800
Maple (average).....	800	160	500	1,400	1,200
Oak, red............	900	160	500	1,200	1,200
white..........	1,000	200	600	1,750	1,400
Pine, longleaf.......	1,000	125	240	1,400	1,200
loblolly.......	....	100	200	1,000	1,000
shortleaf......	....	100	200	1,200	1,100
yellow, (Ark., etc.)........	800	100	200	1,200	1,000
Spruce..............	800	100	200	1,200	1,000
Cast iron.............	3,000	2,500	12,000		6,000
Wrought iron.........	12,000	9,500	12,000		12,000
Steel, structural.......	15,000	10,000	12,000		16,000
rivet............	....	8,000 10,000	18,000 (Bearing)		....
Brickwork (in lime)...	....	....	110		....
Brickwork (in Portland cement)............	....	....	250		....
Concrete (Portland cement)............	....	....	350		....

Table 8 is a modified extract from a table of allowable working stresses compiled by Ricker from building ordinances. A few additions are given. A few changes also have been made to agree with recent building ordinances.

The **factor of safety** is defined as the ratio of the ultimate strength to the working stress. This value varies for the different materials and for the kind of loading. Variable loads produce higher stresses than steady loads of the same magnitude. Suddenly applied loads and shocks produce higher stresses than variable loads of the same magnitude. Therefore the factor of safety for variable loads is usually taken about one-half greater than that for steady loads, and for sudden loads or shocks it is two or three times that for steady loads. In specifications and building ordinances the allowable stresses are usually given and also the tests for the materials specified. For such cases a factor of safety has been considered.

PROBLEMS

1. What must be the height of a brick tower if the compressive unit-stress on the lowest brick is one-tenth of its ultimate strength?

2. Determine the shearing unit-stress tending to shear off the head of a $1\frac{1}{2}$-inch wrought iron bolt under a tension of 10,000 pounds, if the head is $\frac{3}{4}$ inch deep.

3. A wrought iron plate $\frac{1}{2}$ inch thick requires a force of 80,000 pounds to punch a round hole $\frac{3}{4}$ inch in diameter through it. Find the ultimate shearing strength of the plate.

4. What force is required to punch a 1-inch hole in a $\frac{1}{2}$-inch structural steel plate?

5. In a tension test of a 0.19 per cent carbon steel specimen the diameter was 0.5 inch, and the gauge length was 1.25 inches. Each scale division on the extensometer represented $\frac{1}{125,000}$ inch. P is the load in pounds and e is the reading on the extensometer in scale divisions. The following readings were made: $P = 2000$, $e = 60$; $P = 6000$, $e = 180$; $P = 8000$, $e = 240$;

$P = 8000+$, $e = 290$; $P = 8100$, $e = 610$. The maximum load was 15,600 pounds, the corresponding length between punch marks was 1.59 inches. The load at rupture was 12,000 pounds, and the corresponding length was 1.76 inches. The diameter at the fracture was 0.313 inch.

(a) Calculate the unit-stress for each load.

Ans. 30,600 lb. per sq. in. 40,800 lb. per sq. in.

(b) Calculate the unit elongations for each load.

Ans. 0.00115, 0.00154.

(c) Plot the stress-deformation diagram.

(d) What is the elastic limit? *Ans.* 40,800 lb. per sq. in.

(e) What is the reduction of area? *Ans.* 60.7 %.

(f) What is the modulus of elasticity?

Ans. 26,600,000 lb. per sq. in.

6. A wrought iron rod 2 inches square and 10 feet long lengthened 0.02 inch by suspending a load from its lower end. Determine the load.

7. How much will a 100-ft. steel tape, $\frac{1}{2}$ inch wide and $\frac{1}{50}$ inch thick, stretch under a pull of 40 pounds? *Ans.* 0.16 in.

8. A vertical wooden bar 50 feet long and 6 inches square carries a load of 18,000 pounds at its lower end. Find the unit-stress at the upper end and the elongation of the bar due to the combined weight of bar and load.

9. Determine the elongation of a 1-inch wrought iron rod 10 feet long, under a tensile load of 20,000 pounds.

10. How many $\frac{1}{2}$-inch square rods of strong steel would be required for the suspension of a platform loaded with 15 tons, if the stretching of the rods is limited to one-half their elongation at the elastic limit? Each rod carries equal shares of the load.

11. What shearing load will a rivet $\frac{7}{8}$ inch in diameter safely carry? A rivet $\frac{3}{4}$ inch in diameter?

12. What should be the depth of the head of a bolt $\frac{7}{8}$ inch in diameter to carry safely the shear?

13. What must be the bearing area to carry safely a load of 20,000 pounds on a Washington fir beam?

14. What should be the sectional area of a steel member $(a - x)$

of the truss shown in Fig. 8? If $(a-y)$ is a short compression longleaf pine member, what should be its section?

15. Member $(a - y)$, Fig. 9, is a steel rod $1\frac{1}{2}$ inches in diameter stressed to its safe working stress. What should be the corresponding sectional area of the short white cedar compression members $(a-x)$?

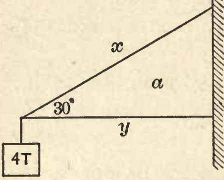


FIG. 8.

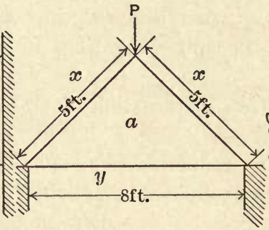


FIG. 9.

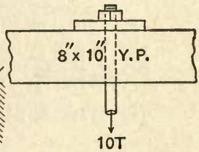


FIG. 10.

16. Design a cast iron washer for the member shown in Fig. 10

17. What must be the cross section of specimens of the following materials in order that the unit-stress may be about one-third the elastic limit under a load of 30,000 pounds in tension? (a) Wrought iron; (b) Steel.

18. In a compression test of a 6-inch concrete cube the load was 107,000 pounds at rupture. What was the shearing unit-stress along a plane inclined 30° to the axis? What was the maximum shearing stress developed?

Ans. 1290 lb. per sq. in. 1490 lb. per sq. in.

19. What will be the elongation at the elastic limit and at rupture of an 8-inch specimen of the following materials? (a) Wrought iron; (b) Structural steel; (c) Hard steel.

20. What should be the sectional area of a steel rod if it is to take a tension load of 70,000 pounds?

21. If a cast iron specimen 1×2 inches in sectional area breaks under a tensile load of 42,000 pounds, what load will probably break a cast iron rod 2 inches in diameter?

CHAPTER III

DIRECT STRESSES — APPLICATIONS

23. Simple Cases of Direct Stresses. The simplest cases of direct stresses are such as exist in eyebars, belts, ropes, cables, tension members in trusses, etc. For such members the unit-stress developed is obtained by dividing the load the member carries by the sectional area of the member. There are other cases for which the stresses developed are practically direct stresses although the line of action of the load may not be along the axis of the member that carries the load. Such cases will be considered in this chapter.

24. Stresses in Thin Cylinders. When a thin cylinder is under interior pressure, as a steam boiler, water pipe, etc., the forces tending to burst the cylinder act normally to the inside surface, Fig. 11. These forces develop internal tensile stresses in the metal of the cylinder. In order to determine the magnitude of these stresses, imagine the plane AB, Fig. 11, passed perpendicular to the page and containing the axis of the cylinder. The portion ABD is in equilibrium under the forces and stresses acting upon it, and if that half of the cylinder were filled with some solid substance the interior forces acting upon it would be normal to the plane AB; the resisting stresses also would be normal to that plane. The internal stresses actually developed in the cylinder are the same as would be developed under the imaginary condition. Fig. 12 is a free-body diagram of the part ABD. Let r be the radius, t the thickness, and l the length of cylinder, Q the interior pressure per unit of area, and f the resisting unit-stress, which is approximately uniform over the resisting area. Then the force tending to rupture the cylinder is $2\,Qlr$, and the

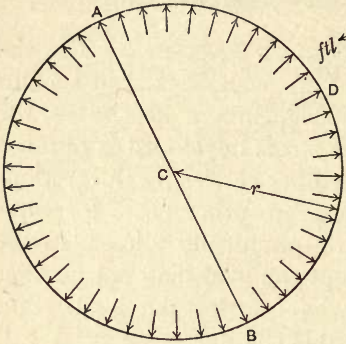


FIG. 11.

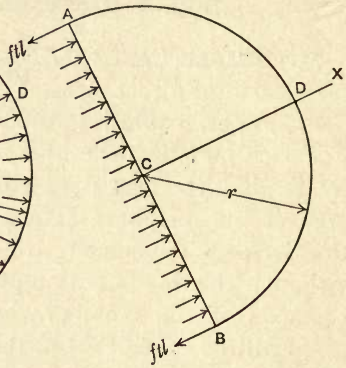


FIG. 12.

resisting stress on one side is ftl and on both sides is $2\,ftl$. Then for equilibrium

$$2\,Qlr = 2\,ftl,$$

$$\therefore \quad Qr = ft.$$

This formula is true for thin cylinders.

25. STRESSES IN A HOOP. If hoops are heated and then shrunk onto cylinders, the unit-stress can be obtained by the application of the formula for the modulus of elasticity, $E = \frac{Pl}{Ae} = \frac{fl}{e}$, if the difference between the normal diameter of the hoop when cool and the one to which it is shrunk can be obtained. The effect is the same as if the hoop is stretched from its normal diameter d_1 to the diameter d of the hoop when shrunk on the cylinder. The difference in the length of the hoop will be $\pi(d - d_1)$ and the unit elongation will be $\pi(d - d_1) \div \pi d_1 = \frac{d - d_1}{d_1}$ or $\frac{d - d_1}{d}$ approximately. For steel tires a common rule is to make $\frac{d - d_1}{d}$ about $\frac{1}{1500}$. Actually the final diameter d of the hoop and cylinder will be slightly less than the original diameter of the cylinder, as the metal of the cylinder will deform under the pressure due to the hoop. The amount of this deformation will depend upon the ratio of the moduli of elasticity of the materials of the cylinder and the hoop, and upon the thickness of each.

26. Stresses Due to Change in Temperature. When a material is heated it will expand if free, and when cooled it will contract. If $t°$ is the change in temperature and c is the coefficient of expansion, or change per unit of length for one degree rise or fall, the change per unit of length will be $\epsilon = ct°$. If the

TABLE 9

COEFFICIENTS OF EXPANSION PER DEGREE FAHR.

Material.	Coefficient of expansion.
Masonry	.0000050
Cast iron	.0000062
Wrought iron	.0000067
Steel	.0000065

member is brought back to its original length by an external force the unit-stress developed will be $f = \epsilon E = ct°E$. If, instead of being allowed to change in length and then being brought back to its original length when a change in temperature takes place, the specimen is rigidly held in its original position, a unit-stress of $f = ct°E$ will be developed. The effect is the same as if the specimen were allowed to change in length and then were brought back to its original length by an external force. Table 9 gives the values of the coefficient of expansion for each degree of change in temperature Fahr.

27. STRESSES IN THIN SPHERES. Internal pressure in domes or other thin spheres tends to cause rupture around a circumference, (Fig. 13). By using the same nomenclature as for thin cylinders, the force tending to push off the dome is $Q\pi r^2$ and the stress resisting this force is $f2\pi rt$,

$$\therefore \quad Qr = 2ft.$$

This formula also applies when an interior pressure acts upon a cylinder head.

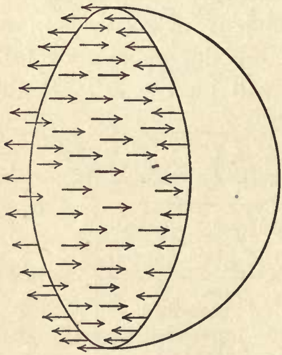


FIG. 13.

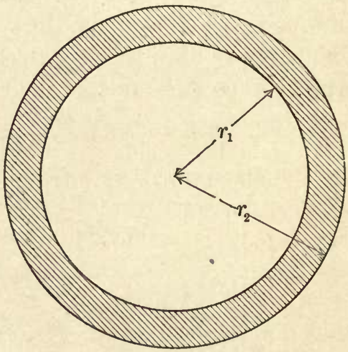


FIG. 14.

28. THICK CYLINDERS UNDER INTERIOR PRESSURE. If the metal is thick in comparison with the diameter of the pipe the stresses developed are not direct stresses and are not uniform throughout the thickness of the metal of the wall of the cylinder, and the formulas of the previous paragraphs cannot be used for such cases, (Fig. 14). Various expressions for the value of the maximum stress developed have been deduced. For the case when there are longitudinal and transverse stresses due to the interior pressure Lamé* gives the formula for the maximum tensile stress developed, which comes on the inner surface of the pipe, where r_1 is the internal radius and r_2 is the external radius,

$$f = \frac{Q\,(r_2^2 + r_1^2)}{(r_2^2 - r_1^2)}.$$

Calvarino* gives for the same case,

$$f = \frac{Q\,(4\,r_2^2 + r_1^2)}{(r_2^2 - r_1^2)}.$$

Birnie* gives the formula for thick guns when no longitudinal stress is developed,

$$f = \frac{Q\,(4\,r_2^2 + 2\,r_1^2)}{3\,(r_2^2 - r_1^2)}.$$

Any of these formulas may be used to investigate or design thick pipes or guns.

29. CYLINDERS UNDER EXTERIOR PRESSURE. Recent experiments on the collapse of tubes under exterior pressure indicate that for a length of tube greater than six times its diameter the rupturing pressure is independent of the length. The formulas deduced by Carman and Carr in the University of Illinois Engineering Experiment Station Bulletin No. 5 are as follows: for thin tubes where $\frac{t}{d}$ is less than 0.025,

$$Q = K\left(\frac{t}{d}\right)^3. \qquad (1)$$

And for thick tubes or $\frac{t}{d}$ greater than 0.03,

$$Q = K'\frac{t}{d} - C. \qquad (2)$$

* See "Strength of Materials," by A. Morley.

In these formulas t is the thickness of the walls, d is the outside diameter, Q is the external pressure in pounds per square inch causing collapse, and K, K', and C are experimental constants. Table 10 gives the values of these constants.

TABLE 10

VALUES OF K, K', AND C FOR PIPES UNDER EXTERIOR PRESSURE

Material.	K.	K'.	C.
Cold-drawn seamless steel......	50,200,000	95,520	2,090
Brass...........................	25,150,000	93,365	2,474
Lap-welded steel..............	....	83,270	1,025

EXAMPLES

1. What internal pressure will probably rupture a cast iron pipe 8 inches in diameter and $\frac{1}{4}$ inch thick?

$$Q\,4 = 20{,}000 \times \tfrac{1}{4},$$
$$Q = 1250 \text{ lb. per sq. in.}$$

2. If a steel rail of sectional area 8.8 square inches is subjected to a drop in temperature of 50° Fahr. and is prevented from shortening, what is the force exerted upon it if the initial force was zero?

The unit-stress developed is

$$.0000065 \times 50 \times 30{,}000{,}000 = 9750 \text{ lb. per sq. in. in tension.}$$

The total force is $8.8 \times 9750 = 85{,}800$ lb.

PROBLEMS

1. What is the maximum tensile unit-stress in a pipe 24 inches in diameter, the plate being $\frac{3}{8}$ inch thick, and the internal pressure 80 pounds per square inch? *Ans.* 2560 lb. per sq. in.

2. What internal pressure will rupture a 12-inch steel pipe $\frac{3}{8}$ inch thick? *Ans.* 3750 lb. per sq. in.

3. What should be the thickness of the lower plates of a steel stand-pipe 20 feet in diameter carrying a water pressure of 80 pounds per square inch? Use a unit-stress of 16,000 pounds per square inch and the efficiency of the joint 75 per cent.

Ans. About 0.8 in.

4. What stress is developed in a spherical steam dome 10 inches in diameter, $\frac{1}{4}$ inch thick, under a steam pressure of 120 pounds per square inch?

5. What external pressure will cause a 2-inch cold-drawn steel tube $\frac{1}{8}$ inch thick to collapse?

6. What internal pressure will burst a wrought iron cylinder of 48 inches inside diameter and $\frac{3}{8}$ inch thickness?

7. Determine the thickness of a wrought iron steam pipe 18 inches inside diameter to resist a pressure of 200 pounds per square inch with an allowable stress of 6000 pounds per square inch.

8. What should be the minimum thickness of a cast iron sphere 12 inches inside diameter to withstand safely a steady internal pressure of 200 pounds per square inch?

9. What internal pressure will burst a cast iron sphere 24 inches inside diameter and $\frac{1}{2}$ inch thick?

10. A short wrought iron bar $1\frac{1}{2}$ inches in sectional area has its ends fixed immovably between two walls with no stress when the temperature is 50° Fahr. What pressure will be exerted on the walls when the temperature is 120° Fahr.?

11. Steel railroad rails, each 30 feet long, are laid at a temperature of 40° Fahr. What space must be left between them in order that their ends shall just meet when the temperature is 100° Fahr.? If the rails had been laid with their ends in contact, what would be the unit-stress in them at 100° Fahr.?

12. A wrought iron tie rod 20 feet in length and 2 inches in diameter is screwed up to a tension of 10,000 pounds in order to tie together two walls of a building. Find the stress in the rod when the temperature falls 30° Fahr. Also when it rises 20° Fahr.

CHAPTER IV

RIVETED JOINTS

30. **Riveted Joints.** In pipes, tanks, and boilers made of rolled plates, the plates are usually connected by rivets, and stress is transmitted from one plate to the other through the rivets. Such joints may be called **boiler, tank,** or **pipe joints.** Connections of bridge members, and connections between the members of roof trusses, columns, beams, etc., are also made by means of rivets. Such joints may be called **structural** joints. Wherever pieces of metal are connected by rivets the design should give the most efficient and economical connection consistent with the given conditions. A joint will fail at its weakest part, and the most efficient design will have all parts of the joint of equal strength.

Although the actual stresses developed in a riveted joint may be complex, the usual method is to simplify the calculation by assuming conditions giving direct tensile, compressive, and shearing stresses. As such the stresses are easily computed. The treatment of boiler joints and of structural joints is essentially the same.

31. **Kinds of Riveted Joints.** Riveted joints may also be classified according to the method of connecting the plates and the number of rows of rivets used. In Fig. 15 are shown two styles of **lap joints;** the main plates overlap each other and are connected by the

rivets. Fig. 15 (*a*) and (*b*) represent a single riveted lap joint, and (*c*) and (*d*) represent a lap joint with two rows of rivets, with the rivets staggered. Fig. 18 shows two styles of **butt joint**; the edges of the main plate almost or wholly butt against each other, and the connection is made through cover plates. Fig. 18 (*b*) shows a butt joint with a single cover plate, and Fig. 18

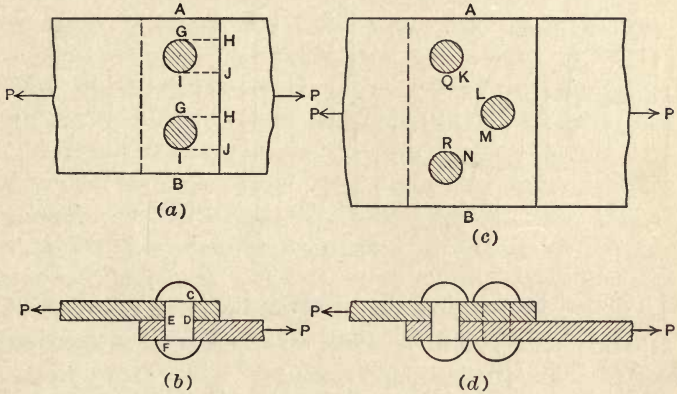


FIG. 15.

(*c*) shows one with a double cover plate. There are other styles of riveted joints, but the general method of treatment is the same for all kinds.

32. METHODS OF FAILURE OF RIVETED JOINTS. The three principal ways in which riveted joints may fail are (1) (for tension loads) by tension in the plates along *AB*, Fig. 15; (2) by crushing of the rivets along *CD* or *EF*; (3) by shear of the rivets along *ED*. Besides these, failure may occur, (4), by shearing of the plate along *GH* and *IJ*; (5) by bending of the rivets; (6) by bending of the plates, thus allowing the rivet heads to shear off or the rivets to fail in tension; (7) by failure of the plate in tension along KL and MN for staggered rivets. Failure by shear in the plates, number (4), is avoided by making the lap GH large enough to insure safety. A rule sometimes followed is to make the lap one and one-half times the diameter of the rivet. Bending of the rivets and of the plates, numbers (5) and (6), may occur in a single-riveted lap joint. Failure number (7)

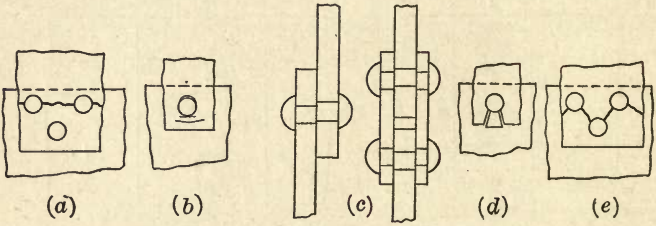


FIG. 16.

is avoided by making the distance between the rows of rivets large enough so that $KL + MN$ is somewhat greater than QR.

Fig. 16 indicates the various ways in which a riveted joint may fail.

(*a*) shows a failure due to tensile stress in the plate.
(*b*) shows a failure due to bearing stress in the rivet.
(*c*) shows a failure due to shearing stress in the rivet.
(*d*) shows a failure due to the shearing stress in the plate between the edge of the plate and the rivet hole.
(*e*) shows a failure by tension along a staggered line.

33. COMPUTATION OF UNIT-STRESSES DEVELOPED IN RIVETED JOINTS. The calculation of the unit-stresses developed in a riveted joint is made by assuming that the stress is uniformly distributed over the particular area which is in tension, compression, or shear, and hence that the unit-stress is $f = \frac{P}{A}$, where P is the total load coming on the area A which resists tension, compression, or shear. In order to determine the tensile stress developed, the area of the section subjected to tension should be obtained. To determine the compressive or bearing stress, the area subjected to bearing should be found. For bearing it is assumed that the stress on one rivet is uniformly distributed over an area equal to that obtained by multiplying the thickness of the plate by the diameter of the rivet. This area is the projection of the rivet on the thickness of the plate.

If the entire stress transmitted by a rivet is taken by a section of the rivet at one face of the plate the rivet is said to be in **single shear,** and the resisting area is equal to the cross-sectional area of the rivet. If the stress transmitted is taken by sections at two faces of a plate the rivet is said to be in **double shear,** and the resisting area is equal to twice the cross-sectional area of the rivet. The method of calculating the stresses will be given for a few cases.

34. **SINGLE-RIVETED LAP JOINT.** Let Fig. 17 represent a single-riveted lap joint in which the load P is to be transmitted from one plate to the other by n rivets in one row. The load must be transmitted by tension through the plate past the row of rivets. The greatest tensile unit-stress developed will come along the section AB. Let f_t be the tensile unit-stress developed, t the thickness of the plate, d' the diameter of the rivet hole, and b the width of the plate. Then the smallest area in tension taking the load is $t(b - nd')$, and the unit-stress is $f_t = \frac{P}{t(b - nd')}$

and $$P = f_t t (b - nd'). \qquad (1)$$

The load brings compression on the side of the rivet as shown in Fig. 17 (*b*). If d is the diameter of the rivet, the total area upon which the load is assumed to be distributed in compression or bearing is ntd, and the bearing unit-stress developed is $f_c = \frac{P}{ntd}$

and $$P = f_c ntd. \qquad (2)$$

The load tends to shear the rivets along the plane CD between the two plates. If f_s is the shearing unit-

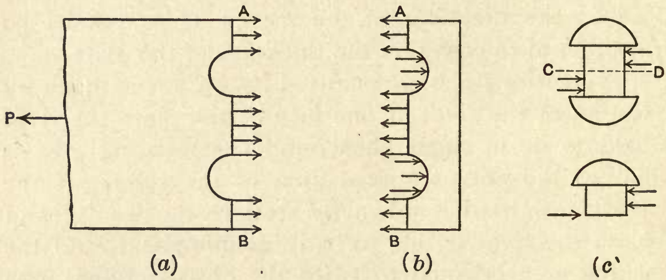


FIG. 17.

stress, and A the sectional area of each rivet, the total resisting area in shear is nA, and the shearing unit-stress is $f_s = \frac{P}{nA}$

and $$P = f_s nA. \qquad (3)$$

By use of the three formulas just developed the unit-stresses existing in a single-riveted lap joint under a given load can be calculated, or the load a given joint will carry can be determined, or a joint can be designed to carry a given load. If the plates connected are of different thickness the smaller value of t should be used in the formulas.

35. DOUBLE-RIVETED LAP JOINT. The equations representing the relation between the load transmitted and the unit-stress developed for a double-riveted lap joint are similar to those given for the lap joint with a single row of rivets.

For this case let n be the number of rivets in one row, and n_1 the total number of rivets, then the formulas become

$$P = f_t t (b - nd'), \qquad (1')$$
$$P = f_c n_1 t d, \qquad (2')$$
$$P = f_s n_1 A. \qquad (3')$$

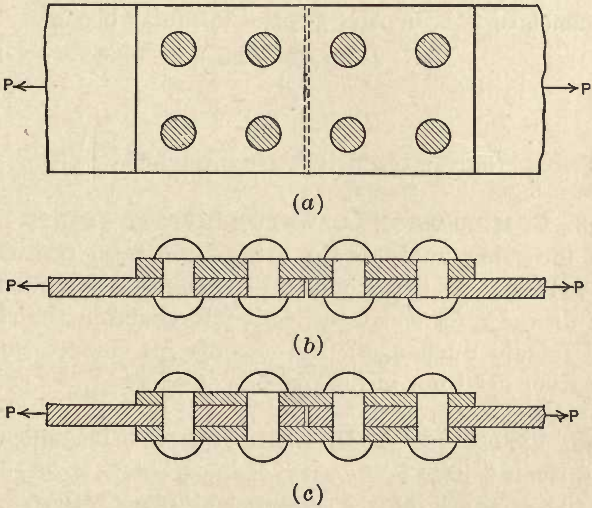


FIG. 18.

36. Lap Joint with More than Two Rows of Rivets. If there are more than two rows of rivets the assumption is generally made that the load is distributed evenly among the rivets, and the load each rivet carries is obtained by dividing the total load by the number of rivets carrying the load. With the same nomenclature as in the last article the same formulas hold for this case (n would usually be the number of rivets in the outside row).

37. BUTT JOINT. Fig. 18 (b) shows a butt joint with a single cover plate, for which kind of joint the formulas of Art. 34 may be applied, if n refers to the number of rivets on one side of the seam. The similarity between this joint and a lap joint is readily seen by considering the cover plate with one side of the main plate.

For a butt joint with two cover plates, using the same nomenclature as in Art. 35, the formulas become

$$P = f_t t(b - nd'), \qquad (1'')$$

$$P = f_c n_1 d, \qquad (2'')$$

$$P = f_s n_1\, 2\, A. \qquad (3'')$$

Two sections of each rivet are brought into shear.

38. COMPRESSION LOADS FOR RIVETED JOINTS. For the foregoing analyses the loads have been considered in tension. If the member is a compression member the formulas for shear and for compression in the rivets will remain unchanged; the formula for compression in the rivet need not ordinarily be considered.

39. EFFICIENCY OF RIVETED JOINTS. The **efficiency of a riveted joint** is the ratio of the strength of the joint to the strength of the unpunched plate. In figuring the strength working stresses are usually used in the formulas to determine the load P. When a joint is designed, its strength under tension in the plate calculated by formulas (1) Art. 34, 35, and 37, the bearing strength of the rivets calculated by formulas (2), and the shearing strength of the rivets calculated by formulas (3) should be obtained and the smallest value of the load P taken as the **strength** of the joint; for if a greater load is put on the joint one of the safe stresses would be exceeded.

This value divided by the working strength of the unpunched plate is the **efficiency.**

In boiler lap joints, for the pitch p, which is the distance from center to center of adjacent rivets in one row, the strength of the unpunched plate is $f_t tp$. Therefore, the efficiency for tension is

$$\epsilon_t = \frac{f_t t\,(p - d')}{f_t tp} = \frac{p - d'}{p}.$$

The efficiency for compression in the rivets is

$$\epsilon_c = \frac{f_c ntd}{f_t tp} = \frac{f_c nd}{f_t p}.$$

And the efficiency for shear in the rivets is

$$\epsilon_s = \frac{f_s nA}{f_t tp}.$$

The actual efficiency of the joint is the smallest one of the above values. Similar expressions for the various other types of riveted joints may be deduced.

Table 11 gives the range of values for efficiency for the types of boiler joints listed for ordinary design.

TABLE 11

EFFICIENCY OF JOINTS

Kind of joint.	Efficiency, per cent.
Single-riveted lap joint.......	50–65
Double-riveted lap joint......	65–75
Single-riveted butt joint......	65–75
Double-riveted butt joint.....	70–80

Triple-riveted joints are frequently used and high efficiencies are obtained.

40. DESIGN OF RIVETED JOINTS. For use with ordinary thickness of plates in structural work $\frac{3}{4}$-inch and $\frac{7}{8}$-inch rivets are the prevailing sizes. For light work

$\frac{1}{2}$-inch and $\frac{5}{8}$-inch rivets are used. In specially heavy sections larger rivets are used, $1\frac{1}{2}$-inch rivets being occasionally used.

The pitch of the rivets to be used in a design depends upon the kind of joint used and the purpose of the joint. For tanks and boilers the joint must be tight as well as strong; therefore the spacing should be small. For structural members strength is the main object to be accomplished.

The diameter of the rivet hole is somewhat larger than the diameter of the rivet. In boiler joints the diameter of the rivet hole is usually assumed equal to or $\frac{1}{16}$ inch larger than that of the rivet, and in structural joints it is assumed $\frac{1}{8}$ inch larger.

The ideal joint for strength and economy would be the one that would be of equal strength in tension in the plate, bearing in the rivets, and shear in the rivets. For this to be the case the three values of the allowable resisting stresses as calculated by the three formulas, Art. 34, 35, and 37, would be the same; therefore, for single-riveted lap joints,

$$f_t t\,(b - nd') = f_c ntd,$$

and

$$f_c ntd = f_s nA,$$

and

$$f_t t\,(b - nd') = f_s nA.$$

Similar equations can be written for the other types of riveted joints.

In practice it is not usually necessary or practicable to make such ideal joints, and the resulting efficiency will be somewhat less than that of the ideal joint.

Limitations of the size of rivet, conditions for tightness of joints, convenience for shop work, and many other items may prevent making joints of equal strength in tension, bearing, and shear, and small variations from the ideal conditions will not materially decrease the efficiency.

For the design of boiler, tank, or pipe joints the following procedure is a convenient one and is recommended.

(a) Decide upon the working stresses for tension, shear, and bearing, and calculate $f_s : f_t$, and $f_c : f_t$.

(b) Select the type of joint to suit the conditions.

(c) In the first calculations assume the efficiency, calculate the necessary thickness, and then select a commercial thickness of plate.

(d) Determine the limiting size of rivet for shear or bearing. The general limiting size for any style of joint may well be expressed in terms of the thickness of the plate. For example, for $f_s : f_t = 2 : 3$ and $f_c : f_t = 3 : 2$ in the design of riveted lap joints any diameter of rivet less than $2.87\,t$ will not involve a question of bearing strength. Also in ordinary butt joints having double cover plates, the bearing strength of the rivet will not need consideration if the diameter of the rivet does not exceed $1.43\,t$.

(e) Select a working size of rivet within this limit.

(f) With t and d determined calculate the pitch by equating the strength in tension and the strength in shear or bearing, using shear or bearing according to which controls the strength for the type of joint used. Only in special or unusual types of joints will tension and shear govern.

(g) Calculate the efficiency of the joint and the stresses in the rivets and plate to see that the working stresses are not exceeded.

Practice is not uniform in regard to the values of the allowable unit-stresses to be used in design. The following values may be used in solving problems in this course: f_s equals 8000 pounds per square inch, f_t equals 12,000 pounds per square inch, and f_c equals

18,000 pounds per square inch. The resulting ratios are $\frac{f_s}{f_t} = \frac{f_t}{f_c} = \frac{2}{3}$. The assumption made for the bearing area of the rivets is only approximate and experiments show high values for the ultimate crushing strength figured on that basis; therefore a high value for the crushing unit-stress can be assumed.

EXAMPLES

1. Select two channels for the lower chord of a truss in which the maximum stress is 49,300 pounds in tension. Also determine the number of $\frac{3}{4}$-inch rivets, if $\frac{3}{8}$-inch gusset plates are used, for the connection. Use $f_t = 15{,}000$ pounds per square inch, $f_c = 18{,}000$ pounds per square inch, and $f_s = 8000$ pounds per square inch.

The net sectional area required is

$$A_t = 49{,}300 \div 15{,}000 = 3.29 \text{ square inches.}$$

By use of a hand book, the section is found first.

Try two 2-inch by 2-inch by $\frac{1}{2}$-inch angles.

The gross area is

$$A_g = 2 \times 1.75 = 3.5 \text{ square inches.}$$

By counting the diameter of the hole $\frac{1}{8}$-inch larger than that of the rivet, the effective tension area will be reduced by the amount

$$A_1 = 2 \times \tfrac{7}{8} \times \tfrac{1}{2} = .875 \text{ square inch.}$$

The effective area then would be

$$A_e = A_g - A_1 = 3.500 - .875 = 2.625 \text{ square inches.}$$

This is too small. Try two $3\frac{1}{2}$-inch by 3-inch by $\frac{5}{16}$-inch angles.

$$A_g = 2 \times 1.94 = 3.88 \text{ square inches.}$$

$$A_1 = 2 \times \tfrac{7}{8} \times \tfrac{5}{16} = .547 \text{ square inch.}$$

The effective area then is

$$A_e = A_g - A_1 = 3.88 - .55 = 3.33 \text{ square inches.}$$

This is a little in excess of the required area; therefore use two $3\frac{1}{2}$-inch by 3-inch by $\frac{5}{16}$-inch angles.

To determine the number of rivets necessary for bearing, the required bearing area is $A_c = \frac{49,300}{18,000} = 2.74$ square inches. The greatest bearing stress will be developed between the gusset plate and the rivet, since the thickness of the two angles is greater than that of the gusset plate. The bearing area for one rivet is

$$dt = \tfrac{3}{4} \times \tfrac{3}{8} = .281 \text{ square inch.}$$

Therefore, the number of rivets required for bearing is

$$n_c = \frac{2.74}{.281} = 10 \text{ rivets.}$$

To determine the number of rivets necessary for shear, the required area for shear is

$$A_s = \frac{49,300}{8000} = 6.16 \text{ square inches.}$$

Each rivet is in double shear and the total area of each rivet in shear is

$$2\,A = 2 \times .442 = .884 \text{ square inch.}$$

Therefore, the number of rivets required for shear is

$$n_s = \frac{6.16}{.884} = 7 \text{ rivets.}$$

Since bearing requires 10 rivets that number must be used. The shearing stress then is below the allowable.

2. Calculate the unit-stresses developed in a triple-riveted lap joint of a boiler 4 feet in diameter carrying 110 pounds per square inch pressure, if the pitch is 3 inches, the thickness of the plate $\frac{1}{2}$ inch, and the diameter of the rivets $\frac{3}{4}$ inch. What is the efficiency of the joint?

The load transmitted through the three rivets in the pitch length of 3 inches is

$$P = \frac{QDp}{2} = \frac{110 \times 48 \times 3}{2} = 7920 \text{ pounds.}$$

The tension area carrying the load is found by assuming the diameter of the rivet hole the same as that of the rivet.

$$\therefore\ f_t = \frac{P}{A_t} = \frac{7920}{1.125} = 7040 \text{ pounds per square inch.}$$

The bearing area is

$$A_c = 3\,td = 3 \times \tfrac{1}{2} \times \tfrac{3}{4} = 1.125 \text{ square inches.}$$

$$\therefore \quad f_c = \frac{7920}{1.125} = 7040 \text{ pounds per square inch.}$$

The shearing area is

$$A_s = 3 \times .442 = 1.326 \text{ square inches.}$$

$$\therefore \quad f_s = \frac{7920}{1.326} = 5970 \text{ pounds per square inch.}$$

By using for the allowable unit-stresses f_s = 8000 pounds per square inch, f_t = 12,000 pounds per square inch, and f_c = 18,000 pounds per square inch, the efficiency of the joint can be calculated. The load the unpunched plate would carry in the pitch length is

$$P = 12{,}000 \times \tfrac{1}{2} \times 3 = 18{,}000 \text{ pounds.}$$

The load the punched plate will safely carry in the pitch length is

$$P_t = 12{,}000 \times 1.125 = 13{,}500 \text{ pounds.}$$

The load the rivets will carry in compression is

$$P_b = 18{,}000 \times 1.125 = 20{,}250 \text{ pounds.}$$

The load the rivets will carry in shear is

$$P_s = 8000 \times 1.326 = 10{,}600 \text{ pounds.}$$

The allowable load will be the least of these three values, which is 10,600 pounds, and the efficiency is

$$\epsilon = \epsilon_s = \frac{10{,}600}{18{,}000} = 59 \text{ per cent.}$$

The efficiencies for tension and bearing are higher. A larger rivet would give a higher efficiency.

PROBLEMS

1. A column bracket consists of a 6-inch × 6-inch × $\frac{3}{8}$-inch angle, and is riveted to the column, which is a 12-inch, 30-pound channel, whose thickness is 0.513 inch. It carries a load of 20,000 pounds and is riveted to the column with 5 rivets $\frac{7}{8}$ inch

in diameter. Determine the unit-stresses developed in bearing and shear.

2. If two plates 4 inches wide and $\frac{3}{8}$ inch thick are connected by four $\frac{7}{8}$-inch rivets in two rows, what load will the joint safely carry?

3. Determine the required number of rivets for a joint to carry 20,000 pounds, using $\frac{1}{2}$-inch plates. What is the efficiency of the joint?

4. Determine the required number of rivets for a joint to carry 25,000 pounds, using $\frac{3}{4}$-inch plates. What is the efficiency of the joint?

5. Design an angle bracket to be riveted to a column which consists of two 12-inch, 30-pound channels latticed together with the channel flanges extending outwards. The bracket is to support one end of a simple beam which carries a total uniform load of 60,000 pounds. Use rivets $\frac{7}{8}$ inch in diameter. Neglect bending in outstanding leg of the bracket angle.

6. Design a splice to connect two plates 10 inches wide and $\frac{3}{4}$ inch thick which are subjected to a tension of 78,000 pounds. Use two splice plates and rivets which are $\frac{7}{8}$ inch in diameter.

7. Select two angles to carry a tension load of 24,500 pounds and determine the number of rivets necessary if $\frac{3}{8}$-inch gusset plates are used. Use the allowable stresses given in example 1.

Ans. 2 – 2 in. × 2 in. × $\frac{5}{16}$ in. ∠s · $n = 5$.

8. Two $\frac{3}{8}$-inch plates are connected by three $\frac{3}{4}$-inch rivets. What tension load will the joint safely carry? If two of the rivets are in one row what should be the width of the plates?

Ans. 10,600 lb. 4.1 in.

9. In a butt joint with a double cover plate the main plates are $\frac{1}{2}$ inch thick and the cover plates are each $\frac{5}{16}$ inch thick. Design the joint to take a tension load of 20 tons. If the load is in compression how will the design be changed?

10. In a boiler 60 inches in diameter carrying steam pressure at 120 pounds per square inch the plate is $\frac{1}{2}$ inch thick, the rivets are $\frac{7}{8}$ inch in diameter and the pitch is $2\frac{1}{2}$ inches. The joint is a double-riveted lap joint. What are the tensile, shearing, and the compressive unit-stresses coming in the joint? What is the efficiency of the joint?

11. Determine the efficiency of a double-riveted lap joint where $t = \frac{7}{16}$ inch, $p = 3\frac{7}{16}$ inches, and the diameter of the rivet is $\frac{15}{16}$ inch.

12. Determine the efficiency of a single-riveted, two-strap butt joint, if $t = \frac{3}{8}$ inch, $p = 2$ inches, and the diameter of the rivet is $\frac{13}{16}$ inch.

13. Determine the pitch for a double-riveted, two-strap butt joint in which $t = \frac{1}{2}$ inch, and the diameter of the rivets is $\frac{15}{16}$ inch, so that the strength of the joint against tearing the plates between the rivet holes shall equal the compressive strength of the rivets. What is the efficiency of this joint?

14. Find the thickness of plates for a boiler shell 8 feet in diameter to carry a pressure of 160 pounds per square inch, if the efficiency of the joint is 80 per cent, and the stress in the plates is 5 tons per square inch.

15. Determine the efficiency of a single-riveted lap joint if $t = \frac{1}{2}$ inch, $d = \frac{7}{8}$ inch, and $p = 2\frac{9}{16}$ inches.

16. Calculate the efficiency of a double-riveted lap joint if $t = \frac{7}{16}$ inch, $d = 1$ inch, and $p = 3\frac{5}{8}$ inches.

17. Determine the pitch for a single-riveted, two-strap butt joint in which $t = \frac{5}{8}$ inch and $d = 1\frac{1}{16}$ inches, so that the strength of the joints against tearing the plates between the rivet holes shall equal the compressive strength of the rivets. Determine also the efficiency of the joint.

18. Design a double-riveted, two-strap butt joint for $\frac{5}{16}$-inch plates and find its efficiency.

CHAPTER V

BEAMS

EXTERNAL FLEXURAL FORCES

41. **DEFINITIONS.** Flexure occurs in a member when the load has a component normal to the axis of the member which causes the member to bend. A **beam** is a bar subjected to flexure. Usually the applied forces are normal to the axis of the beam, as when a horizontal bar resting on supports at its ends sustains vertical loads along its length. However, the term is also applicable when the direction of the applied forces is not at right angles to the axis. The loads on a beam cause it to bend and thus produce internal stresses which resist the bending. These stresses are called **flexural stresses.** The curve assumed by the axis of the beam under load is the **elastic curve.** The following treatment considers the beam to be horizontal and the loads vertical, but with slight modifications it may be adapted to beams in any position and with loads in any direction. The X-axis will be taken to coincide with the axis of the beam before bending. The Y-axis will be taken at right angles to the X-axis through the left support or left end.

A **cantilever beam** is one which has one end free and the other end fixed in such a manner that the tangent to the elastic curve at the fixed end remains horizontal. The elastic curve and the beam itself may be spoken of as being horizontal at the fixed end.

A **simple beam** is one which rests upon two end supports.

An **overhanging beam** has one or both of its supports away from the ends of the beam.

A **continuous beam** is one that rests on more than two supports.

The end of a beam is said to be **fixed** if it is restrained in such a manner that the elastic curve remains horizontal when the load is applied. It follows from the definitions above that a cantilever beam has one support, a simple beam has two supports, and a continuous beam has more than two supports.

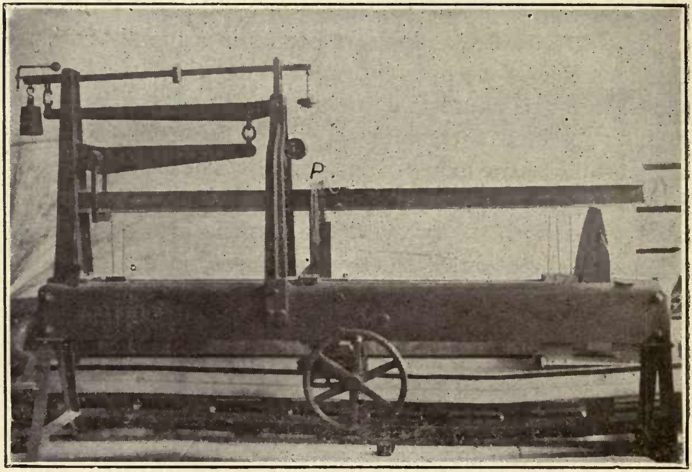


FIG. 19. Concentrated load.

42. METHODS OF LOADING BEAMS. According to the distribution of the loads, a beam may carry concentrated, uniform, and nonuniform or varying loads. When the load is transmitted to the beam through a comparatively small area it is said to be **concentrated.** Fig. 19 shows a concentrated load at the center of the span. If the load is distributed evenly over the beam it is a **uniform**

load. Fig. 20 shows a load that is practically uniform. If the load is distributed over the beam and is not of the same intensity throughout, it is said to be **non-uniform** or **varying.**

According to the method of application, loads are said to be dead or live loads. A **dead load** is one that the member always supports, such as its weight or

FIG. 20. Uniform load.
(Loaned by the Leonard Construction Company, Chicago.)

loads due to the weight of other portions of the structure of which the member is a part. **Live loads** are those that come upon the member temporarily, such as a train passing over a bridge, a crowd of people assembled in an auditorium, or machinery or a stock of goods. The loads shown in Fig. 19 and 20 are live loads.

43. FORCES ACTING ON A BEAM AS A WHOLE. The external forces acting on a beam are in equilibrium. The loads supported by the beam are usually known.

The forces supporting the beam — the reactions — may not be known at the start, but when possible should be determined before making other calculations. The loads and the reactions of the supports form a system of parallel forces. From theoretical mechanics the conditions of equilibrium for such a system are that there shall be no resultant force and no resultant moment. These conditions are expressed in two equations:

$$\Sigma F = 0, \quad (1)$$

$$\Sigma M = 0. \quad (2)$$

These formulas are used in determining the reactions.

ILLUSTRATIVE EXAMPLE

Let it be required to determine the reactions on a beam 8 feet long, carrying a uniform load of 4000 pounds and a concentrated load of 6000 pounds 3 feet from the left support. (See Fig. 21.)

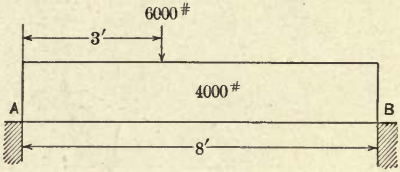


FIG. 21.

Let R_1 be the reaction of the left support and R_2 the reaction of the right support, then

$$\Sigma F = R_1 + R_2 - 4000 - 6000 = 0,$$

$$R_1 + R_2 = 10{,}000 \text{ pounds},$$

$$\Sigma M_a = R_2 \times 8 - 4000 \times 4 - 6000 \times 3 = 0,$$

$$R_2 = \frac{34000}{8} = 4250 \text{ pounds, and}$$

$$R_1 = 10{,}000 - 4250 = 5750 \text{ pounds}.$$

As a check take moments about the right reaction,

$$\Sigma M_b = 6000 \times 5 + 4000 \times 4 - R_1 \times 8 = 0,$$

$$R_1 = 5750 \text{ pounds}.$$

The line of action of the distributed load is taken through its center of gravity.

44. FORCES ACTING ON A PORTION OF A BEAM. INTERNAL STRESSES. In Fig. 22 is shown a beam with its loads. An effect of loading the beam is to produce internal stresses in the beam at all sections. To study the nature of these stresses imagine the beam to be cut along the vertical section AB. Then in order that equilibrium in the left portion of the beam be maintained with the external loads and reactions acting upon it, forces such as V_1, H_1, and H_2 must be supplied.

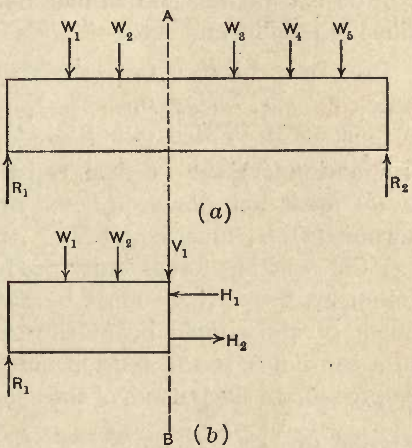


FIG. 22.

It is evident that before the section was cut internal stresses must have existed at this section, which acted upon the left portion of the beam. For present purposes these stresses may be considered to be replaced by V_1, H_1, and H_2 which will later be found to be the resultants of the internal stresses. All the forces shown in Fig. 22 (*b*) are external with respect to the left portion, but when the whole beam is considered, V_1, H_1, and H_2 are replaced by internal stresses. The reaction and the loads on the left portion of the beam tend to cause motion upward (or downward) and rotation in the clockwise direction about an axis in the section. Both of these tendencies are counteracted by the internal stresses at this section, or by their resultants, V_1, H_1, and H_2. The right portion of the beam could be treated in a similar manner, but the direction of V_1, H_1, and H_2 would be opposite to that shown for the left portion, and the magnitude would be the same.

The system of forces acting to one side of the section of the beam is coplanar, nonparallel, nonconcurrent. The equations of equilibrium are,

$$\Sigma F_y = 0, \qquad (1)$$

$$\Sigma F_x = 0, \qquad (2)$$

$$\Sigma M = 0. \qquad (3)$$

To satisfy equation (1) the vertical resisting force V_1 in Fig. 22 (*b*) must act downward (or upward). To satisfy equation (2) H_1 must equal H_2. And to satisfy equation (3) the resisting forces must produce an anti-clockwise moment, hence there must be compression in the top fibers of the simple beam shown. In other cases, as in a cantilever beam, tension may exist in the top and compression in the bottom of the beam.

45. **VERTICAL SHEAR.** From Art. 44 it is seen that the external forces to the left of the section tend to cause the left portion of the beam to slip upward (or downward) past the portion on the right. Whenever this is the case vertical shear is said to exist at the section. **Vertical shear** is the force that tends to move the left portion of a beam past the right portion or to cut the beam along a vertical plane. The magnitude of this force is measured by the summation of the vertical forces acting on the beam to the left of the section. (The vertical forces acting on the beam to the right of the section could be used as well.)

If V represents the vertical shear at a section the distance x from the left support,

$$V = \Sigma R_L - \Sigma W_L,$$

in which ΣR_L represents the sum of all the reactions to the left of the section, and ΣW_L is the sum of all the loads to the left of the section. The magnitude of V is equal to that of the resultant of all the external forces acting on the left portion of the beam.

46. Sign and Unit of Vertical Shear. The sign of the vertical shear depends upon the relative values of the reactions and the loads to the left of the section. If ΣR_L is the greater, the sign is plus, in which case the left portion tends to slip in the upward or positive direction past the right portion. The unit of vertical shear is the same as that used for force, and is usually the pound.

47. The Values of the Vertical Shear at the Section AB, Distant x from the Left End or Origin for Cantilever and Simple Beams. (1) For a cantilever beam with a concentrated load W at the end (Fig. 23 (1)),

$$V = -W.$$

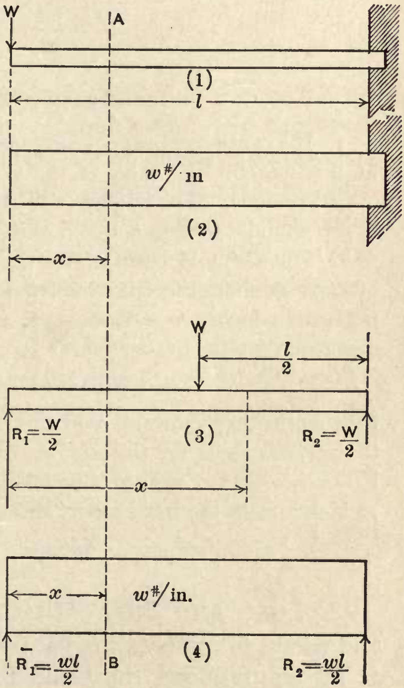


FIG. 23.

(2) For a cantilever beam with a uniform load of w pounds per inch of length (Fig. 23 (2)),

$$V = -wx.$$

(3) For a simple beam with a concentrated load W in the center of the span (Fig. 23 (3)),

$$V = \frac{W}{2} \text{ to the left of the center}$$

or $$V = -\frac{W}{2} \text{ to the right of the center.}$$

(4) For a simple beam with a uniform load of w pounds per inch of length (Fig. 23 (4)),

$$V = \frac{wl}{2} - wx.$$

ILLUSTRATIVE EXAMPLES

1. If a cantilever beam of 8-ft. span carries a load of 500 pounds at the free end, the value of the vertical shear at any section is -500 pounds. The left reaction is zero and the load to the left is 500 pounds; therefore $V = 0 - 500 = -500$ pounds.

2. For a simple beam of 10-ft. span carrying a uniform load of 6000 pounds, the left reaction is 3000 pounds. If x is expressed in inches $w = 6000 \div 120 = 50$ pounds per inch and the expression for the vertical shear is

$$V = 3000 - 50x.$$

By substituting values of x in this equation we obtain values of the vertical shear for the sections considered; thus at 25 inches, $V = 3000 - 1250 = 1750$ pounds. At the distance 6 feet or 72 inches from the left support the vertical shear is

$$V = 3000 - 3600 = -600 \text{ pounds.}$$

48. LOAD AND SHEAR DIAGRAMS. It is convenient and useful to indicate the value of the load and the shear at all points along the beam by means of vertical distances measured from a horizontal axis. These vertical distances are called ordinates. With the length of the beam and the loading known, the axis OX, Fig. 24, can be drawn to scale to represent the length of the beam. Thus, if the beam is 12 feet or 144 inches long, and, if it is convenient to make the axis OX 3 inches long, the scale of the length would be 1 inch equals 48 inches. From the axis in Fig. 24 at any point A a

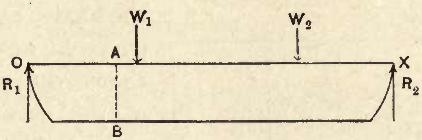


FIG. 24.

length AB can be erected perpendicular to OX to represent the intensity of load at that point. If the load at that point is 240 pounds per foot of length or 20 pounds per inch of length, the ordinate AB can be made $\frac{1}{5}$ inch, in which case 1 inch would represent a load of 100 pounds per inch. In the same way the value of the intensity of load at all other points along the beam can be represented by ordinates. The continuous curve connecting the ends of these ordinates is called the **load curve,** and the whole figure the **load diagram,** because the intensities of the load are shown at every point along the beam. The locations of the concentrated loads are indicated by arrows. Positive values are measured to the right of and up from the origin O, negative to the left of and down from the origin O. Loads act down and consequently are negative.

Knowing the loading, the reactions can be calculated and the values of the vertical shear can be obtained for all sections along the beam by the formula

$$V = \Sigma R_L - \Sigma W_L.$$

By a procedure similar to that used in making the load diagram the values of the vertical shear at all points along the length of the beam can be indicated by ordinates from a reference axis OX. Thus, Fig. 25 is the shear diagram for the loading shown in Fig. 24. In the shear diagram the distance from the axis OX to the curve at any point A' shows the value $A'B'$ of the shearing force for that section of the beam.

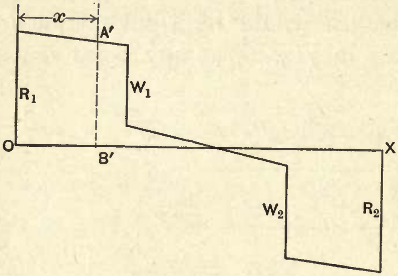


FIG. 25.

49. RELATION BETWEEN THE LOAD AND THE SHEAR. For distributed loads there is a definite relation between the load and the shear. To deduce this relation let Fig. 26 (*a*) be the load diagram and Fig. 26 (*b*) the shear diagram for a given case. Let u be a small length (called an element of length) measured along the OX axis, and let w be the average load per unit of length over this portion. Let x be the distance from the left support or origin, and V the vertical shear at the left end of the element of length. The load over this length is wu, and is equal to the small shaded area in the load diagram. Then the difference

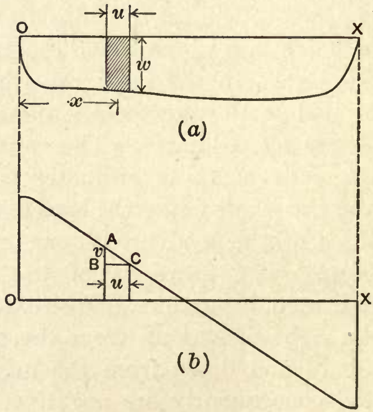


FIG. 26.

in the vertical shear at the two sides of the small length is

$$v = wu.$$

50. THE RATE OF CHANGE OF THE VERTICAL SHEAR. The rate at which the vertical shear changes is equal to the amount of change divided by the length in which the change is made, and is

$$\frac{v}{u} = \frac{wu}{u} = w.$$

u may be taken so small that AC in Fig. 26 (b) approaches a straight line. When u is made indefinitely small AC coincides with the tangent to the shear curve. It is also seen that $\frac{v}{u}$ equals tan ACB, which represents the slope of the shear curve at the given section when u is indefinitely small. w is the intensity of the load at that section. Therefore, **the rate of change of the vertical shear at any section of a beam equals the intensity of the load at that section.** It is also represented by the slope of the shear curve at that section.

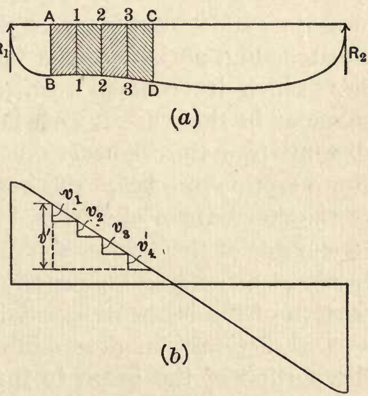


FIG. 27.

51. RELATION BETWEEN THE LOAD AND SHEAR DIAGRAMS. The relation given in Art. 49 affords a convenient graphical method of determining the change in the vertical shear between any two sections AB and CD of a beam, Fig. 27. Divide the length AC into several small lengths. Then v_1 equals area $AB11$, v_2 equals area 1122, v_3 equals area 2233, etc. The total change v' between the two sections is the summation of all the v's between them. This summation is equal to the area under the load curve between the sections, which is the total load between the two sections. Therefore, **the change in the vertical shear between any two sections of a beam equals the area under the load curve between the two sections.** (Distributed loads.)

52. **BENDING MOMENT. Moment** of a force is the product of the force and the perpendicular distance from the line of action of the force to the origin of moments. The moment measures the tendency of the force to cause the object acted upon by the force to turn about an axis through the origin of moments. Fig. 28 indicates a force F acting upon an object which it tends to turn about the point O. The moment in this case is Fa where a is the perpendicular distance from the origin O to the line of action of the force.

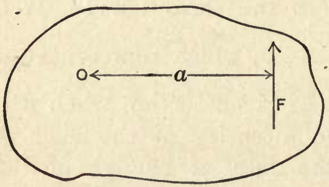


FIG. 28.

For a section of a beam the **bending moment** is the sum of the moments of all external forces acting on the beam to one side of the section about an axis in the section. A result of a bending moment is a tendency to cause the portion of the beam considered to rotate about an axis in the section. In determining the bending moment, **the portion of the beam to the right or the one to the left of the section may be used with the same results.** It is the common convention to use the left portion of the beam and this convention will be followed in this book. The bending moment at any section then is equal to the summation of the moments of all the forces acting upon the beam to the left of the section about an axis in the section. Since the external forces are made up of the reactions and loads, the bending moment M equals the moments of the reactions minus the moments of the loads to the left of the section, or expressed in a formula (see Fig. 29),

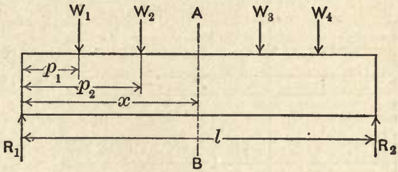


FIG. 29.

$$M = \Sigma R_L x - \Sigma W (x - p),$$

in which $\Sigma R_L x$ is the summation of the moments of the reactions, and $\Sigma W (x - p)$ is the summation of the moments of the loads to the left of the section.

53. SIGN AND UNIT OF BENDING MOMENT. When there is compression in the top fibers of the beam at a section the bending moment is positive at the section; when there is tension in the top fibers the sign of the bending moment is negative. The radius of curvature of the elastic curve is positive for a positive bending moment and negative for a negative bending moment. The sign of the bending moment due to a force is positive when the force itself would produce compression in the top fibers. Hence in all cases the bending moment of a reaction is positive, and that of a load is negative. The unit in which the bending moment is measured will depend upon the units of force and length employed. The pound-inch* is in most common use for beams and will be employed here.

* The term "inch-pound," which is also used for the unit of moment, does not make a distinction between the unit of work and the unit of moment.

54. The Values of the Bending Moment at the Section AB, Distant x from the Left End or Origin for Cantilever and Simple Beams.

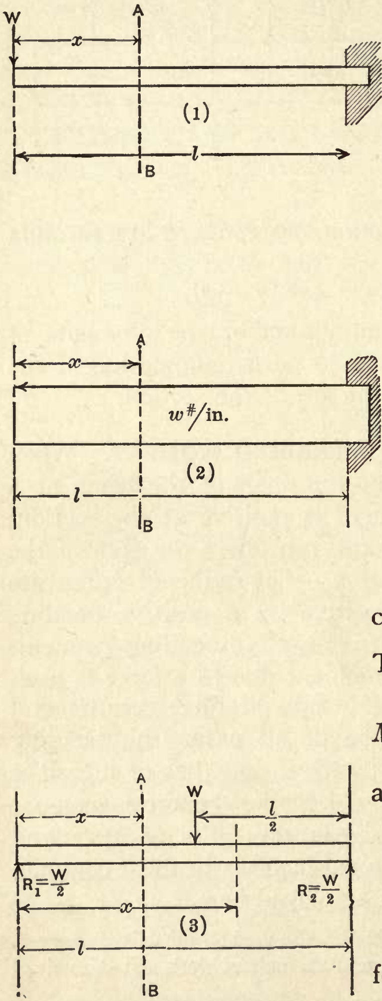


FIG. 30.

(1) Cantilever beam, concentrated load W at the end

$$M = -Wx.$$

Left reaction is zero (Fig. 30 (1)).

(2) Cantilever beam, uniformly distributed load of w pounds per inch

$$M = -wx\frac{x}{2} = -w\frac{x^2}{2}.$$

The left reaction is zero. The load to the left of the section is wx and its arm is $\frac{x}{2}$ (Fig. 30 (2)).

(3) Simple beam, concentrated load W at center. The left reaction is $\frac{W}{2}$,

$$M = \frac{W}{2}x \text{ for the left half,}$$

and

$$M = \frac{W}{2}x - W\left(x - \frac{l}{2}\right)$$

$$= \frac{Wl}{2} - \frac{Wx}{2}$$

for the right half (Fig. 30 (3)).

(4) Simple beam, uniform load of w pounds per inch.

The left reaction is $\frac{wl}{2}\cdot$

$$M = \frac{wl}{2}x - wx\frac{x}{2}$$

$$= \frac{wlx}{2} - \frac{wx^2}{2}.$$

The load to the left of the section is wx and its arm is $\frac{x}{2}$ (Fig. 30 (4)).

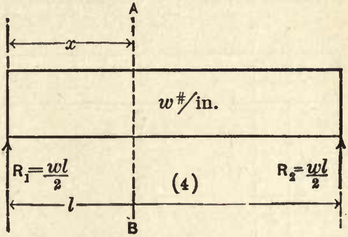


FIG. 30.

ILLUSTRATIVE EXAMPLES

1. For a cantilever beam of 9-ft. span carrying a load of 2000 pounds at the free end the bending moment at the section distant x from the free end is $M = -2000\ x$. If x is 50 inches, $M = -2000 \times 50 = -100{,}000$ lb.-in.

2. For a simple beam of 20-ft. span with a concentrated load of 10,000 pounds at the center the left reaction is 5000 pounds and the bending moment at 8 feet or 96 inches is

$$M_{96} = 5000 \times 96 = 480{,}000 \text{ lb.-in.}$$

At 16 feet the bending moment is

$$M_{192} = 5000 \times 192 - 10{,}000 \times 72 = 240{,}000 \text{ lb.-in.}$$

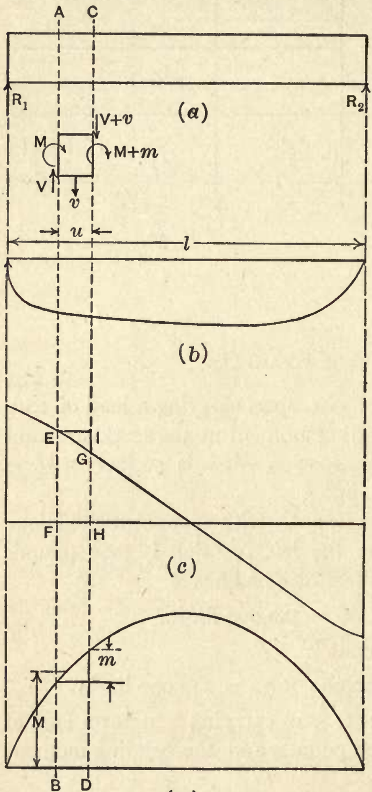
3. For a simple beam of 12-ft. span carrying a uniform load of 100 pounds per inch $R_1 = 7200$ pounds and the bending moment at the center is

$$M_{72} = 7200 \times 72 - \frac{100 \times 72 \times 72}{2} = 259{,}200 \text{ lb.-in.}$$

55. BENDING-MOMENT DIAGRAMS. The bending-moment diagram shows the value of the bending moment at all points along the beam. Fig. 31 (d) is such a diagram. OX represents to scale the length of the beam, and the ordinate M represents the bending moment at the section AB. The values of the bending moment may be calculated by the formula

$$M = \Sigma R_L x - \Sigma W(x - p),$$

or by the method of the following article.



(d)
FIG. 31.

56. RELATION BETWEEN THE VERTICAL SHEAR AND THE BENDING MOMENT. At a section distant x from the left support the bending moment is M, and at a section distant $x + u$ it is $M + m$, where u is an element of length along the OX-axis and m is the difference in the bending moment at the two sections. In the free-body diagram of the element of length of the beam between the two sections shown in Fig. 31 (*a*) M is the bending moment of the external forces to the left of the section AB. This moment is transferred to the element of the beam, and the bending moment transferred from this element to the right portion is $M + m$. The increase m is due to the external forces acting on the small portion of the beam. The external forces with respect to this portion acting upon it are V the vertical shear at the left, wu the weight, where w is the weight per unit of length, and $V + v$ the vertical shear at the right, as shown in the diagram. Taking moments about an axis in the section CD, there results

$$M + m = M + Vu - wu\frac{u}{2} = M + Vu - w\frac{u^2}{2};$$

$$\therefore \quad m = Vu - w\frac{u^2}{2}.$$

u may be taken so small that the load over this elementary length may be considered uniform. From Fig. 31 (c) it is seen that Vu is the area of the rectangle $EFHI$ and $w\frac{u^2}{2}$ is that of the triangle EGI.

Therefore, $m = EFHI - EGI$, which is equal to the area between the axis and the shear curve. If the sections are taken far apart the distance between the two sections should be divided into a large number of elementary lengths, and the total change in the bending moment will equal the summation of all the elementary changes. This leads to the conclusion that **the change in the bending moment between two sections equals the area under the shear curve between the two sections.** Having the vertical shear diagram drawn and knowing the value of the bending moment for any section of the beam, that for any other section may be obtained by getting the area under the shear curve between the two sections and adding it to the known moment. Areas above the axis are positive and those below are negative.

57. THE RATE OF CHANGE OF THE BENDING MOMENT. From the equation for the change of the bending moment between two sections, $m = Vu - w\frac{u^2}{2}$ the expression for the rate at which the bending moment changes at any section may be deduced by allowing u to become so small that the two sections AB and CD will be consecutive; then $w\frac{u^2}{2}$ is so small compared to Vu that it may be considered equal to zero. Therefore the rate of change is

$$\frac{m}{w} = \frac{Vu}{u} = V.$$

That is, **the rate of change of the bending moment at any section equals the vertical shear at that section.**

58. THE MAXIMUM VERTICAL SHEAR AND BENDING MOMENT. The greatest shearing stress will be at the section for which the vertical shear is the greatest and the greatest tensile and compressive moment stresses will be where the bending moment is the greatest. In any kind of beam the greatest shear occurs just to one side of a support. Because beams usually fail at the section of maximum bending moment, that section is called the **danger section.** From the relation existing between the shear area and the bending moment there is found a simple method of locating the danger section. By Art. 56 the area under the shear curve between two sections represents the change in the bending moment between those two sections. As we go along the beam the bending moment increases as long as the shear area is positive, and when the area becomes negative the moment grows less (Fig. 32). Shear area above the axis is positive and that below is

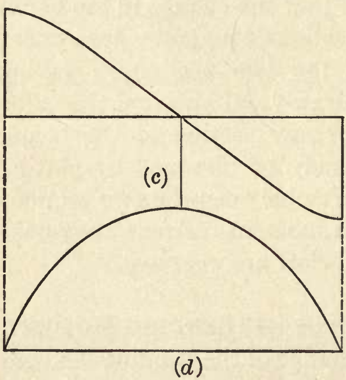


FIG. 32.

negative, the sign of the shear changing where the shear curve crosses the axis. Therefore, **the maximum bending moment in a beam occurs where the shear curve crosses the axis, i.e., where the vertical shear is zero.** This point may be obtained by plotting the shear curve. It may also be obtained from the equation of the vertical shear for the portion of the beam in which the shear passes through zero, by equating V to zero and solving for x. Note that the equation representing V must be for the portion of the beam in which the shear actually does pass through zero. Whenever the shear passes through zero at a concentrated load, the maximum bending moment will be under that load.

Since the bending moment is zero at both supports of a simple beam the shear area above the axis is equal to that below the axis.

59. LOAD, SHEAR, AND MOMENT DIAGRAMS FOR CANTILEVER AND SIMPLE BEAMS. MAXIMUM SHEAR AND MOMENT. For all cases let x be the distance from the origin to the section AB considered, W the total load on the beam, l the span, V_m the maximum shear, and M_m the maximum moment.

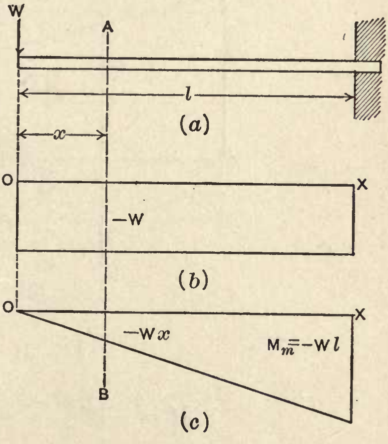


FIG. 33.

(1) For a cantilever beam with a load at the end (Fig. 33),

$$V = -W, \quad V_m = -W, \quad M = -Wx.$$

The area in shear diagram is negative and equals $-Wx$. The maximum moment occurs at the wall and equals the entire area in the shear diagram which is

$$M_m = -Wl.$$

(2) For a cantilever beam with a uniform load (Fig. 34). The load per unit of length is

$$-w = -\frac{W}{l}.$$

$$V = -\frac{W}{l}x, \quad V_m = -W,$$

which comes at the wall.

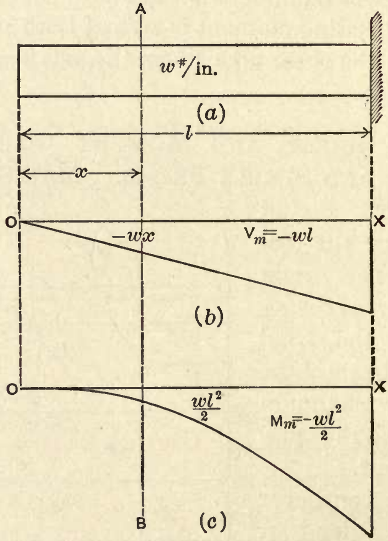


FIG. 34.

$$M = -W\frac{x^2}{2} = -\frac{W}{l}\frac{x^2}{2}.$$

This is equal to the shear area to the left of the section.

The maximum moment comes at the wall and equals the entire area under the shear curve and is

$$M_m = -W\frac{l^2}{2} = -\frac{Wl}{2}.$$

(3) For a simple beam with a concentrated load at the center of the span (Fig. 35),

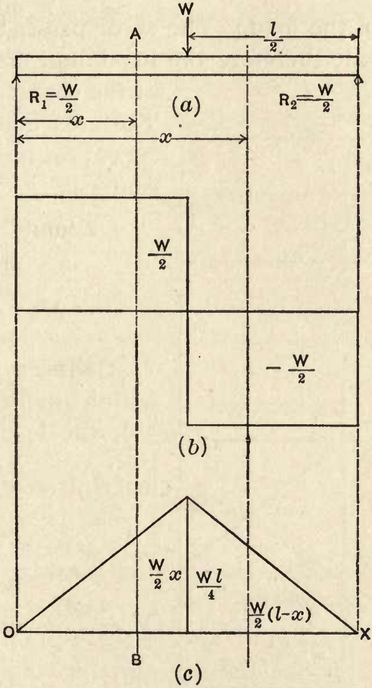


FIG. 35.

$$R_1 = \frac{W}{2},$$

$$V = \frac{W}{2} \text{ to the left of the load,}$$

$$V = -\frac{W}{2} \text{ to the right of the load.}$$

$$V_m = \frac{W}{2},$$

$$M = \frac{W}{2}x \text{ to the left of the load}$$

$$M = \frac{W}{2}x - W\left(x - \frac{l}{2}\right) = \frac{W}{2}(l - x)$$

to the right of the load. The shear passes through zero under the load; therefore the maximum moment occurs at the center of the beam. The area under the shear curve to the left of the center is

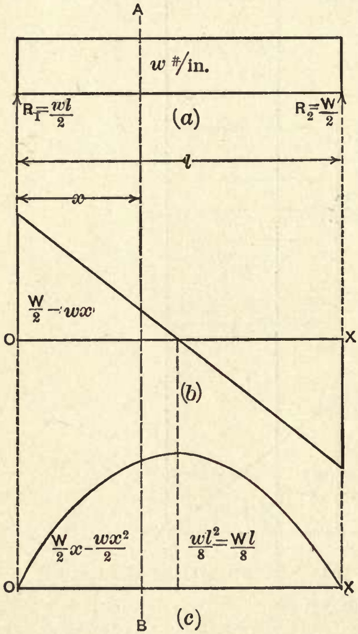


FIG. 36.

$$\frac{W}{2} \times \frac{l}{2} = \frac{Wl}{4},$$

$$M_m = \frac{Wl}{4}.$$

(4) For a simple beam with a uniform load (Fig. 36), the load per unit of length is $-w = -\frac{W}{l}$.

$$R_1 = \frac{wl}{2} = \frac{W}{2}.$$

$$V = \frac{wl}{2} - wx = \frac{W}{2} - \frac{W}{l}x,$$

$$V_m = \frac{W}{2}.$$

The average ordinate in the shear area to the section at the distance x from the left end is

$$\left(\frac{wl}{2} + \frac{wl}{2} - wx\right) \div 2 = \frac{wl}{2} - \frac{wx}{2};$$

therefore the area under the curve to the section equals

$$\frac{wlx}{2} - \frac{wx^2}{2},$$

$$M = \frac{wlx}{2} - \frac{wx^2}{2} = \frac{Wx}{2} - \frac{Wx^2}{2l}.$$

The shear passes through zero at the center of the span; therefore the maximum moment occurs at that section and equals the shear area to the left of the center.

$$M_m = \frac{wl}{2} \times \frac{l}{2} \times \frac{1}{2} = \frac{wl^2}{8} = \frac{Wl}{8}.$$

(5) For a simple beam with a concentrated load at

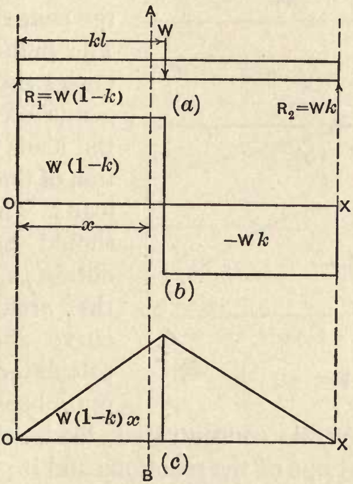


FIG. 37.

any point distant kl from the left support, in which k is less than 1 (Fig. 37),

$$R_1 = W(1 - k),$$
$$V = W(1 - k) \text{ to the left of load,}$$
$$V = -Wk \text{ to the right of load,}$$
$$M = W(1 - k)x \text{ to the left of load,}$$
$$M = W(1 - k)x - W(x - kl)$$
$$= Wk(l - x) \text{ to the right of load.}$$

The shear passes through zero under the load. The area to the load is

$$W(1 - k)\,kl,$$

$$M_m = W(1 - k)\,kl.$$

(6) A uniform and a concentrated load on a simple beam (Fig. 38). For cases of this kind the shear will be a maximum at one of the supports and will pass through zero under the concentrated load or between that load and the center of the beam. The point at which it passes through zero depends on the ratio of the loads and the position of the concentrated load. The shear curve should be plotted to obtain that point, then the area under that curve above the axis calculated for the maximum bending moment.

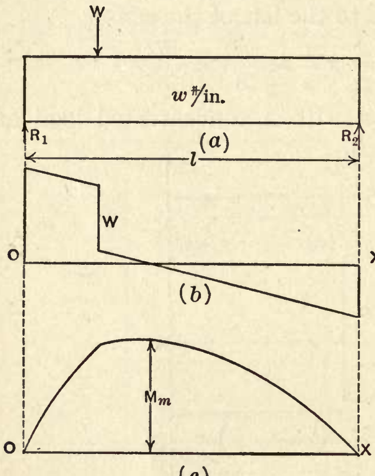


FIG. 38.

(7) For several concentrated loads the maximum shear occurs at one of the reactions and it passes through zero at one of the loads (Fig. 39). The point of zero shear may be obtained by plotting the shear curve. Then the maximum moment may be obtained by calculating the area in the shear diagram above the axis.

(8) For the case of several concentrated loads and a uniform load the shear may pass through zero at one of the concentrated loads or between any two of them, the position of zero shear depending upon the relative values of the loads and their positions (Fig. 40).

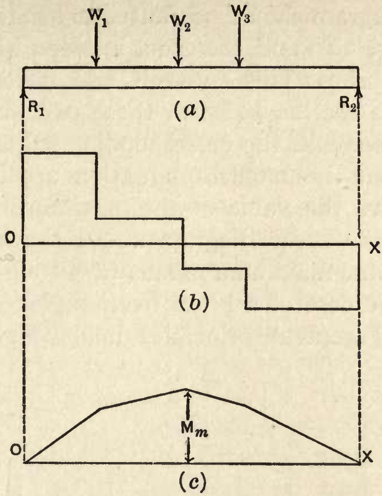


FIG. 39.

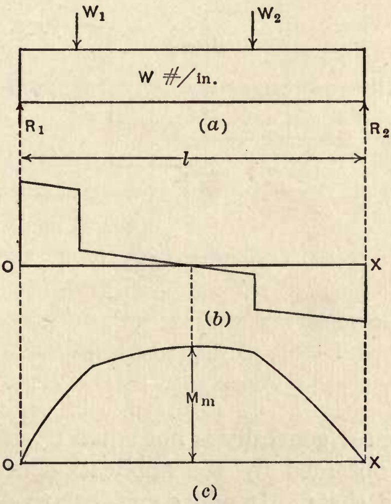


FIG. 40.

The shear diagram should be plotted to locate the section of zero shear. To find the exact location when the section of zero shear falls between two loads the shear formula for a section between these two loads may be equated to zero and the corresponding value of x determined. Then the moment equation applied for that point will give the value of the maximum moment, or the area in the shear diagram above the axis may be obtained for the maximum moment.

(9) For the case of a beam overhanging one or both supports the general principles hold (Fig. 41). The

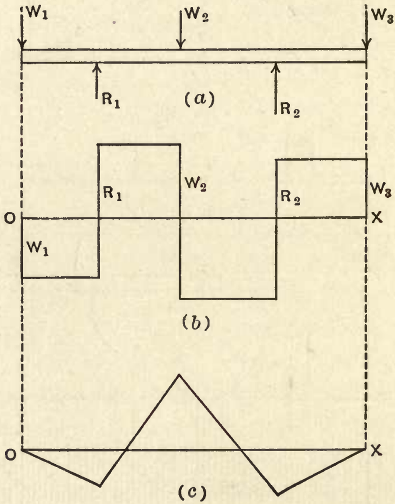


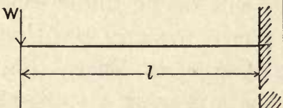
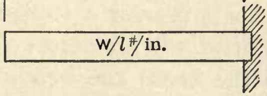
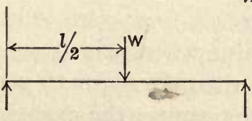
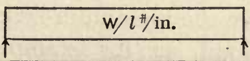
FIG. 41.

maximum shear generally is not equal to a reaction, but it may be obtained by the application of the general equation for shear. It will come just to one side of a support. The shear passes through zero at the supports, and also between them, consequently the maximum bending moment may occur at any one of the three points. The moment must be obtained for all three and the greatest one taken for the maximum.

60. RELATIVE STRENGTH OF CANTILEVER AND SIMPLE BEAMS. The shearing unit-stresses developed in a beam are directly proportional to the vertical shear, and the unit-stresses resisting the bending moment are directly proportional to the bending moment; therefore the shearing strength of a certain type of beam is inversely proportional to the maximum vertical shear developed in that type, and the strength to resist bending is inversely proportional to the maximum bending moment developed. Table 12 gives the values of the maximum vertical shear and the maximum bending moment, and the relative strengths in shear and bending for simple and cantilever beams.

TABLE 12

RELATIVE STRENGTHS IN SHEAR AND BENDING

Kind of beam.	Maximum vertical shear.	Maximum bending moment	Relative strength in shear.	Relative strength in bending.
	W	$-Wl$	1	1
	W	$-\frac{Wl}{2}$	1	2
	$\frac{W}{2}$	$\frac{Wl}{4}$	2	4
	$\frac{W}{2}$	$\frac{Wl}{8}$	2	8

ILLUSTRATIVE EXAMPLES

Calculate the maximum bending moment developed in a beam of 12-ft. span carrying a load of 800 pounds, (a) when used as a cantilever with the load concentrated at the end; (b) if the load is uniformly distributed on the cantilever; (c) when used as a simple beam with the load concentrated at the center; (d) if the load is uniformly distributed over the simple beam.

(a) $M_m = -800 \times 12 \times 12 = -115{,}200$ lb.-in. at the wall.

(b) $M_m = \dfrac{800 \times 12 \times 12}{2} = -57{,}600$ lb.-in. at the wall.

(c) $M_m = \dfrac{800 \times 12 \times 12}{4} = 28{,}800$ lb.-in. at the center.

(d) $M_m = \dfrac{800 \times 12 \times 12}{8} = 14{,}400$ lb.-in. at the center.

61. MOVING CONCENTRATED LOADS ON A BEAM. When several concentrated loads pass over a beam the beam must be strong enough to take the greatest shear and the greatest bending moment caused by the loads. Hence it is necessary to determine the maximum shear and moment developed by the system of loads. The greatest vertical shear will be developed at the support when one of the loads is very near it.

The greatest bending moment will occur under one of the loads, since the moment curve for any position of the loads consists of a series of straight lines. For simple beams, when one of the loads is over a support the bending moment under that load will be zero. As the system of loads passes over the beam the bending moment under each load will increase from zero, when at the support, to a maximum value when it is at some point between the supports, and then decrease to zero at the other support. Let us determine the position of the system of loads that will give the maximum bending moment under a particular load, as, for example,

W_2 in Fig. 42. Let R be the resultant of all the loads and its line of action be at the distance $\bar{x}$ from load W_2

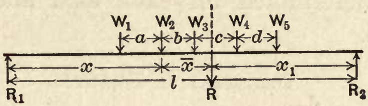


FIG. 42.

and x_1 from the right support. Let the load be at the distance x from the left support; then

$$R_1 = \frac{Rx_1}{l} \text{ and } x_1 = l - x - \bar{x}.$$

The moment under the load then is

$$M_2 = \frac{Rx_1}{l}x - W_1a = \frac{R}{l}xl - \frac{R}{l}x^2 - \frac{R}{l}\bar{x}x - W_1a$$

$$M_2 = \frac{R}{l}x^2 + \frac{R}{l}(l - \bar{x})\,x - W_1a.$$

By algebra it can be shown* that the value of x to give the greatest value of a function of the form

$$ax^2 + bx + c \text{ is } -\frac{b}{2a}.$$

By substituting in this formula for x,

$$x = \frac{\frac{R}{l}(l - \bar{x})}{2\frac{R}{l}} = \frac{l - \bar{x}}{2},$$

$$2x = l - \bar{x} \text{ or transposing } x,$$

$$x = l - \bar{x} - x = x_1.$$

This shows that **the position of one of a system of moving concentrated loads, when the greatest bending moment occurs under that load, is such that its distance from the left support is the same as the distance of the center of gravity of all the loads from the right support.**

* See "Higher Algebra," by John F. Downey, page 245.

In order to obtain the maximum bending moment produced by the system of moving loads the maximum should be determined for each load and the highest value taken.

ILLUSTRATIVE EXAMPLE

Let it be required to obtain the maximum bending moment produced in a beam of 21-ft. span by a system of three moving loads of 4 tons, 2 tons, and 3 tons spaced 3 feet and 4 feet apart respectively. (Fig. 43.)

The resultant is 9 tons. Its line of action with respect to the loads may be obtained by taking moments about any load. The

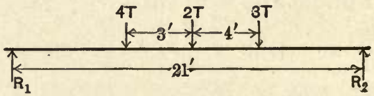


FIG. 43.

line of action of the resultant passes through the 2-ton load. For the maximum moment coming under the 4-ton load its distance from the left end must be the same as the distance of the resultant of the three loads from the right end. $\bar{x}$ for this load is 3 feet; therefore, the distance of the load from the left support is $x = \frac{l - \bar{x}}{2} = \frac{21 - 3}{2} = 9$ feet. Placing the loads in this position, obtaining the left reaction by taking moments about the right support, and using the resultant instead of each load separately, there results

$$R_1 = \frac{9}{21} \times 9 \times 2000 = 7710 \text{ pounds.}$$

The moment under this load is

$$M_1 = 7710 \times 9 \times 12 = 833{,}000 \text{ pound-inches.}$$

The maximum moment will come under the 2-ton load when it is at the center of the beam, since $\bar{x}$ for this load is zero; then

$$R_1 = 4.5 \times 2000 = 9000 \text{ pounds.}$$

$$M_2 = (9000 \times 10\tfrac{1}{2} - 8000 \times 3)\,12 = 846{,}000 \text{ lb.-in.}$$

For the 3-ton load $\bar{x} = -4$ feet,

$$x = \frac{21 - (-4)}{2} = 12\tfrac{1}{2} \text{ feet.}$$

$$R_1 = \frac{12.5}{21.0} \times 9 \times 2000 = 10,710 \text{ pounds.}$$

$$M_3 = (10,710 \times 12.5 - 8000 \times 7 - 4000 \times 4)12 = 744,000 \text{ lb.-in.}$$

The maximum moment for this system of loads then is developed under the 2-ton load and is 846,000 pound-inches.

EXAMPLES

1. Given a simple beam, 15-ft. span, with a load varying uniformly from zero at the left end to 1000 pounds per lineal foot at the right end, and a concentrated load of 9000 pounds 6 feet from the left end. Determine the reactions. (See Fig. 44.)

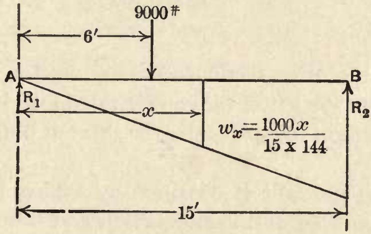


FIG. 44.

The average distributed load is $(0 + 1000) \div 2 = 500$ pounds per foot. The total distributed load is $500 \times 15 = 7500$ pounds. The center of gravity of this load is $\frac{2}{3} \times 15 = 10$ feet from the left support. Taking moments about the right support,

$$\Sigma M_B = 9000 \times 9 + 7500 \times 5 - R_1 \times 15 = 0,$$

$$\therefore \quad R_1 = \frac{9000 \times 9 + 7500 \times 5}{15} = 7900 \text{ pounds.}$$

Taking the summation of the forces to obtain R_2,

$$\Sigma F = 7900 + R_2 - 9000 - 7500 = 0; \quad \therefore \quad R_2 = 8600 \text{ pounds.}$$

For checking on R_2 take moments about R_1,

$$\Sigma M_A = R_2 \times 15 - 9000 \times 6 - 7500 \times 10 = 0;$$

$$\therefore \quad R_2 = \frac{9000 \times 6 - 7500 \times 10}{15} = 8600 \text{ pounds.}$$

2. Calculate the shear and bending moment at sections 2 feet apart and draw the shear and moment diagrams for a simple beam of 14-ft. span with a uniform load of 14,000 pounds and a concentrated load of 7000 pounds 5 feet from left support.

$$R_1 = \tfrac{7}{14} \times 14{,}000 + \tfrac{9}{14} \times 7000 = 7000 + 4500 = 11{,}500 \text{ pounds.}$$

$$R_2 = 21{,}000 - 11{,}500 = 9500 \text{ pounds.}$$

Check $\quad R_2 = \tfrac{7}{14} \times 14{,}000 + \tfrac{5}{14} \times 7000 = 9500$ pounds.

From the definition $\quad V = R_1 - \Sigma W_L$,

$$V_0 = 11{,}500 - 0 = 11{,}500 \text{ pounds.}$$

$$V_2 = 11{,}500 - 2000 = 9500 \text{ pounds.}$$

$$V_4 = 11{,}500 - 4000 = 7500 \text{ pounds.}$$

$$V_5' = 11{,}500 - 5000 = 6500 \text{ pounds.}$$

$$V_5'' = 11{,}500 - 5000 - 7000 = -500 \text{ pounds.}$$

$$V_6 = 11{,}500 - 6000 - 7000 = -1500 \text{ pounds.}$$

The values at the other sections were obtained in the same way. The shear just to the left of the concentrated load is different from that just to the right; consequently the shear at both sections must be calculated.

The bending moment is obtained by taking the moment of the reaction about an axis in the section and subtracting from it the moment of the loads to the left of the section about the same axis.

$$M_0 = 0,$$

$$M_2 = 11{,}500 \times 24 - 2000 \times 12 = 252{,}000 \text{ pound-inches,}$$

$$M_4 = 11{,}500 \times 48 - 4000 \times 24 = 456{,}000 \text{ pound-inches,}$$

$$M_5 = 11{,}500 \times 60 - 5000 \times 30 = 540{,}000 \text{ pound-inches,}$$

$$M_6 = 11{,}500 \times 72 - 6000 \times 36 - 7000 \times 12 = 528{,}000 \text{ pound-inches.}$$

The bending moments at the other sections were obtained in the same way. The shear passes through zero at 5 feet from the left support; therefore the maximum moment occurs at that point.

In Fig. 45 are drawn the load, shear, and moment diagrams for this beam.

3. In Example No. 2 obtain the bending moment at sections 5 feet and 8 feet from the left support by use of the shear diagram.

x feet.	*V* pounds.	*M* pound-inches.
0	11,500	0
2	9,500	252,000
4	7,500	456,000
5′	6,500	540,000
5″	− 500	540,000
6	−1,500	528,000
8	−3,500	468,000
10	−5,500	360,000
12	−7,500	204,000
14	−9,500	0

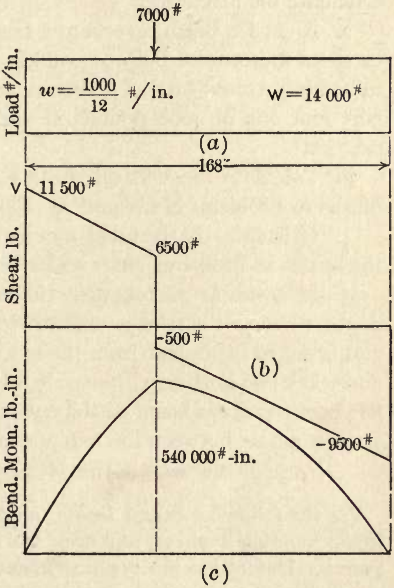


FIG. 45.

Let the first section be taken under the load; then the base of the trapezoid is $5 \times 12 = 60$ inches; the average ordinate is $(11,500 + 6500) \div 2 = 9000$ pounds. The area under the curve is $9000 \times 60 = 540,000$ pound-inches $= M_5$. At 8 feet the negative area must be taken from the positive. As obtained above, the positive area is 540,000 pound-inches. The base of the negative area is $3 \times 12 = 36$ inches. The average ordinate is $(-500 - 3500) \div 2 = -2000$ pounds, therefore, the area is $36 \times (-2000) = -72,000$ pound-inches.

$$M_8 = 540,000 - 72,000 = 468,000 \text{ pound-inches.}$$

PROBLEMS

1. A simple beam of 12-ft. span carries a uniform load of 6000 pounds, a concentrated load of 3000 pounds 4 feet from the left support, and one of 6000 pounds 9 feet from the left support. Calculate the reactions. *Ans.* $R_1 = 6500$ lb.

2. A 24-ft. beam overhangs the right support 4 feet. It carries a uniform load of 900 pounds per foot of length in addition to a concentrated load of 10,000 pounds 9 feet from the left support and one of 4000 pounds at the right end. Determine the reactions.

3. Calculate the vertical shear at points 2 feet apart for the beams in Problems No. 1 and 2. Plot the shear diagrams.

4. Calculate the bending moment at points 2 feet apart for the beams in Problems No. 1 and 2. Plot the moment diagrams.

5. If a simple rectangular timber beam of 14-ft. span and depth 12 inches carries a uniform load of 1000 pounds per foot and is cut in two 4 feet from the left support, what vertical force, and what two horizontal forces, if 8 inches apart, must act on the left portion of the beam at the cut section in order to replace the stresses acting between the two portions before the beam was cut? *Ans.* $V = -3000$ lb. $H = 30{,}000$ lb.

For the following beams locate the danger section, calculate maximum bending moment, and draw the load, shear, and moment diagrams. Determine the bending moment by getting the area in the shear diagram.

6. A 15-ft. simple beam having a uniform load of 400 pounds per foot and concentrated loads of 6000 pounds and 5000 pounds at 5 feet and 8 feet respectively from the left support.

7. A 20-ft. simple beam having concentrated loads of 2000 pounds, 4000 pounds, and 3000 pounds at 4 feet, 6 feet, and 12 feet respectively from the left support.

8. Simple beam of 20-ft. span carrying a uniform load of 120 pounds per foot and concentrated loads of 400 pounds, 600 pounds, and 600 pounds at 4 feet, 6 feet, and 16 feet respectively from the left support.

9. Cantilever beam of 8-ft. span with a load of 10,000 pounds at the end. *Ans.* $M_m = 960{,}000$ lb.-in.

10. Cantilever beam of 10-ft. span with a uniform load of 15,000 pounds. *Ans.* $M_m = 900{,}000$ lb.-in.

11. Simple beam of 12-ft. span with a concentrated load of 9000 pounds at the middle. *Ans.* $M_m = 324{,}000$ lb.-in.

12. Simple beam of 15-ft. span with a uniform load of 15,000 pounds. *Ans.* $M_m = 337{,}500$ lb.-in.

13. Simple beam of 14-ft. span with a concentrated load of 12,000 pounds 4 feet from left support.

14. Simple beam of 40-ft. span with a uniform load of 20 tons, concentrated load of 100 tons at the center.

15. Simple beam of 16-ft. span with a uniform load of 32,000 pounds, and a concentrated load of 16,000 pounds at 4 feet from left support. *Ans.* $M_m = 1{,}200{,}000$ lb.-in.

16. Simple beam of 18-ft. span with a concentrated load of 9000 pounds 4 feet from left support, one of 7000 pounds 8 feet from left support, one of 12,000 pounds 13 feet from left support.

17. Simple beam of 12-ft. span with a uniform load of 18,000 pounds, two equal concentrated loads of 10,000 pounds each at the one-third points.

18. Overhanging beam 18 feet long overhanging the left support 4 feet, with a uniform load of 1500 pounds per foot.

19. Simple beam, same as in Problem No. 14, with an additional load of 5000 pounds at the center of the beam.

20. A simple beam of 20-ft. span weighing 12 pounds per lineal foot, with a load of 240 pounds 5 feet from the left end.

21. A simple beam of 20-ft. span with concentrated loads of 2000 pounds 4 feet from the left end, and 1000 pounds 18 feet from the left end, and also a uniform load of 100 pounds per lineal foot.

22. An overhanging beam 12 feet long overhanging the right end 2 feet carrying a uniform load of 2000 pounds per foot on entire beam.

23. An overhanging beam 13 feet long overhanging the right support 3 feet, carrying a uniform load of 1000 pounds per foot between supports, and a uniform load of 500 pounds per foot on the overhanging end.

24. Two loads each of 3000 pounds, 5 feet apart, roll over a simple beam of 15-ft. span. Find the position of these loads for the maximum bending moment and find its value.

Ans. $6\frac{1}{4}$ ft. from left support.

25. Two wagon wheels, 8 feet apart, roll over a simple beam of 24-ft. span. If the load on each wheel is 2000 pounds, find their position for the maximum bending moment and determine its value. *Ans.* M_m = 200,000 lb.-in.

26. Compute the maximum bending moment due to two loads of 1000 pounds each and 5 feet apart rolling over a 25-ft. simple span.

27. Determine the maximum bending moment produced in a beam of 24-ft. span by a system of three rolling loads of weight 10,000 pounds, 5000 pounds, and 12,000 pounds if the distance between the first and second is 6 feet and the distance between the second and third is 5 feet.

CHAPTER VI

BEAMS—INTERNAL FLEXURAL STRESSES

62. Forces and Stresses. In Chapter V the external forces acting on beams were considered. These external forces tend to cause the beam to rupture, and that tendency is resisted by the internal stresses set up in the beam. The nature, distribution, and magnitude of these stresses will be considered in this chapter.

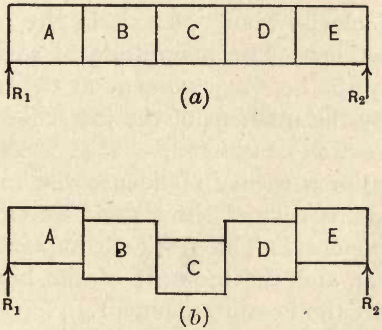


Fig. 46.

The vertical shear tends to rupture the beam along a vertical plane by causing one portion to slip past the other. To illustrate this effect imagine a beam built up of several blocks glued together as indicated in Fig. 46 (*a*). If the glue becomes soft and sticky, the inner portions will slide down past the others. Or otherwise stated the outer portions will slide up past the inner portions. The magnitude of the tendency of one portion to slip past the one to the right is measured by the vertical shear at the plane between the two portions. This vertical shear is resisted by internal resisting stresses acting vertically at the section considered.

The bending moment at a section tends to cause the end of the beam to rotate about an axis in that section. Thus, at the section between the portions A and B of the beam in Fig. 47 the external forces tend to cause A

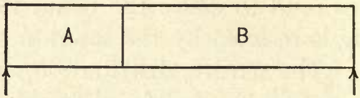


FIG. 47

to rotate clockwise about an axis in the plane between the two portions. The magnitude of this tendency is measured by the bending moment at the section. This is resisted by the moment of the internal stresses which act on the section considered.

In the common theory of flexure the internal resisting stresses are divided into their vertical and horizontal components. The vertical components resist the vertical shear and the moment of the horizontal components resist the bending moment.

63. **RESISTING SHEAR. THE SHEAR FORMULA.** As already defined the vertical shear V at a section is the algebraic sum of the external forces acting on the portion of the beam to the left of the section; and this vertical shear is resisted by the stresses in the fibers between the two portions of the beam. This stress, which resists the vertical shear, represented in Fig. 48 by V', is called the **resisting shear.** If the maximum shearing unit-stress in the cross section is s the average value will be ks, where k is a constant depending upon the

shape of the cross section, and if A is the area of the cross section the resisting shear equals ksA. Therefore, since equilibrium exists, the vertical shear equals the resisting shear and the shear formula is

$$V = ksA. \qquad (1)$$

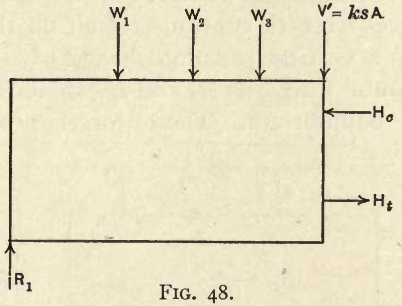


FIG. 48.

64. THE VALUE OF k IN THE SHEAR FORMULA. As shown in Chapter XIII, the intensity of the shearing unit-stress is not the same at all points of the cross section. The maximum stress from the shear formula is $s = \frac{V}{kA}$.

For a rectangular beam $s = \frac{3}{2}\frac{V}{A}$, and $k = \frac{2}{3}$.

For a circular beam $s = \frac{4}{3}\frac{V}{A}$, and $k = \frac{3}{4}$.

For I-beams and built-up sections it is approximately assumed that the maximum stress is equal to that obtained by dividing the shear by the area of the web; then the maximum stress is

$$s = \frac{V}{A_1}, \text{ where } A_1 \text{ is the area of the web.}$$

If k for this case is desired it is equal to $\frac{A_1}{A}$. The assumption is only approximately correct but the values are near enough the true ones to be used in design.

65. RESISTING MOMENT. If a beam were cut through at a section, as at AB, Fig. 49, and the same external forces were to continue to act on the left portion, besides a vertical resisting shear ksA, forces equal to the horizontal forces, as H_c and H_t, should be supplied to produce equilibrium. These forces are the horizontal components of the internal stresses acting on the given section of the beam and make up the horizontal resisting forces which produce the resisting moment.

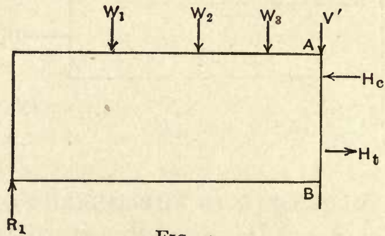


FIG. 49.

66. ASSUMPTIONS FOR THE RESISTING MOMENT. The moment formula to be derived is for beams made of materials that have the same modulus of elasticity in tension and compression. The formula is true only for stresses less than the elastic limit of the material. It is assumed that a transverse plane section of the beam before bending remains a plane section after bending. From these assumptions the nature, distribution, and magnitude of the stresses producing H_c and H_t may be found.

67. Distribution of the Fiber Stresses. Two sections of the beam, as *AB* and *CD* in Fig. 50 (*a*), parallel before the beam is bent, assume positions shown in Fig. 50 (*b*) after bending occurs. All fibers of the beam, except those along the surface *OX*, will be lengthened or shortened, thus having stresses developed in them. The surface *OX* along which no tensile or compressive stresses are developed is the **neutral surface.** Its intersection *IJ* with a transverse section is the

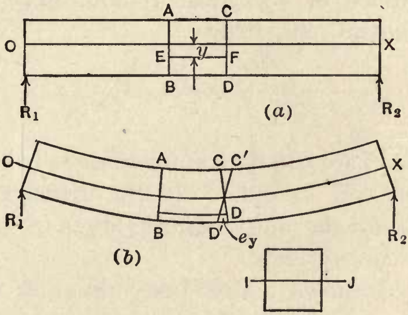


FIG. 50.

neutral axis. Since the modulus of elasticity E is considered constant, all other fibers will have a stress proportional to their deformation. Take the fiber *EF* at a distance y from the neutral surface. Draw $C'D'$ parallel to *AB* in Fig. 50 (*b*); then e_y represents the deformation of that fiber in the length *EF* after bending. The deformation e_y divided by the length *EF* gives the unit deformation. If this be multiplied by the modulus of elasticity of the material the result will give the unit-stress f_y coming on the fiber. The direction of the stress is the same as that of the deformation which is along the line *EF* and acts normal to the section.

Since the greatest deformation occurs in the fiber farthest from the neutral axis the greatest stress will be in that fiber.

It is seen that the deformation of a fiber is proportional to its distance from the neutral surface, consequently the stress is proportional to the distance of the fiber from the neutral axis. The ratio of the maximum unit-stress in compression to the maximum unit-stress in tension is the same as the ratio of the distance of the most remote fiber in compression from the neutral axis to the distance of the most remote fiber in tension from the neutral axis, hence

$$\frac{f_c}{f_t} = \frac{c_c}{c_t},$$

where f_c and f_t are the maximum stresses in compression and tension, and c_c and c_t are the distances from the neutral axis to the most remote fibers in compression and tension respectively.

Fig. 51 (*a*) shows a free-body diagram of the left portion of a beam, in which the resisting stresses in tension and compression are indicated in intensity by the length of the vectors representing them. Fig. 51 (*b*) is an end view of the section, in which the stresses acting normal to the section (also normal to the plane of the paper) are indicated by crosses for compression and by circles and dots for tension. The intensity of the stress is indicated by the weight of the lines.

68. POSITION OF THE NEUTRAL SURFACE AND THE NEUTRAL AXIS. Let Fig. 51 (*a*) represent a portion of a beam under load and Fig. 51 (*b*) represent the cross section. Let the maximum fiber stress developed be f, which comes on the fiber most remote from the neutral axis IJ. Call the distance of this fiber from the neutral axis c, then the unit-stress developed on a fiber at the distance y is $\frac{y}{c}f$, and the total stress acting upon the small area a is $\frac{y}{c}fa$. The sum of the horizontal stresses

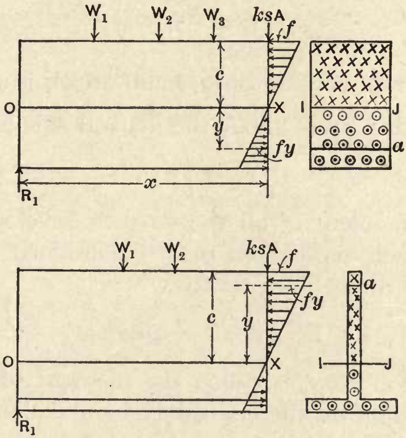


FIG. 51.

acting upon the entire cross section is equal to the algebraic sum of all stresses on the elements of areas. Since these are the only horizontal forces acting upon the left portion of the beam this sum is equal to zero.

$$\Sigma \frac{y}{c} fa = \frac{f}{c} \Sigma\, ay = 0,$$

$$\frac{f}{c} \Sigma\, ay = \frac{f}{c} A\bar{y} = 0,$$

$$\therefore \quad \bar{y} = 0.$$

$\Sigma\, ay$ is the moment of the area with respect to the neutral axis and $\bar{y}$ is the distance of the centroid of the cross section from the neutral axis. Therefore, **the neutral surface passes through the centroid of the cross section of a beam.**

69. The Moment Formula. In Fig. 51 let c be the distance of the most remote fiber from the neutral axis, f the tensile or compressive unit-stress developed on that fiber. The unit-stress on a fiber at a distance y from the neutral axis is

$$f_y = \frac{y}{c} f.$$

The total stress coming on a small area a is $a \frac{y}{c} f$. The moment of this stress about the neutral axis is

$$a \frac{y}{c} f y = \frac{f}{c} a y^2.$$

To get the moment of all the stresses developed in the section all such expressions must be summed up, giving the resisting moment equal to

$$\Sigma \frac{f}{c} a y^2 \quad \text{or} \quad \frac{f}{c} \Sigma a y^2.$$

The expression $\Sigma a y^2$ is called the moment of inertia of the cross section for the neutral axis and is denoted by I. (See Appendix A.) By substituting I for $\Sigma a y^2$ the resisting moment becomes $\frac{fI}{c}$, and as equilibrium exists the bending moment M equals the resisting moment $\frac{fI}{c}$; therefore,

$$M = \frac{fI}{c}. \qquad (2)$$

In the moment formula f is the maximum tensile or compressive unit-stress existing in the section for which the bending moment is M. This stress is developed in the fiber most remote from the neutral axis, which is at the distance c from that axis. A stress obtained by the use of the moment formula is called a **fiber stress.** The quantity $\frac{I}{c}$ depends upon the size and shape of the cross section of the beam and is called the **section factor** or the **section modulus.**

70. UNITS. The unit of f depends upon those of M and $\frac{I}{c}$. If M be in ton-inches, f is in tons per square inch. If M is in pound-inches, f is in pounds per square inch. The unit of s depends upon those of V and A. The units used will be pounds per square inch for stresses, pounds for loads, square inches for areas, pound-inches for moments, $(\text{inches})^4$ for moments of inertia, $(\text{inches})^3$ for section factors, and inches for distances.

ILLUSTRATIVE EXAMPLE

As an illustration of the application of this formula let it be required to determine the maximum fiber stress developed at a certain section of a beam where $M = 115{,}200$ pound-inches as calculated by the principles of Chapter V. $I = 21.8$ inches^4 and $c = 3$ inches as calculated by the methods of Appendix A. Then

$$f = \frac{Mc}{I} = \frac{115{,}200 \times 3}{21.8} = 15{,}850 \text{ lb. per sq. in.}$$

71. TOTAL HORIZONTAL COMPRESSIVE AND TENSILE STRESSES. From Art. 68, the stress on an element of area a at the distance y from the neutral axis is

$$f_y a = \frac{y}{c} fa.$$

To obtain H_c, which is the resultant of the compressive stresses, a summation of the stresses on the area in compression must be made. Therefore,

$$H_c = \frac{f}{c} \Sigma\, ay,$$

$$H_c = \frac{f}{c} \bar{y}' A',$$

in which $\bar{y}'$ is the distance from the neutral axis to the centroid of the area A' which is in compression.

A similar expression is obtained for the resultant of the tensile stresses which is

$$H_t = \frac{f}{c}\bar{y}_1'A_1'.$$

If the bending moment is positive the compression area is above the neutral axis. In the formula

$$H_c = \frac{f}{c}\bar{y}'A',$$

$\frac{f}{c}\bar{y}'$ is the unit-stress on a fiber at the centroid of the area A'. Therefore, **the resultant of the horizontal compressive stresses equals the stress on the centroid of the area in compression multiplied by the area in compression.** There is a similar principle for tension. This principle is general and can be applied to any portion of a cross section of a beam when the stresses are below the elastic limit.

The line of action of the resultant of the compressive or of the tensile stresses is obtained by the principle of moments for a system of forces, which is, **the moment**

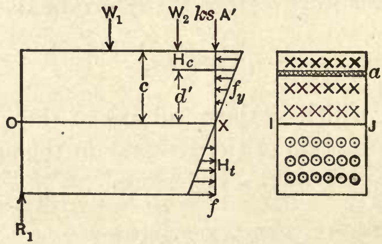


FIG. 52.

of the resultant of a system of forces about any axis is equal to the summation of the moments of all the forces about the same axis. Let d' be the distance from the neutral axis to the resultant of the compressive stresses, then (see Fig. 52),

$$H_c d' = \sum \frac{y}{c} fay,$$

$$\frac{f}{c}\bar{y}'A'd' = \frac{f}{c}\Sigma ay^2,$$

$$\therefore \quad d' = \frac{\Sigma ay^2}{\bar{y}'A'} = \frac{I'}{G'}$$

as $H_c = \frac{f}{c}\bar{y}'a'$, and I' is the moment of inertia of the area in compression about the neutral axis, and G' is the moment of that area about the same axis. A similar expression is obtained for the distance to the center of tension.

As H_c and H_t are equal they produce a couple. The arm between the center of compression and the center of tension multiplied by H_c or H_t equals the resisting moment.

For a rectangular beam of depth d, $\frac{I'}{G'} = \frac{d}{3}$. The distance between the center of compression and the center of tension, then, is $\frac{2}{3}d$.

72. THE THREE PROBLEMS. In each one of the two fundamental beam formulas, the shear formula $V = ksA$, and the moment formula $M = \frac{fI}{c}$, there are three variables: V, s, and A in the former and M, f, and $\frac{I}{c}$ in the latter. Any one of the three variables in each equation may be determined if the other two are known. This gives rise to three problems that may be investigated by the use of the shear and moment formulas.

PROBLEM I. **Investigation of Beams.** Given a beam with its load, to calculate the maximum unit-stresses.

By the principles developed in Chapter V the values of the maximum vertical shear and the bending moment may be calculated. A and $\frac{I}{c}$ may be determined from the dimensions of the cross section of the beam, by the methods of Appendix A. To obtain the maximum shearing stress, $s = \frac{V}{kA}$ may be used, and to obtain the fiber stress, $f = \frac{Mc}{I}$ may be used.

ILLUSTRATIVE EXAMPLE

Calculate the maximum shearing stress and the maximum fiber stress developed in a longleaf pine beam of 10-ft. span, breadth 4 inches, depth 8 inches, when carrying a concentrated load of 12,000 pounds 4 feet from the left support.

$$R_1 = 720 \text{ pounds}, \quad V_m = 720 \text{ pounds},$$
$$S_m = \tfrac{3}{2}(720 \div 32) = 33.7 \text{ pounds per square inch},$$
$$M_m = 720 \times 48 = 34{,}560 \text{ pound-inches},$$
$$I = \frac{4 \times 8 \times 8 \times 8}{12} = \frac{512}{3} \text{ inches}^4,$$
$$c = 4 \text{ inches},$$
$$\therefore f = \frac{34{,}560 \times 4 \times 3}{512} = 810 \text{ pounds per square inch.}$$

PROBLEM II. **Safe Loads for Beams.** By the use of the shear and moment formulas, the load which a given beam will safely carry may be obtained. $M = \frac{fI}{c}$ gives the value of the maximum allowable bending moment, from which values the load may be selected. After determining the load by use of the moment formula, the beam should be investigated for the maximum shearing stress developed by that load by use of the shear formula $V = ksA$. The allowable shearing stress or bending moment as calculated should not be exceeded.

Illustrative Example

What uniform load will a 10-inch, 25-pound I-beam carry when used as a simple beam of 16-ft. span with an allowable fiber stress of 16,000 pounds per square inch? From Table 20 which gives values of section factors of I-sections, $\frac{I}{c} = 24.4$ inches³. The maximum allowable resisting moment is $16{,}000 \times 24.4 = 390{,}400$ pound-inches. The maximum bending moment for the uniformly distributed load on a simple beam occurs at the center and is $\frac{Wl}{8}$.

$$\therefore \quad \frac{W \times 192}{8} = 390{,}400,$$

$$W = 16{,}270 \text{ pounds.}$$

To get the approximate shearing stress, the area of the web is $10 \times 0.31 = 3.1$ square inches. $\therefore\ s = (16{,}270 \div 2) \div 3.1 = 2630$ pounds per square inch, which is safe.

Problem III. **The Design of Beams.** The loading of a beam and the maximum allowable stress may be specified, to select or design a beam to carry the load safely. The design of beams is the problem most generally met with by the engineer and architect. It admits of many solutions, and the designer must use his best judgment in choosing the form and size of the cross section to be used. The material most remote from the neutral axis is all that is stressed to the maximum, while that at the neutral axis has no fiber stress in it. The material is most efficiently used when the largest proportion of it is stressed nearly to the maximum stress, and obviously this condition exists in a section having a large part of the material well away from the neutral axis. Necessarily there must also be such a distribution of the material as will insure safety against shear, buckling, and twisting. In steel I-beams there is a large portion of the material near the outside fiber, and yet the web is large enough to resist the shear.

Generally, rupture is due to bending rather than to shear, and occurs at the danger section. In the determination of the safe loads for beams and the design of beams, the moment formula $M = \frac{fI}{c}$ is the governing formula, the shear formula $s = \frac{V}{kA}$ being used afterward to investigate the beam for the shearing unit-stress. If the shearing unit-stress developed is too large another design must be made, but this is seldom necessary, except for short deep beams.

ILLUSTRATIVE EXAMPLE

Design a square loblolly pine cantilever beam for a span of 8 feet with a concentrated load of 500 pounds at the free end. From the table of allowable stresses (No. 8) f = 1000 pounds per square inch. $M_m = - 500 \times 96 = - 48{,}000$ pound-inches. (The negative sign may be neglected, as that simply means that the stress in the top fibers is tension.) If b is the breadth of the section,

$$\frac{I}{c} = \frac{b^3}{6},$$

$$\therefore \quad 48{,}000 = 1000 \times \frac{b^3}{6},$$

$$b^3 = 288,$$

$$b = 6.61 \text{ inches.}$$

The maximum shearing unit-stress for this load is

$$s = \frac{3\,V}{2\,A} = \frac{3}{2} \times \frac{500}{43.7} = 17.2 \text{ lb. per sq. in.,}$$

which is safe. While the above beam satisfies the condition of the problem it is not a standard section and probably would be replaced by a 6-inch by 8-inch beam, for which case the fiber stress is

$$f = \frac{48{,}000 \times 4 \times 12}{6 \times 8 \times 8 \times 8} = 750 \text{ lb. per sq. in.}$$

The maximum shearing stress is

$$s = \frac{3}{2} \times \frac{500}{48} = 15.6 \text{ lb. per sq. in.}$$

73. MODULUS OF RUPTURE. The moment formula $M = \frac{fI}{c}$ is not applicable to beams of material for which the stress is not proportional to the deformation, or for non-homogeneous beams, or for beams under stresses greater than the elastic limit of the material. However, it is frequently used to calculate a nominal unit-stress developed in a beam when the bending moment is great enough to cause failure. The unit-stress thus calculated is called the **modulus of rupture.** This usually lies between the ultimate compressive strength and the ultimate tensile strength of the material. Table 13 gives average values for the modulus of rupture.

TABLE 13

MODULUS OF RUPTURE

Material.	Modulus of rupture lb. per sq. in.
Timber................	7000 to 9000
Cast iron..............	35,000

74. MAXIMUM STRESS DIAGRAMS. The value of the maximum shearing unit-stress for a section can be obtained by dividing the vertical shear for the section by kA, the product of the sectional area and a constant k depending upon the shape of the section. The value of the maximum fiber stress for the section can be obtained by dividing the bending moment at that section by $\frac{I}{c}$, the section modulus. If the shear and moment diagrams are drawn, the values of V and M may be taken directly from the diagrams; thus, in Fig. 53 (*b*), CD divided by kA (assumed constant) gives the shearing unit-stress. The stress is always proportional to the ordinate CD. C_1D_1 in Fig. 53 (c) is the bending moment at the section which divided by $\frac{I}{c}$ gives the maximum fiber stress, which comes in the extreme fiber of the beam at that section. The fiber stress is pro-

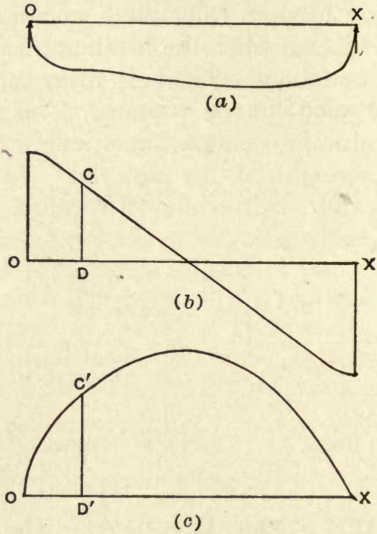


FIG. 53.

portional to the ordinate of the moment curve. Thus for a beam of uniform section the ordinates to the vertical shear curve represent the maximum shearing stresses, and the ordinates to the bending moment curve represent the maximum fiber stresses existing in the beam. Consequently the shear and moment diagrams may be considered stress diagrams. The above reasoning is for beams of constant section, but with modifications similar reasoning may be applied to beams in which the section is not uniform.

75. Beams of Uniform Strength. If a beam is of uniform section the maximum fiber stress occurs only in the outside fiber at the section of the greatest bending moment. The stress varies with the bending moment along the length of the beam. In order to have the most efficient beam all the material in it should be stressed to the allowable stress, and to approach this state, besides keeping the material near the outside surface, the cross section is sometimes made to vary with the bending moment. This is done in plate girders where extra cover plates are added toward the center of the span.

Examples

1. If a 4-inch by 6-inch by 0.4-inch channel is used as a simple beam of 8-ft. span with a concentrated load of 2000 pounds three feet from the left support, (a) what is the maximum fiber stress developed? (b) What is the stress developed on a fiber 2 inches from the top and 2 feet from the right support?

$$R_1 = 1250 \text{ pounds.}$$

(a) The maximum moment is under the load and is

$$M_m = 1250 \times 3 \times 12 = 45{,}000 \text{ pound-inches.}$$

The centroid is 1.29 inches from the back (see Fig. A_5, Appendix A). Therefore

$$c = 4 - 1.29 = 2.71 \text{ inches,}$$
$$A = 5.28 \text{ square inches,}$$
$$I = 8.41 \text{ inches}^4,$$
$$f = \frac{Mc}{I} = \frac{45{,}000 \times 2.71}{8.41} = 14{,}500 \text{ lb. per sq. in.}$$

(b) $M_{72} = 1250 \times 8 \times 12 - 2000 \times 3 \times 12 = 18{,}000 \text{ lb.-in.}$

$$y = 2.71 - 2 = .71 \text{ inch.}$$

$$f_y = \frac{18{,}000 \times 0.71}{8.41} = 1520 \text{ lb. per sq. in.}$$

2. What uniform load will a 4-inch by 6-inch yellow pine timber safely carry when used as a simple beam of 10-ft. span?

$$\frac{I}{c} = \frac{\frac{bd^3}{12}}{\frac{d}{2}} = \frac{bd^2}{6} = \frac{1 \times 6 \times 6}{6} = 24 \text{ inches}^3.$$

The allowable stress is 1000 pounds per square inch. (Table 8),

$$\frac{fI}{c} = 1000 \times 24 = 24{,}000 \text{ pound-inches},$$

which is the allowable resisting moment.

The maximum bending moment is

$$\frac{Wl}{8} = \frac{W \times 10 \times 12}{8} = 24{,}000,$$

$$W = 1600 \text{ pounds}.$$

The load per lineal foot is $\frac{1600}{10}$ = 160 pounds per foot. Investigating for the shearing stress, $s = \frac{3}{2} \times \frac{800}{24}$ = 50 pounds per square inch.

3. Using the results of Example No. 2, what is the total compressive stress at a section 30 inches from the left end, and where is the line of action of the resultant?

$$R_1 = 800 \text{ pounds}.$$
$$M_{30} = 800 \times 30 - \frac{160}{12} \times 30 \times 15 = 18{,}000 \text{ lb.-in.}$$

The area in compression is 12 square inches. The stress on the centroid of this area is

$$f_y = \frac{18{,}000 \times 1.5}{72} = 375 \text{ lb. per sq. in.}$$

The total compressive stress is $12 \times 375 = 4500$ pounds. The line of action of the resultant is $\frac{1}{6}$ the depth from the top or 1 inch from the top or 2 inches above the neutral surface.

4. Design a simple cypress timber beam to carry a uniform load of 8000 pounds on a span of 12 feet.

$$M_m = \frac{Wl}{8} = \frac{8000 \times 144}{8} = 144{,}000 \text{ lb.-in.}$$

The beam must be large enough to take this moment without exceeding the allowable stress which is 1000 pounds per square inch. (Table 8),

$$\frac{M}{f} = \frac{144{,}000}{1000} = 144 \text{ inches}^3 = \frac{I}{c}.$$

An indefinite number of cross sections will satisfy this, but the one chosen should be economical and suited for the purpose. It must be wide enough to prevent lateral bending. If b is assumed equal to $\frac{d}{2}$ then

$$\frac{I}{c} = \frac{bd^2}{6} = \frac{d^3}{12} = 144, \; d^3 = 1728, \; d = 12 \text{ inches, and } b = 6 \text{ inches.}$$

The maximum shearing stress is

$$s = \tfrac{3}{2} \times \tfrac{4000}{72} = 83\tfrac{1}{3} \text{ lb. per sq. in.,}$$

which is safe. Other dimensions to the nearest even inch above the actual size required are 2 inches by 18 inches; 4 inches by 16 inches; 10 inches by 10 inches; or three, 3 inches by 10 inches. The beam 2 inches by 18 inches would be too narrow and deep unless well braced laterally. The one 4 inches by 16 inches might be chosen in some cases if well braced laterally. The one 10 inches by 10 inches would not be economical unless the two inches vertical distance saved would be more valuable than the extra material in the beam of this size.

PROBLEMS

1. If a 15-inch, 42-pound I-beam carries a uniformly distributed load of 39,270 pounds on a span of 16 feet, (a) What is the maximum shearing stress? (b) Draw the load, shear, and moment diagrams. (c) Calculate the maximum fiber stress. (d) What is the resultant of the horizontal compressive or tensile

stresses at the danger section? (e) What is the rate of change of the vertical shear at any section? (f) What is the rate of change of the bending moment at the left support, at the quarter point, and at the center of the span?

Ans. (c) 16,000 lb. per sq. in.

2. (a) What uniform load will a simple rectangular Washington fir beam of breadth 8 inches and depth 12 inches carry on a span of 12 feet? (b) With the calculated load on the beam what is the maximum shearing stress? (c) What is the value of the fiber stress 4 inches from the top of the beam and 4 feet from the right support? (d) What is the maximum shearing stress on that section? (e) What is the resultant of the horizontal tensile stresses at that section? (f) Where is the line of action of the resultant of the tensile stresses?

3. In a table giving the safe load in pounds uniformly distributed for rectangular beams of white pine, cedar, and spruce for each inch in thickness the following values are given: span 10 feet, depth 14 inches, load 1524 pounds; span 16 feet, depth 21 inches, load 2144 pounds; span 25 feet, depth 22 inches, load 1255 pounds. What unit-stress was allowed in compiling that table?

Ans. 700 lb. per sq. in.

4. With an allowable unit-stress of 1100 pounds per square inch what will be the allowable uniform loads per inch thickness of the beam for the following spans and depths? Span 8 feet, depth 12 inches; span 11 feet, depth 14 inches; span 20 feet, depth 24 inches.

5. Solve Problem No. 4 if the allowable stress is 1200 pounds per square inch. *Ans.* 2376 lb.

6. A 12-inch, 40-pound I-beam of a span of 20 feet is used to carry a uniform load of 500 pounds per foot and a concentrated load of 5000 pounds 4 feet from the left end. What is the maximum stress developed?

7. Compute the maximum unit-stress in a 2×8 inch joist carrying loads of 240 pounds 3 feet from the left end and of 360 pounds 4 feet from the right end of a simple span of 12 feet.

8. Determine maximum fiber stress in a 6-inch by 12-inch simple beam of 12-ft. span which carries a uniform load of

100 pounds per foot and three concentrated loads of 1300 pounds, 1500 pounds, and 1000 pounds at 3 feet, 5 feet, and 8 feet respectively from the left support.

9. Compute the maximum fiber stress in a 15-inch, 42-pound I-beam, over a simple span of 30 feet carrying a uniform load of 500 pounds per foot and two concentrated loads of 5000 pounds and 10,000 pounds at 3 feet and 23 feet respectively from the left support.

10. A rectangular, cantilever timber beam of 12-ft. span, 4 inches broad, and 8 inches deep carries a uniform load of 50 pounds per lineal foot. Find the maximum fiber stress.

11. A cantilever white cedar beam of 5-ft. span has a rectangular section 2 inches broad and 3 inches deep. Find the total uniform load it can safely carry.

12. Find the uniform load per lineal foot which a wooden cantilever beam 6 feet in length, of rectangular section 2 inches broad and 3 inches deep, can carry with a maximum fiber stress of 800 pounds per square inch.

13. A 15-inch, 42-pound I-beam is carrying a total uniform load of 30,000 pounds on a simple span of 20 feet. Compute the intensity of stress at a point 3 inches below the top flange face and 6 feet from the left support.

14. Determine the total amount of horizontal compressive stress at the section of maximum bending moment in a 6-inch by 12-inch wooden beam carrying a uniform load of 4000 pounds per foot on a simple span of 12 feet.

15. Design a cypress beam of 18-ft. span to carry a load that varies uniformly from zero at the ends to a maximum of 1800 pounds per foot at the center.

16. Select the proper I-beam 18 feet long which overhangs both supports 3 feet that will carry concentrated loads of 5000 pounds at the left end, 10,000 pounds at the center, and 8000 pounds at the right end.

17. Determine the maximum fiber stress at the sections indicated in an 8-inch by 12-inch simple beam of 12-ft. span which carries a uniform load of 800 pounds per foot: (a) 2 feet from the supports, (b) at the quarter points, (c) at the center.

Ans. 906 pounds per square inch.

18. A 6-inch by 12-inch cantilever beam of 9-ft. span carries a load of 500 pounds per foot. Calculate the fiber stress (a) 2 inches from the top at the wall, (b) on the bottom fiber at the wall, (c) on the top fiber at the middle. *Ans.* 1687 lb. per sq. in.

19. A 2-inch by 4-inch maple timber 10 feet long is to be used as a simple beam. What central load will it safely carry (a) when it is laid flat, and (b) when placed on edge? What is to be learned from the results?

20. What should be the depth of a rectangular shortleaf pine beam of 18-ft. span and 4 inches broad to sustain a uniformly distributed load of 1000 pounds?

21. If a 4-inch by 4-inch timber is to carry a load of 50 pounds per foot what will be the length as a simple beam to give the maximum fiber stress of 1200 pounds per square inch?

22. Design a simple yellow pine beam of 12-ft. span carrying concentrated loads of 1, 2, and 3 tons at distances of 3, 6, and 7 feet respectively from the left support, and a uniform load of $\frac{3}{4}$ ton.

23. Design a rectangular cast iron cantilever beam to carry a load of 3000 pounds at the end of a 4-ft. span.

24. A rectangular hemlock cantilever beam 8 feet long and 6 inches deep is to support a load of 200 pounds at the free end. What should be its width if the weight of the beam is neglected?

25. A load of 500 pounds is rolled over a simple beam of 20-ft. span. Find the position of this load for the maximum bending moment, compute its value, and design a longleaf pine beam to take the load.

26. A round pin carries a load of 10,000 pounds at the center. It may be considered as a simple beam of 6-in. span. Find the diameter of the pin if the fiber stress is not to exceed 15,000 pounds per square inch or the shear to exceed 7500 pounds per square inch.

27. Two loads 4000 pounds and 2000 pounds 6 feet apart roll over a simple beam 12 feet long. Find the position of the loads for the maximum moment and determine its value. Design a shortleaf pine beam to carry this load.

28. Four loads, 1000, 2000, 3000, and 4000 pounds and spaced 2, 3, and 5 feet respectively, roll over a simple beam of 16-ft. span. Determine position for maximum bending moment, and determine its value. Design a loblolly pine beam to carry this load.

The following problems are to be solved by the use of a steel company's handbook.

Find the missing terms by several methods (loads do not include weight of beam). $f = 16{,}000$ pounds per square inch.

No.	Length, feet.	Uniform load, lb. per ft.	First con. load, lb.	Dist. to first con. load, ft.	Second con. load, lb.	Dist. to second con. load, ft.	I-beam.
			Simple Beams				
29.......	10	2,000	0	0	0	0	?
30.......	12	?	0	0	0	0	10″ 25#
31.......	14	0	28,000	7	0	0	?
32.......	14	0	?	7	0	0	9″ 21#
33.......	16	1,000	10,000	8	0	0	?
34.......	16	?	20,000	8	0	0	15″ 90#
35.......	16	2,000	?	8	0	0	15″ 85#
36.......	18	0	12,000	4	7,000	9	?
37.......	18	0	9,000	3	?	9	7″ 20#
38.......	20	1,000	20,000	5	0	0	?
39.......	20	1,000	?	6	0	0	12″ 31#
40.......	24	?	12,000	8	0	0	15″ 80#
41.......	20	1,000	5,000	6	6,000	10	?
42.......	20	?	4,000	8	5,000	12	12″ 100#
			Cantilever Beams				
43.......	8	4,000	0	0	0	0	?
44.......	10	?	0	0	0	0	10″ 25#
45.......	12	0	10,000	12	0	0	?
46.......	12	2,000	10,000	12	0	0	?

CHAPTER VII

STRESSES IN SUCH STRUCTURES AS CHIMNEYS, DAMS, WALLS, AND PIERS

76. Kinds of Stresses. For structures that sustain a side thrust and a direct weight, as chimneys, dams, etc., there is a combination of direct and flexural stresses. The treatment given in this chapter is based upon the assumption that the direct and flexural stresses act independently of each other, and that the side thrust does not cause appreciable deflection. It is also assumed that the stress is proportional to the deformation and that the material of the structure is elastic. For bearing on soil this assumption may be only approximately true.

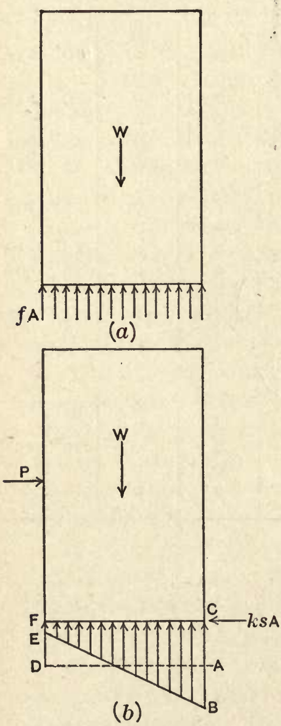


Fig. 54.

When there is no side thrust and the vertical section is symmetrical, the total weight above a horizontal section is resisted by direct compressive stresses on the sections and the unit-stress is $f = \frac{W}{A}$ (Fig. 54 (*a*)). A normal force *P* produces flexural stresses. A shearing stress of $s = \frac{P}{kA}$ acts along the section. The bending moment due to the load increases the compression on the opposite side from the force and decreases it on the nearer side (Fig. 54 (*b*)). This lateral pressure P may be due to the wind, water, embankments, etc. It may not be horizontal, in which case the horizontal and vertical components of the resultant of the weights and lateral forces should be taken as producing the flexural stresses and the direct stresses respectively.

77. ECCENTRIC LOADS ON SHORT PRISMS. Let the load W have the eccentricity e (Fig. 55 (*a*)). This load may be replaced by its components, W, W_1, and W_2 shown in Fig. 55 (*b*), W_1 and W_2 being taken as acting along the axis of the prism and equal in magnitude to W. The two equal and opposite forces W and W_2 form a couple, the moment of which is We. The effect of this couple is to produce a bending or moment stress. In Fig. 55 (*c*) W and W_2 are replaced by the equivalent moment C.

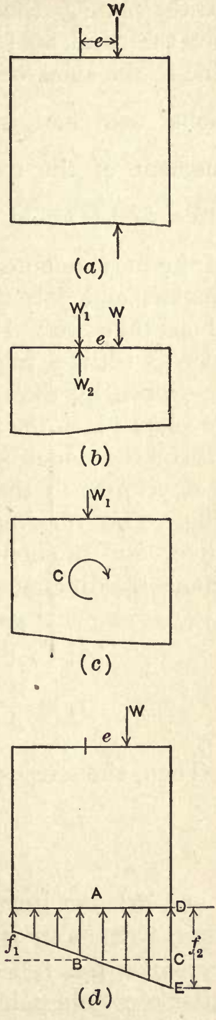


FIG. 55.

The stresses developed by the eccentric load W will be the same as those produced by its components which are the axial load W and the couple We. The stress due to the axial load is the same at all points of the sectional area and is $\frac{W}{A}$. The stress developed by the moment of the couple may be considered as a fiber stress and is equal to $\frac{Mc}{I} = \frac{Wec}{I}$, where c is the distance of the most remote fiber from the centroidal axis of the section, and I is the moment of inertia of the section about that axis. If r is the radius of gyration, $I = Ar^2$. Fig. 55 (d) is a free-body diagram showing the stresses developed by the eccentric load. The maximum stress developed is in the fiber most remote from the centroidal axis on the side nearer the load while the minimum stress is developed in the most remote fiber on the opposite side. The maximum stress equals the sum of the direct and moment stresses, and the minimum stress equals the difference. Therefore,

$$f_2 = \frac{W}{A} + \frac{Wec}{I} = \frac{W}{A}\left(1 + \frac{ec}{r^2}\right),$$

$$f_1 = \frac{W}{A} - \frac{Wec}{I} = \frac{W}{A}\left(1 - \frac{ec}{r^2}\right).$$

Then, the stress developed in the outside fiber is

$$f = \frac{W}{A}\left(1 \pm \frac{ec}{r^2}\right).$$

78. **Eccentricity of a Load that Will Produce Zero Stress in the Outside Fiber.** If the eccentricity is increased f_2 becomes greater and f_1 becomes smaller. After a certain point is passed f_1 reverses if the material will take tension. If W is a compression load, just before the tensile stresses act, f_1 becomes zero. To

obtain the eccentricity e_1 that will make f_1 zero, equate f_1 to zero. Then

$$f_1 = \frac{W}{A}\left(1 - \frac{e_1 c}{r^2}\right) = 0,$$

$$e_1 = \frac{r^2}{c},$$

where e_1 is the greatest eccentricity the load may have before tension is produced on the side away from the compression load.

For a rectangle the eccentricity to give zero stress in the outside fiber will now be found,

$$r^2 = \frac{I}{A} = \frac{bd^3}{12\,bd} = \frac{d^2}{12} \quad \text{and} \quad c = \frac{d}{2},$$

$$\therefore \quad e_1 = \frac{\frac{d^2}{12}}{\frac{d}{2}} = \frac{d}{6}.$$

Therefore, as long as a compression load is kept on the middle third of a solid rectangular prism the stress over the entire area will be compression.

For a circle,

$$r^2 = \frac{\frac{\pi d^4}{64}}{\frac{\pi d^2}{4}} = \frac{d^2}{16} \quad \text{and} \quad c = \frac{d}{2},$$

$$\therefore \quad e_1 = \frac{\frac{d^2}{16}}{\frac{d}{2}} = \frac{d}{8}.$$

Therefore, as long as the load is kept on the middle quarter of a solid circular prism all the stress will be compression.

79. THE KERN. If the line of action of the load falls within a certain part of the cross section of a prism, all stresses in the section will be of one kind; and if the line of action of the load falls outside of that area the stresses will be partly tension and partly compression. This area is called the **kern** or **kernel.** In a solid circular prism the kern is a circular area whose diameter is one-fourth the diameter of the prism. And for a rectangular prism the kern is a diamond-shaped figure whose diagonals are one-third the lateral dimensions of the prism. (See Fig. 56.)

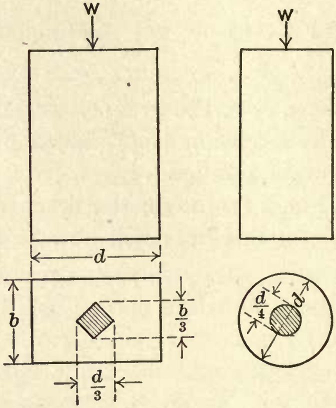


FIG. 56.

80. CASE OF ECCENTRIC LOADS CAUSED BY A COMBINATION OF THE WEIGHT OF THE MATERIAL AND LATERAL PRESSURE. Call the weight of the material above the section *AB* being considered, *W*, and call the lateral pressure on the prism above the section *P*, Fig. 57. The magnitude and direction of the resultant, *R*, of these two forces depend upon the forces. Its line of action passes through the intersection of *W* and *P* and intersects the section *AB* in some point *C* usually not at the centroid, thus producing an eccentric load on the section *AB*. The eccentricity *e* is *DC* and can be calculated by taking moments about the centroidal axis at *D*. Resolve *R* into its vertical and horizontal components, *Y* and *H*, at *C* where it intersects the cross section for which the stresses are to be found. (Fig. 57 (*b*)).

H produces a shearing unit-stress along the plane of magnitude $s = \frac{H}{kA}$, and Y is an eccentric load producing the stress

$$f = \frac{Y}{A}\left(1 \pm \frac{ec}{r^2}\right)$$

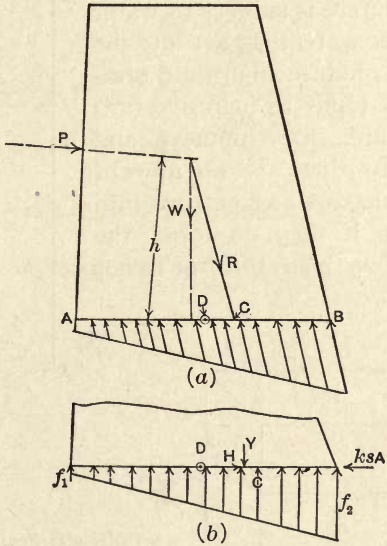


FIG. 57.

in which the plus sign is used to obtain the maximum compressive stress, f_2, and the minus sign is used to obtain the minimum stress, f_1.

81. EFFECT WHEN THE LINE OF ACTION OF THE RESULTANT FALLS OUTSIDE OF THE KERN. If the resultant of all loads above a section of a prism under compression falls outside the kern, the minimum stress

$f_1 = \frac{Y}{A}\left(1 - \frac{ec}{r^2}\right)$ becomes negative or tension if the material will take tension, Fig. 58. For materials that will not take tension, such as masonry, the joints on the side opposite the eccentric load will tend to open, but failure will not necessarily follow. If the structure is subject to water pressure the water may get into the cracks and produce an upward pressure which tends to help overturn the structure. In chimneys and walls where there is no upward pressure due to water getting into the cracks if they do form, the tendency for them to form is not so objectionable. The safe limit for the compressive stress should not be exceeded by the maximum stress developed.

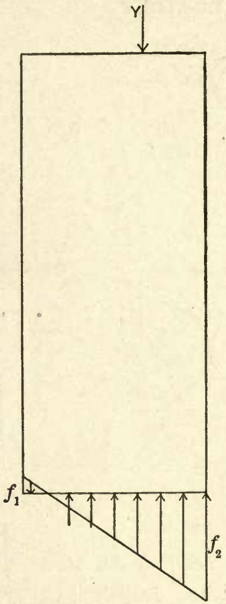


FIG. 58.

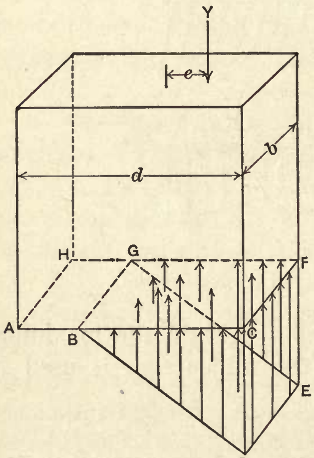


FIG. 59.

82. THE MAXIMUM STRESS WHEN THE LINE OF ACTION OF THE RESULTANT FALLS OUTSIDE THE MIDDLE THIRD FOR RECTANGULAR PRISMS WHICH TAKE NO TENSION. The stresses will be distributed as shown in Fig. 59 forming a wedge-shaped prismatic stress volume *BCDEFG*, there being no stress on the area *ABGH*. The vertical component, Y, of the resultant equals the summation of the vertical resisting stresses. The line of action of Y passes through the centroid of the prism which is $\frac{1}{3}(CB)$ from CF. If b is the breadth CF, the area over which the resisting stress is distributed is

$$A' = 3\left(\frac{d}{2} - e\right)b.$$

If d is the depth AC, and e the eccentricity, Y is at the distance $\frac{d}{2} - e$ from the edge CF. The length BC is $3\left(\frac{d}{2} - e\right)$. If f is the maximum compressive stress, the average on the stressed area is $\frac{f}{2}$ and the total resisting stress is

$$\frac{fA'}{2} = \frac{f}{2}3\left(\frac{d}{2} - e\right)b = \frac{3}{4}fb(d - 2e),$$

$$\therefore \quad Y = \tfrac{3}{4}fb(d - 2e),$$

$$f = \frac{4Y}{3b(d - 2e)}.$$

EXAMPLES

1. Find the maximum stress at the foot of a stone wall 20 feet high and 4 feet thick when there is a wind pressure of 35 pounds per square foot; also when there is a wind pressure of 45 pounds per square foot if the masonry weighs 150 pounds per cubic foot.

Consider a portion of the wall l feet long,

$$Y = W = 150 \times 20 \times 4 \times l = 12{,}000 \times l \text{ pounds},$$

$$H = P = 35 \times 20 \times l = 700 \times l \text{ pounds},$$

$$s = \frac{3 \times 700 \times l}{2 \times 4 \times 144 \times l} = 1.8 \text{ pounds per square inch.}$$

The eccentricity is found by taking moments about a centroidal axis of the base.

$$e = \frac{7000 \times l \times 10}{12{,}000 \times l} = \frac{7}{12} \text{ foot. This is in the middle third,}$$

$$\therefore \quad f_2 = \frac{Y}{A} + \frac{Yec}{I} = \frac{12{,}000 \times l}{48 \times 12 \times l} + \frac{12{,}000\, l\, 7 \times 12 \times 24 \times 12}{12 \times 12\, l \times 48 \times 48 \times 48}$$

$$= 21 + 18 = 39 \text{ pounds per square inch.}$$

For second part,

$$Y = 12{,}000 \times l; \quad H = 900 \times l; \quad e = \frac{900 \times 10}{12{,}000} = \frac{9}{12} \text{ foot.}$$

This is outside the middle third.

$$3\left(\frac{d}{2} - e\right) = 3\,(24 - 9) = 45 \text{ inches.}$$

$$\therefore \quad A' = 45 \times 12 \times l = 540 \times l \text{ square inches.}$$

$$f = \frac{2\,Y}{A'} = \frac{2 \times 12{,}000 \times l}{540 \times l} = 44 \text{ pounds per square inch.}$$

2. What should be the thickness of a rectangular wall 15 feet high to resist a wind pressure of 40 pounds per square foot without any tension in the windward side, if the material weighs 140 pounds per cubic foot?

Let d be the thickness.

The weight of each lineal foot is $W = 15 \times 1 \times 140\, d = 2100\, d$ pounds.

The wind pressure for each lineal foot is $H = 15 \times 1 \times 40 = 600$ pounds,

$$\therefore \quad e_1 = \frac{600 \times 7.5}{2100\, d} = \frac{15}{7\, d}.$$

For zero stress $e_1 = \frac{d}{6}$.

$$\therefore \quad \frac{d}{6} = \frac{15}{7\, d}; \quad d^2 = \frac{90}{7} = 12.86,$$

$$d = 3.59 \text{ feet.}$$

PROBLEMS

1. Find the maximum and minimum unit-stress in a rod 2 inches in diameter under a tension load of 16,000 pounds if it is applied at a point $\frac{1}{3}$ inch from the center of the cross section.

2. What are the dimensions of the kern in a rectangle 3 inches by 8 inches? In a hollow circular chimney of inner diameter 8 feet and outer diameter 10 feet?

3. In a brick wall 20 feet high and 4 feet thick, weighing 115 pounds per cubic foot, (a) What horizontal wind pressure will cause zero stress on the windward side of the base? (b) With that wind pressure what will be the maximum stress? (c) If the wind pressure is 40 pounds per square foot what will be the maximum and minimum stresses? (d) With a wind pressure of 40 pounds per square foot what will be the stress on the windward side 10 feet above the ground?

4. A square compression piece 9 × 9 inches carries an eccentric load of 16,200 pounds so applied that the stress on one edge equals 0. Determine the application point of the resultant load.

5. What must be the thickness of a wall 25 feet high, weighing 120 pounds per cubic foot, if a maximum unit-stress of 47.2 pounds per square inch is developed when the wind pressure is 40 pounds per square foot? *Ans.* 4.5 feet.

6. Would a brick wall 30 feet high, weighing 120 pounds per cubic foot, 3 feet thick at the top and 4 feet thick at the base, with one side vertical, be safe if subject to a wind pressure of 40 pounds per square foot?

CHAPTER VIII

GRAPHIC INTEGRATION *

83. DEFINITIONS. In Chapter V the relations between the load, shear, and moment diagrams are given as follows: the difference between the ordinates at any two sections in the moment diagram represents the area in the shear diagram between the two sections, and the difference between the two ordinates of the shear diagram represents the area in the load diagram between the ordinates at the same sections; thus, in Fig. 60, *LN* in the moment diagram represents the area *EHIG* of the shear diagram, and *EF* in the shear diagram represents the area *ABCD* of the load diagram.

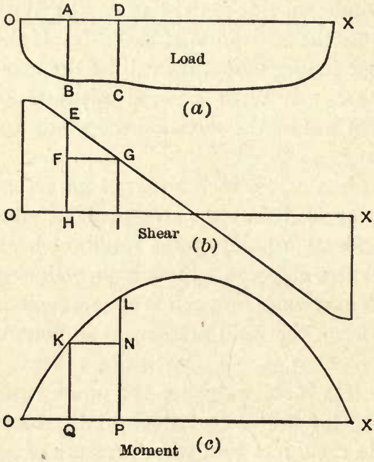


FIG. 60.

The **first integrated curve** is defined as one in which the ordinates represent the area under a given curve. Thus, the moment curve is the first integrated curve of the shear curve, and the shear curve is the first integrated curve of the load curve. Since the moment curve is the

* For students who have had integral calculus and who do not intend to follow the graphical method of determining deflections of beams, this chapter may be omitted.

first integrated curve of the first integrated curve of the load curve, it is called the **second integrated curve** of the load curve. **The second integrated curve** is one in which the ordinates represent areas under the first integrated curve. The integrated curve of the second

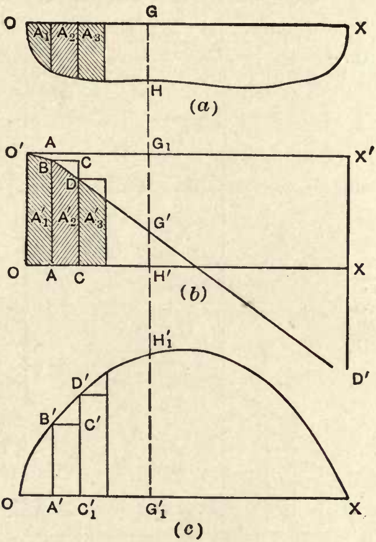


FIG. 61. FIRST METHOD OF GRAPHIC INTEGRATION

integrated curve is the **third integrated curve.** Similarly, the **nth integrated curve** is one in which ordinates represent the area in the $(n - 1)$th integrated curve. The graphical method of deriving the integrated curves from given ones will be deduced before applying them to the theory of beams.

84. THE FIRST METHOD OF OBTAINING THE SECOND INTEGRATED CURVE. In the following methods, if the given areas are not bounded by straight lines, the greater the number of points secured on the resulting curve, the more nearly accurate that curve will be. The equations representing the curves will be deduced by making the number of points secured infinite.

Let Fig. 61 (*a*) be the given curve of which it is desired to obtain the first and second integrated curves. The

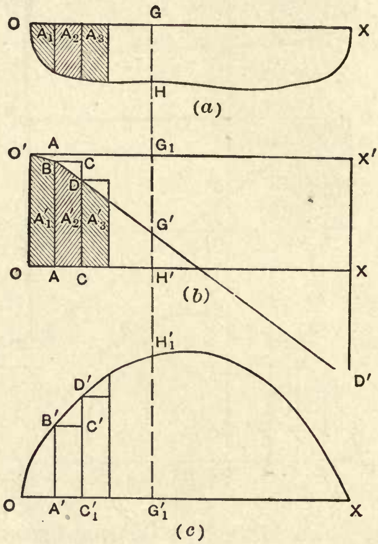


FIG. 61. FIRST METHOD OF GRAPHIC INTEGRATION.

curve is taken below the axis making the area between the curve and the axis negative, which corresponds to the load curves already described. To obtain the first integrated curve, divide the area into a number of parts as indicated, and measure each area by any method. Then from an arbitrarily chosen axis $O'X'$ in (*b*) lay off AB to a selected scale to represent the area A_1, then lay off CD to represent the area A_2. Continue this process for the entire area under curve (*a*), then connect by a continuous curve $O'G'D'$ all the points thus obtained. This is the first integrated curve. Any ordinate to this curve as G_1G' represents the total area accumulated from the left end to the section GH.

In physical problems there is always a constant as OO' to be added to the area under the curve. This constant, which is called the **constant of integration** is the value of the ordinate to the integrated curve at the origin. In the usual case it can be determined by the conditions of the problem. It frequently is zero. In the shear curve, the constant is the vertical shear at the left support as OO' in (*b*). In the moment curve it is the moment at the left support. After determining the value of this constant, draw the axis OX which is the true axis of reference for the integrated curve (*b*); the true value of the function represented in curve (*b*) is then $H'G'$ at the section GH. The area to be considered in the integrated curve is that between the curve and the axis OX.

To obtain the second integrated curve, divide the area between the curve $O'G'D'$ and the axis OX into small parts as indicated in Fig. 61 (*b*), and from some chosen axis OX in (*c*) erect, to some scale, the ordinate $A'B'$ equal to the area A_1' in (*b*), $C'D'$ in (*c*) equal to the area A_2' in (*b*), and so on until the entire area in (*b*) is covered, then the ordinate $G_1'H_1'$ represents the accumulated area in (*b*) from the origin to the section $G'H'$. The constant of integration will depend upon the conditions of the problem. In the illustration it is assumed to be zero.

Curve (*b*) is the first integrated curve of (*a*), and (*c*) is the first integrated curve of (*b*) and a second integrated curve of (*a*). As long as the constant can be determined, a higher integrated curve can be obtained by the foregoing method, the *n*th process giving the *n*th integrated curve.

85. THE SECOND METHOD OF OBTAINING THE SECOND INTEGRATED CURVE. The method given in the previous article cannot be employed unless the constants can be determined independently. For cases when the

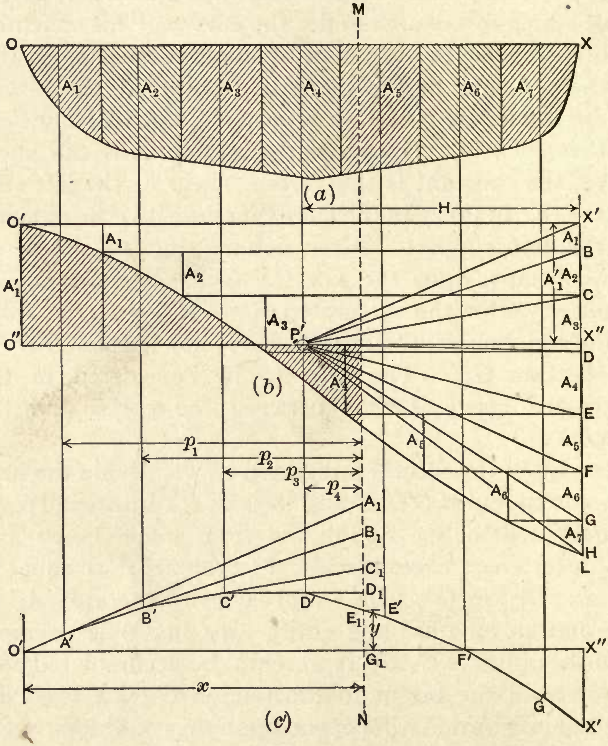


FIG. 62. SECOND METHOD OF GRAPHIC INTEGRATION.

constants cannot be determined a second method must be combined with the first. Draw the first integrated curve Fig. 62 (*b*) in the same manner as in the first method, using the arbitrary axis $O'X'$. Project the areas A_1, A_2, A_3, etc., in (b) to the vertical axis $X'L$ as indicated. If it is desired to obtain the integrated curve for (b) referred to any axis as $O''X''$, thereby assuming a constant of integration $O''O'$, take a **pole** P' on that axis a distance H from the axis $X'L$. H, which is called the **pole distance**, is measured in the same units and to the same scale as distances along the axis OX. Connect P' to the ends of the lengths representing the areas on the line $X'L$, as $P'X'$, $P'B$, $P'C$, etc. These lines are called **rays,** and the polygon $P'X'L$ is called the **ray** or **vector polygon.** A_1 is to be replaced by its components $X'P'$ and $P'B$, A_2 by its components BP' and $P'C$, etc. Draw through the mean ordinate of the area A_1 in (a) a vertical line of indefinite length, as A_1A'. Also draw through the mean ordinate of the area A_2 in (a) the vertical line A_2B', and draw through the mean ordinate of A_3 the line A_3C'. Continue this process for all the elementary areas in curve (a). From the origin O' in (c') draw $O'A'$ parallel to $P'X'$ in (b), and from the point A' in (c') where the line $O'A'$ intersects the vertical line through the mean ordinate of A_1 in (a) draw $A'B'$ parallel to $P'B$ in (b), and from the intersection, B' in (c') of $A'B'$ with the vertical through the mean ordinate of A_2, draw $B'C'$ parallel to $P'C$ in (b). Continue this process until the entire polygon $O'A'B' — E' — X'$ is drawn. This polygon is called the **string** or **funicular polygon** and the lines $O'A'$, $A'B'$, $B'C'$, etc., are called **strings.** The ordinate measured from the horizontal axis $O'X''$ in curve (c') represents the integrated area of diagram (b) between the curve and the axis $O''X''$, i.e., the ordinate of a second integrated curve of (a), The constant $O''O'$ of (b) for the axis $O''X''$ call A_1'. Take any section MN, then y in (c') represents the accumulated or integrated area in (b) from the origin to the section.

Proof: From the similar triangles $P'X'X''$ and $O'A_1G_1$ in the ray and funicular polygons respectively, $\frac{A_1'}{H} = \frac{(A_1G_1)}{x}$; from triangles $P'X'B$ and $A'A_1B_1$, $\frac{A_1}{H} = \frac{(A_1'B_1')}{p_1}$; from triangles $P'BC$ and $B'B_1C_1$, $\frac{A_2}{H} = \frac{(B_1C_1)}{p_2}$; from triangles $P'CD$ and $C'C_1D_1$, $\frac{A_3}{H} = \frac{(C_1D_1)}{p_3}$; from triangles $P'DE$ and $D'D_1E_1$, $\frac{A_4}{H} = \frac{(D_1E_1)}{p_4}$. By clearing the above equations of fractions, the following are obtained:

$$A_1'x = H(A_1G_1), \quad (1)$$

$$A_1p_1 = H(A_1B_1), \quad (2)$$

$$A_2p_2 = H(B_1C_1), \quad (3)$$

$$A_3p_3 = H(C_1D_1), \quad (4)$$

$$A_4p_4 = H(D_1E_1). \quad (5)$$

By subtracting the left members of the last four equations from the left member of the first equation, and the right members of the last four equations from the right member of the first equation there results the following equation:

$$A_1'x - A_1p_1 - A_2p_2 - A_3p_3 - A_4p_4 = H(A_1G_1 - A_1B_1 - B_1C_1 - C_1D_1 - D_1E_1), \text{ and}$$

$$A_1'x - \Sigma Ap = Hy,$$

since $A_1G_1 - A_1B_1 - B_1C_1 - C_1D_1 - D_1E_1 = y$.

It can be seen from curve (*b*) that $A_1'x - \Sigma Ap$ equals the algebraic sum of the area under the curve represented by the shaded portion; therefore, Hy equals the area between the axis and the curve from the origin to the section, and by use of a proper scale y represents the area. Hence, (c') is a first integrated curve of (*b*) and is a second integrated curve of (*a*).

The greater the number of parts into which the area under the curve is divided the more nearly the true curve will the funicular polygon be. When the number of parts becomes infinite the funicular polygon becomes

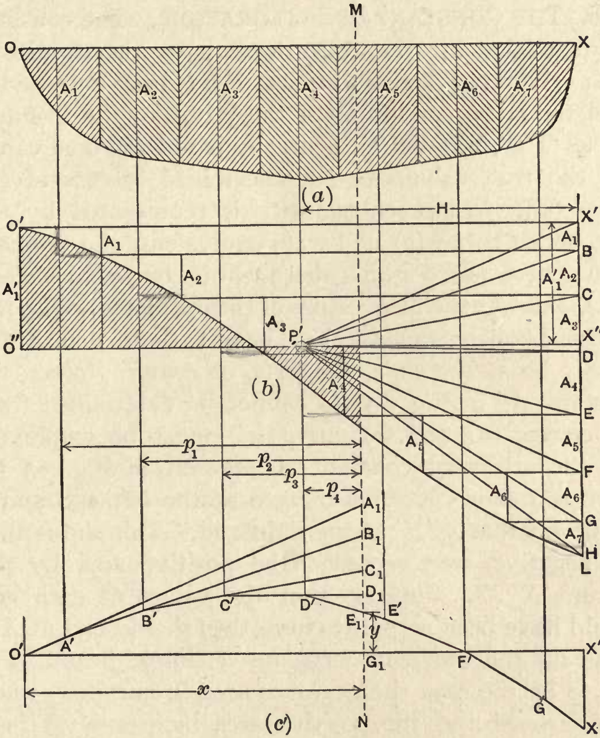


FIG. 62. SECOND METHOD OF GRAPHIC INTEGRATION.

a smooth curve inscribed within the broken funicular polygon shown. This smooth curve is the true integrated curve and can be drawn inscribed in the broken funicular polygon. In Fig. 62 (*b*), by choosing the pole P' on the axis $O''X''$ it assumes the constant of integration equal to $O''O'$, and if this is not the true value, curve (c') is not the true curve, and the process so far is only tentative.

86. THE CONSTANT OF INTEGRATION. The constant of integration in every case depends on the conditions of the given problem, and unless it has been determined from the given conditions of the problem, the assumed one as $O''O'$, Fig. 62 (b), generally is not the true value, but the true value can be determined by use of the curve (c'). Whatever quantity is represented by the area under curve (b) or by its equivalent, the ordinate of the curve (c'), Fig. 63, there will be two points along the X-axis at which the values of the ordinates are known or can be determined. For beams these points will usually be at the ends, supports, or center. When the constant A_1' in Fig. 63 (b) cannot be determined from the curves (a) and (b) curve (c') must be employed. For the arbitrary constant A_1' chosen in Fig. 63 (b) curve (c') shows a value of zero at the left end and a negative value $X''X'$ at the right end. This shows that the negative area exceeded the positive area by the amount $X''X'$. Suppose that the values at each end should have been zero, the curve then should end at X'', as would the moment curve for a simple beam. For this to be the case the negative area in curve (b) must be decreased and the positive area increased. To accomplish this the reference axis must be lowered, thus making the constant larger than the value assumed, A_1'.

The method of obtaining the value of the constant that will make the positive area equal to the negative area in (b) is to draw $O'X'$ the **closing line** of the funicular polygon in (c'), then draw $P'X$ in the vector polygon parallel to $O'X'$ in the funicular polygon. Then through

X draw the horizontal axis OX, giving XX' or OO' in (b) the true value of the constant.

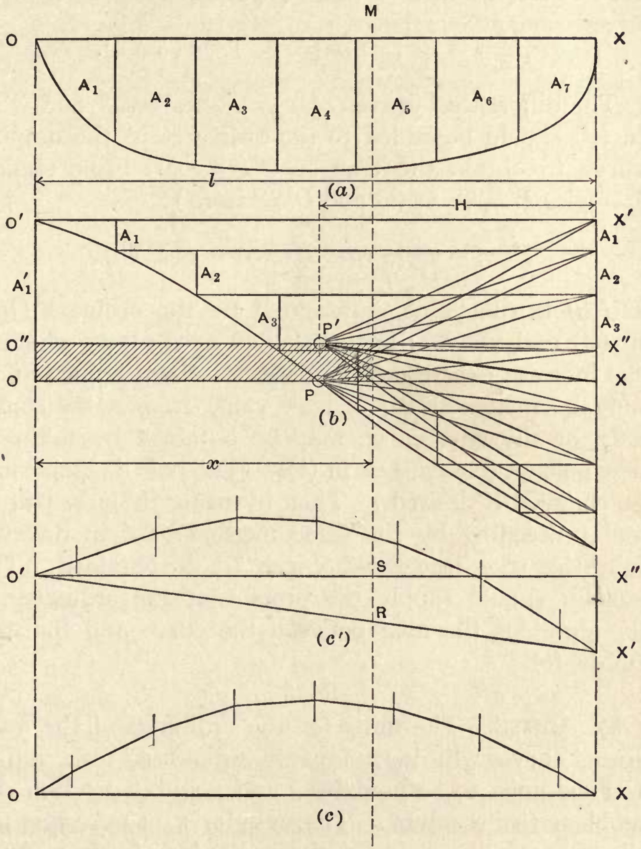


FIG. 63. TO DETERMINE THE CONSTANT OF INTEGRATION.

Proof: By lowering the reference axis in (b) to OX the positive area is increased the amount $O''X''XO$ which equals l multiplied by XX''. This must equal H

multiplied by $X''X'$ in (c') to have the positive and negative areas in (b) equal. From similar triangles $P'X''X$ in (b) and $O'X''X'$ in (c'):

$$\frac{H}{XX''} = \frac{l}{X'X''};\quad H(X'X'') = l(XX'').$$

The difference between the two axes $O'X''$ and $O'X'$ in (c') should be added to the ordinates in the original curve to obtain the true curve. Proof: From similar triangles $P'X''X$ in (b) and $O'SR$ in (c'):

$$\frac{H}{XX''} = \frac{x}{(RS)} \quad \text{or} \quad H(RS) = x(XX'').$$

$H(RS)$ is the value represented by the ordinate (RS) in (c') and $x(XX'')$ is the shaded area in (b) which is the increase over the former value. Each ordinate now may be increased to its true value from a horizontal axis, or the true curve may be obtained by taking a new pole on the axis OX in (b). (The pole distance may be changed if desired.) Then by using the new pole P and proceeding by the same method used in drawing (c'), the true integrated curve (c) is obtained. The student should supply the proof that the ordinates in (c) represent the area between the curve and the axis OX in (b).

87. UNITS. The units for the ordinates of the integrated curves will depend on the units used for x and y, and the units to be used for x and y will depend on the problem to be solved. The unit for x is the same for all curves. The unit for the ordinates of curve (b) is the product of the x unit and the y unit or the unit formed by the product xy; and that for curve (c) is the product of the unit of the ordinate for (b) and the x unit, i.e., the unit formed by the product x^2y. In problems for beams x will represent a length.

EXAMPLES

1. By the method of graphic integration draw the shear and the moment curves for a simple beam of 12-ft. span carrying a concentrated load of 2000 pounds 9 feet from the left support.

The shear diagram is drawn in the usual way, Fig. 64. Selecting the pole P with the pole distance equal to 72 inches the moment

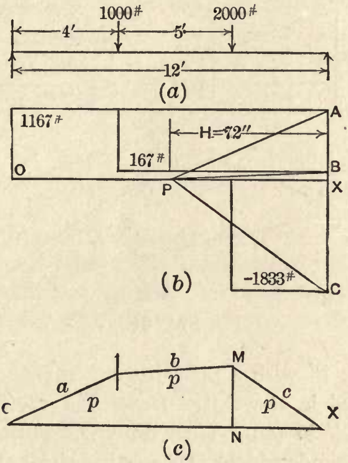


FIG. 64.

curve (c) is drawn, ($p - a$) in (c) is parallel to ($P - A$) in (b), ($p - b$) in (c) is parallel to ($P - B$) in (b), and ($p - c$) in (c) is parallel to ($P - C$) in (b). To get the moment at any section as at MN measure MN, using the same scale as is used in drawing the shearing forces, and multiply by the pole distance, in this case 72 inches. This gives the bending moment represented by MN equal to 66,000 pound-inches.

2. Determine the vertical shear at the left support (constant of integration) by the graphical method for a simple beam of 14-ft. span carrying a uniform load of 500 pounds per foot and a concentrated load of 3500 pounds 5 feet from the left support.

By using the $O'X'$-axis in Fig. 65 (*b*) lay off distances as AB to represent the area in the load diagram to the left of the section. Lay off the concentrated load BC to the same scale in pounds.

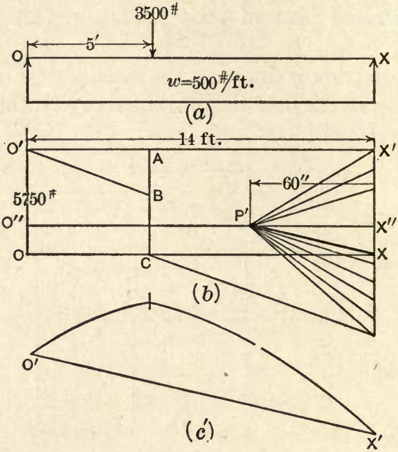


FIG. 65.

Select any pole P' with a pole distance say 60 inches and draw (c') as described in the text. Draw $P'X$ in (*b*) parallel to the closing line $O'X'$ in (c'). Then draw OX horizontal giving OO' equal to 5750 pounds which is the vertical shear at the left support.

PROBLEMS

1. By the graphical methods draw the shear and moment diagrams for the following cases:

(a) Cantilever beam of 9-ft. span, concentrated load of 5000 pounds at the free end and one of 6000 pounds at the center.

(b) Simple beam of 16-ft. span with a total uniform load of 18,000 pounds.

(c) Simple beam of 16-ft. span, uniform load of 1000 pounds per foot and a concentrated load of 10,000 pounds 8 feet from the left support.

2. By the graphical methods determine the vertical shear at the left end for each of the following systems of loading:

(a) Simple beam of 10-ft. span with a uniform load of 400 pounds per foot and a concentrated load of 1000 pounds 3 feet from the left support.

(b) Simple beam of 20-ft. span, concentrated loads of 5000 pounds 7 feet, 2500 pounds 10 feet, and 10,000 pounds 15 feet from left support.

(c) A beam 16 feet long overhanging the right support 4 feet with a uniform load of 1500 pounds per foot.

(d) A cantilever beam of 10-ft. span with a uniform load of 300 pounds per foot and a concentrated load of 800 pounds at the free end.

(NOTE. The value of the moment at the free end is zero, and that at some other point should be calculated and laid off to scale to draw the *closing line*. The vertical scale for the moment curve equals the product of the vertical scale of the shear curve multiplied by the pole distance.)

CHAPTER IX

DEFLECTION OF BEAMS

ELASTIC CURVE

88. Bending. The **elastic curve** is the curve assumed by the neutral surface of a beam under load. The deflection of beams can be obtained only from the elastic curve. For certain kinds of beams the reactions, the maximum shear and the maximum moment can be obtained only by the use of the elastic curve, while for cantilever, simple, and overhanging beams the reactions, the shear, and the moment may be obtained without its use, also for beams fixed at both ends and loaded symmetrically the reactions and shear may be obtained without its use.

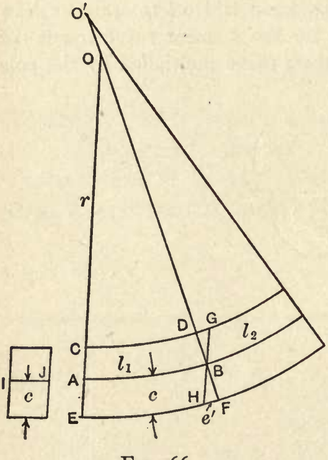


Fig. 66.

89. The Radius of Curvature of Beams. In Fig. 66 let l_1 be an element of length of a beam under load. The loads cause a bending moment M at the section CE. This bending moment may be considered constant over the element of length l_1. The deformation due to the shearing stresses will not be considered in this discussion.

The neutral surface AB, which was straight before the load was applied, is bent to a curve under the influence of the bending moment M. Assume that a normal section of the beam before bending remains a normal section after bending, that the moduli of elasticity of the material in compression and in tension are equal, and that the stresses developed are below the elastic limit. Let CE and DF be two sections normal to the beam and parallel to each other before bending. After bending, their planes will intersect at some point O, the center of curvature of AB. Let the radius of curvature BO be r, and the distance of the most remote fiber from the neutral surface BF be c. Draw GH parallel to CE, then DG is the shortening of the top fiber in the length l_1 and HF is the elongation of the bottom fiber in that length. For a symmetrical section these deformations are equal. Let this deformation be e'. Then the unit deformation is $\frac{e'}{l_1}$, and the unit-stress developed is $f = \frac{e'}{l_1}E$. From the moment formula $f = \frac{Mc}{I}$.

Therefore, $$\frac{e'}{l_1}E = \frac{Mc}{I}, \text{ or } \frac{e'}{l_1} = \frac{Mc}{EI}.$$

Since l_1 is very small the triangles OAB and BHF may be considered to be similar.

Hence, $$\frac{HF}{AB} = \frac{BF}{OB}, \text{ or } \frac{e'}{l_1} = \frac{c}{r}.$$

$$\therefore \quad \frac{c}{r} = \frac{Mc}{EI}, \text{ or } r = \frac{EI}{M}$$

In this equation M is the bending moment for the element of length l_1, r is the radius of curvature of AB, E is the modulus of elasticity of the material, and I is the moment of inertia of the cross section of the beam about the neutral axis. It is seen from the equation that the radius of curvature of a portion of a beam varies inversely with the bending moment.

90. The Slope of the Neutral Surface. The **slope** of a curve at a given point is the measure of the tangent of the angle the curve makes with the horizontal axis. Thus, in Fig. 67, tan α is the slope of the

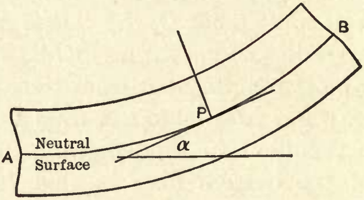


FIG. 67.

curve AB at the point P. In Fig. 68, which is greatly exaggerated, let α be the angle the neutral surface of a beam at A makes with the horizontal. α' is the increase in the angle over the element of length AB. Since the angles are very small for ordinary beams, the tangents or slopes of the angles may be considered equal to the angles themselves without appreciable error; therefore, the increase in the slope is equal to the increase in the angle measured in radians, and the slope and the angle may be interchanged. From the figure it is seen that $\alpha' = \frac{l_1'}{r_1}$; hence, the increase in the slope over the element of length l_1' is

$$\alpha' = \frac{l_1'}{r_1} = \frac{M'l_1'}{EI}, \text{ since } r = \frac{EI}{M}.$$

Similarly the increase in the slope α'' over the element of length BK is $\frac{M''l_1''}{EI}$. The increase in the slope over any element of length has a similar form. The total increase of the slope between any two sections of the

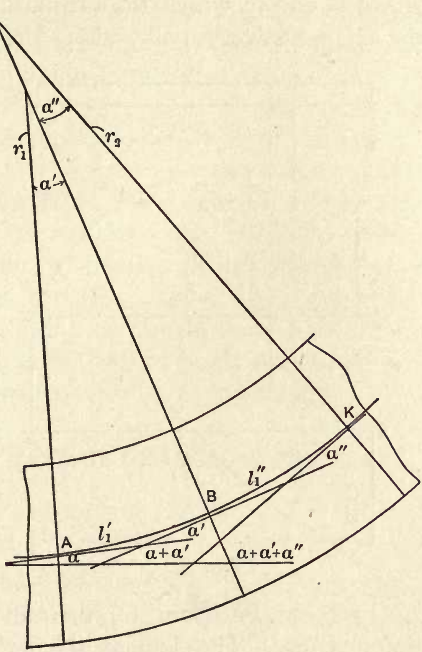


FIG. 68.

beam is obtained by adding all the increases between the sections, which is

$$\frac{M'l_1'}{EI} + \frac{M''l_1''}{EI} + \frac{M'''l_1'''}{EI} + \cdots = \sum \frac{Ml_1}{EI}.$$

If the slope at any section is known, the slope at another section may be found by adding the increase, $\sum \frac{Ml_1}{EI}$, between the two sections.

91. THE SLOPE CURVE. The relation in the last article affords the means of deriving the graphical method of determining the slope curve for a beam. The slope curve (α curve) is one in which the ordinates show the values of the slope at every point along the beam. In

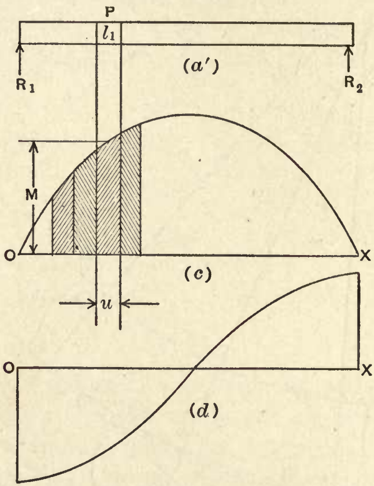


FIG. 69.

Fig. 69 let (a') represent a beam, (c) the moment curve, and (d) the slope curve. The element of length l_1 is measured along the beam and u is measured along the horizontal axis. Since the length l_1 is very small it may be considered a straight line, then $u = l_1 \cos \alpha$ where α is the angle the beam at the point P makes with the X-axis. Since α is almost zero $\cos \alpha$ may be taken equal to 1, and u equal to l_1, and then the increase in the slope between any two sections becomes $\sum \frac{Mu}{EI}$. From Fig. 69 (c) it is seen that Mu equals one of the small shaded portions of the area under the moment curve, and ΣMu is the sum of all the small areas between the sections.

Therefore, **to obtain the increase in the slope of the elastic curve of a beam between two sections, divide the area under the moment curve between the two sections by the product of the modulus of elasticity of the material and the moment of inertia of the cross section of the beam about the neutral axis.**

If the moment of inertia of the cross section is not the same throughout the length of the beam, a modified curve may be obtained from the moment curve by dividing the ordinates in the moment curve by EI for several sections, I being the value of the moment of inertia at the section where the ordinate to the moment curve is measured, and then taking the area under this modified curve for the change in slope between the two sections. When the second method of integration is used, the pole distance may be varied with I.

92. THE RATE OF CHANGE OF THE SLOPE. The rate of change of the slope at a section is $\dfrac{\frac{Mu}{EI}}{u} = \dfrac{M}{EI}$, which is the bending moment for the section divided by the product of the modulus of elasticity of the material and the moment of inertia of the cross section about the neutral axis.

93. THE DEFLECTION OF BEAMS. THE ELASTIC CURVE. In Fig. 70 (*e*) let APB represent the position assumed by a portion of the neutral surface of a beam under load. Divide the length of the curve into the elements l_1', l_1'', l_1''', etc. If at any point P the value of the deflection y is known, that for any other point Q may be determined by calculating the increase in the deflection between the two points. Let the angles made with the horizontal by the lengths l_1', l_1'', l_1''', etc., be α', α'', α''', etc.

Then the increase in y over the length l_1' is $y' = u' \tan \alpha'$ where u' is the horizontal projection of l_1', $y'' = u'' \tan \alpha''$, $y''' = u''' \tan \alpha'''$, etc. The tangent of the angle the elastic curve makes with the horizontal axis is the

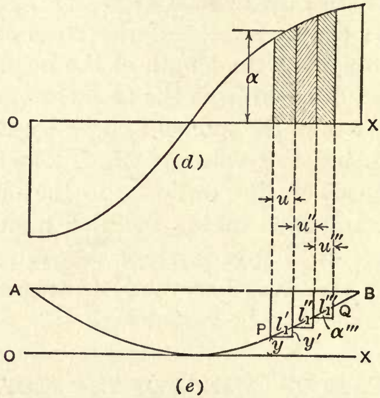


FIG. 70

slope of the curve at that point; therefore, the increase in y between the two points is

$$y' + y'' + y''' + \cdots = u' \tan \alpha' + u'' \tan \alpha'' + u''' \tan \alpha''' + \cdots = u'\alpha' + u''\alpha'' + u'''\alpha''' + \cdots = \Sigma\, u\alpha.$$

From Fig. 70 (d) it is seen that $u\alpha$ equals a small shaded area under the slope curve, and the summation of all the areas, $\Sigma\, u\alpha$, equals the total area under the slope curve between the two points. Therefore, **the increase in the deflection of the elastic curve of a beam between any two points is equal to the area under the slope curve between those two points.** The deflection at the supports is usually known.

94. THE RATE OF INCREASE OF THE DEFLECTION. The rate of increase of the deflection is $\frac{u\alpha}{u} = \alpha$. **The rate of increase in the deflection at a section is equal to the slope of the elastic curve at the section.**

95. RELATIONS BETWEEN THE FIVE CURVES. Combining the relations between the load, shear, and moment curves deduced in Chapter V, with those between the moment, slope, and elastic curves, important results are obtained. Since these five curves are the principal ones for beam problems, and since they form a continuous chain between the load and the deflection of a beam, they will be referred to as the **five curves.** The relation existing between them may be stated as follows:

Between any two sections of the beam:

(1) The increase in the vertical shear equals the area under the load curve between the sections.

(2) The increase in the bending moment equals the area under the shear curve between the sections.

(3) The increase in the slope equals the area under the moment curve between the sections divided by EI.

(4) The increase in the deflection equals the area under the slope curve between the sections.

Thus it follows that the question of determining the elastic curve is one of determining constants of integration and of obtaining areas under curves. These principles will be applied to various kinds of beams, and the constants determined and the areas obtained.

96. THE UNITS FOR THE FIVE CURVES. In the following discussions (*a*) will refer to the load curves, (*b*) the vertical shear curves, (*c*) the bending moment curves, (*d*) the slope curves, and (*e*) the deflection or elastic curves. The modulus of elasticity E will be taken in pounds per square inch, I in (inches)^4. For all the five curves and for the pole distances one inch along the X-axis will represent m inches of length measured parallel to the beam. The scale of ordinates of the curves will be:

Curve (a) 1 inch = w' pounds per inch run. 1 square inch area = $w'm$ pounds.

Curve (b) 1 inch = n square inches from (a) = $nw'm$ pounds. 1 square inch area = $nw'm^2$ pound-inches.

Curve (c) 1 inch = p square inches from (b) = $pnw'm^2$ pound-inches. 1 square inch area = $pnw'm^3$ pound-(inches)^2.

Curve (d) 1 inch $= \dfrac{q \text{ square inches from } (c)}{EI} = \dfrac{qpnw'm^3}{EI}$

which is a ratio. 1 square inch area $= \dfrac{pqnw'm^4}{EI}$ inches.

Curve (e) 1 inch = r square inches from (d) $= \dfrac{rqpnw'm^4}{EI}$ inches.

For an illustration of the method of determining the scales of the curve see Example 1 at the end of the next chapter.

EXAMPLE

1. What will be the increase in the slope from the left end to the middle of a 9-inch, 21-pound I-beam of 12-ft. span with the concentrated load at the center that will produce a maximum fiber stress of 16,000 pounds per square inch?

The maximum moment is developed at the center and is

$$M = \frac{fI}{c} = 16{,}000 \times 18.9 = 302{,}400 \text{ pound-inches.}$$

Since the moment increases directly from zero at the end to the maximum, the moment curve is as drawn in Fig. 71. The area under the moment curve to the left of the center is $\frac{1}{2} \times 72 \times 302{,}400 = 10{,}886{,}400$ pound-inches². Therefore the change in the slope over this length is

$$\frac{10{,}886{,}400}{30{,}000{,}000 \times 84.9} = .0043.$$

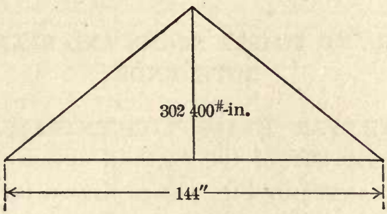


FIG. 71.

PROBLEMS

1. What is the radius of curvature at the ends and at the center of a 3-inch × 4-inch stick 8 feet long used as a simple beam with a load of 350 pounds concentrated at the center?

Ans. 2860 inches.

2. What is the change in the slope from the left end to the center in the beam in Problem No. 1? What is the change in the slope over the first two feet?

3. What is the total change in the slope over the entire length of a cantilever beam of 10-ft. span carrying a concentrated load of 3000 pounds at the end, if the beam is of a standard I-section and the stress does not exceed 12,000 pounds per square inch?

4. What is the rate of increase of slope at every two feet of length along the beams in Problems No. 2 and 3?

CHAPTER X

CANTILEVER AND SIMPLE BEAMS AND BEAMS FIXED AT BOTH ENDS

97. Cantilever Beam, Concentrated Load at the End. By use of the method analyzed in Art. 85 and 86 the curves in Fig. 72 are drawn for a cantilever beam with a load W at the end. From these curves the shear, moment, slope, and deflection at any section may be scaled off directly. Algebraic expressions for these quantities will now be deduced. From the definition of vertical shear,

$$V = -W.$$

Since the bending moment at the left end is zero, that at the section AB is equal to the area under the shear curve between the origin and the section, which is $-Wx$.

$$M = -Wx.$$

The increase in the slope from the left end is equal to the area in the moment diagram from the origin to the section divided by EI, and equals $-\frac{Wx^2}{2EI}$. The free end of the beam deflects. The beam remains horizontal at the wall, thereby making the slope zero at the wall. The total area under the moment curve is $-\frac{Wl^2}{2}$. This divided by EI gives the total change in the slope from one end to the other.

If the slope at the left end of the beam is α_1, it is changed from this value to zero at the right end. Therefore,

$$\alpha_1 - \frac{Wl^2}{2EI} = 0 \quad \text{or} \quad \alpha_1 = \frac{Wl^2}{2EI}.$$

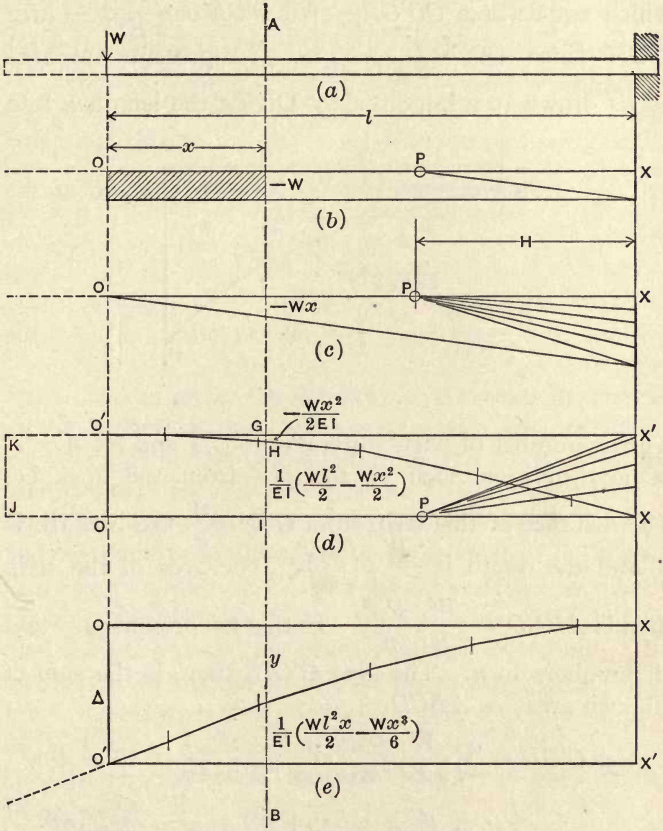


FIG. 72. CANTILEVER BEAM, CONCENTRATED LOAD

To get the slope at the section AB add to α_1 the change from the left end to the section which gives

$$\alpha = \frac{Wl^2}{2EI} - \frac{Wx^2}{2EI} = \frac{W}{2EI}(l^2 - x^2).$$

The change in the deflection of the elastic curve is equal to the area under the slope curve. The change in the deflection is shown in (*d*) by the area $OO'HI$ which equals area $OO'GI$ − area $O'GH$ or $\frac{Wl^2}{2\,EI}x$ − area $O'GH$ since $OO' = \frac{Wl^2}{2\,EI}\cdot$ Fig. 73 represents the area $O'GH$ drawn to a large scale. Divide the length x into

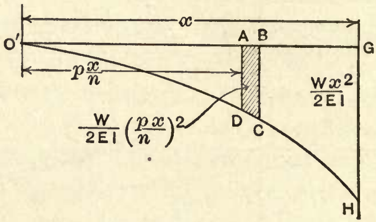


FIG. 73.

a great number of parts n parallel to GH and let $ABCD$ be any division such as the pth from the apex O'. The distance of this strip from O' is $p\frac{x}{n}$, the breadth is $\frac{x}{n}$, and the depth is $\frac{W}{2\,EI}\left(\frac{px}{n}\right)^2$. The area of the strip then is $ABCD = \frac{W}{2\,EI}\,\frac{p^2x^3}{n^3}\cdot$ In this p represents any and all numbers to n. The area $O'GH$, then, is the sum of all such areas as $ABCD$.

$$\text{Area } O'GH = \sum \frac{W}{2\,EI}\,\frac{p^2x^3}{n^3} = \frac{W}{2\,EI}\,\frac{x^3}{n^3}\sum (1^2 + 2^2 + 3^2 + \cdots + p^2 + \cdots + n^2) = \frac{W}{2\,EI}\,\frac{x^3}{n^3}\,\Sigma\,(n)^2.$$

By algebra it can be shown that

$$\Sigma\,(n)^2 = \frac{2\,n^3 + 3\,n^2 + n}{6}.^*$$

* See "Higher Algebra," by John F. Downey, page 373.

$$\text{Area } O'GH = \frac{W}{2\,EI}\,\frac{x^3}{n^3}\left(\frac{2\,n^3 + 3\,n^2 + n}{6}\right)$$

$$= \frac{W}{2\,EI}x^3\left(\frac{1}{3} + \frac{1}{2\,n} + \frac{1}{6\,n^2}\right).$$

Since the side $O'H$ is a continuous curve, as n is increased the assumed broken line will approach the actual curve and they will coincide when n equals infinity. By assuming this to be the case, the actual area $O'GH$ is obtained, and $\frac{1}{2\,n}$ and $\frac{1}{6\,n^2}$ become zero; therefore

$$\text{Area } O'GH = \frac{Wx^3}{6\,EI},$$

and

$$\text{Area } OO'HI = \frac{Wl^2x}{2\,EI} - \frac{Wx^3}{6\,EI}.$$

Since the area under the slope curve represents the change in the deflection the expression for the area $OO'HI$ is the increase in the deflection from the left end to the section AB. In Fig. 72 (*e*) if the X'-axis were used as reference, the deflection at the left end would be zero, and the above equation would give the value of the deflection at any section AB. The axis of reference is usually taken in the position of the neutral surface before any load is on the beam, for which case the deflection at the left end is OO' equal to XX', Fig. 72 (*e*), which equals the total change in the deflection over the entire length of the beam. This change is obtained by letting x equal l in the expression for the area under the slope curve, $\frac{Wl^2x}{2\,EI} - \frac{Wx^3}{6\,EI}$ giving $\frac{Wl^3}{3\,EI}$.

$$OO' = -\frac{Wl^3}{3\,EI}$$

$$\therefore\quad y = \frac{Wl^2x}{2\,EI} - \frac{Wx^3}{6\,EI} - \frac{Wl^3}{3\,EI},$$

which is the equation of the elastic curve of a cantilever beam with a load W at the free end. The maximum deflection occurs at the free end and is

$$\Delta = -\frac{Wl^3}{3EI}.$$

98. CANTILEVER BEAM, CONCENTRATED LOAD AWAY FROM THE FREE END. The solution of the problem for a concentrated load at the end of a cantilever beam can be extended to cover the problem when the load is away from the end. The dotted lines in Fig. 72 indicate the extension of the solution for the previous case to cover this case. The load, shear, and moment curves would be similar to those given. The slope has the constant value OO' from the free end to the load. The additional deflection of the free end equals the area $KJO'O$. The student may deduce the equations for this case.

99. CANTILEVER BEAM, UNIFORM LOAD. In Fig. 74 are drawn the curves for a cantilever beam with a uniform load. The expressions for the values represented by the different curves at the section AB the distance x from the free end will be deduced.

The load per unit of length of the beam equals $-w$.

Vertical shear $V = -wx$.

Bending moment $M = -\frac{wx^2}{2}$.

The slope at the right end is zero as the elastic curve there is horizontal. The area under the moment curve to the section AB is $-\frac{wx^3}{6}$ (see Art. 97). The total area is $-\frac{wl^3}{6}$, hence the total change in the slope is $-\frac{wl^3}{6EI}$, making the slope at the left end α_1 equal to $\frac{wl^3}{6EI}$; therefore,

Slope $$\alpha = \frac{wl^3}{6EI} - \frac{wx^3}{6EI}.$$

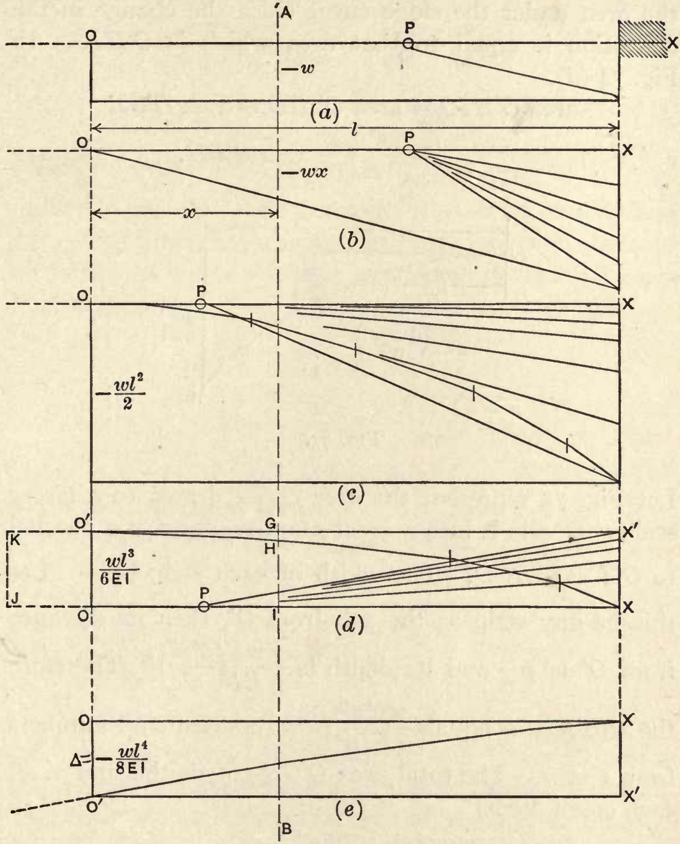


FIG. 74. CANTILEVER BEAM, UNIFORM LOAD.

The deflection at the right end is zero. In order to find the deflection at any section AB it is necessary to obtain the area under the slope curve since the change in the deflection is equal to that area which is $O'HIO$. In Fig. 74 (d),

$$\text{area } O'HIO = \text{area } O'GIO - \text{area } O'GH$$
$$= \frac{wl^3x}{6EI} - \text{area } O'GH.$$

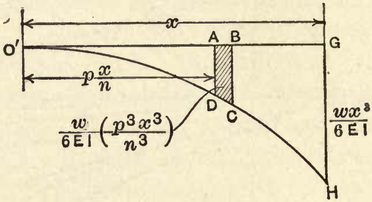


FIG. 75.

Let Fig. 75 represent the area $O'GH$ drawn to a larger scale. Divide it into a great number of strips n parallel to GH as $ABCD$. The width of each strip is $\frac{x}{n}$. Let this be any strip as the pth from O', then its distance from O' is $p\frac{x}{n}$ and its depth is $\frac{w}{6EI}\left(\frac{p^3x^3}{n^3}\right)$. Therefore the area $ABCD$ equals $\frac{wp^3x^4}{6EIn^4}$. p represents all numbers from 1 to n. The total area $O'GH$ equals the sum of all such small areas.

$$\text{Area } \quad O'GH = \sum\frac{wp^3x^4}{6EIn^4} = \frac{wx^4}{6EIn^4}\sum(1^3+2^3+3^3$$
$$+\cdot\cdot\cdot+p^3+\cdot\cdot\cdot+n^3) = \frac{wx^4}{6EIn^4}\Sigma(n)^3.$$

$$\text{From algebra } \Sigma\,(n)^3 = \frac{n^4+2\,n^3+n^2}{4}.*$$

* See "Higher Algebra," by John F. Downey, page 373.

$$\text{Area } O'GH = \frac{wx^4}{6\,EIn^4}\frac{(n^4 + 2\,n^3 + n^2)}{4}.$$

$$= \frac{wx^4}{6\,EI}\left(\frac{1}{4} + \frac{1}{2\,n} + \frac{1}{4\,n^2}\right) = \frac{wx^4}{24\,EI}$$

since $\frac{1}{2\,n}$ and $\frac{1}{4\,n^2}$ reduce to zero when n equals infinity. Therefore area $O'GIO = \frac{wl^3x}{6\,EI} - \frac{wx^4}{24\,EI}$. This equals the change in the deflection from the free end to the section AB. The total change over the entire length is obtained by letting x equal l in the expression for the change. This reduces to

$$\frac{wl^4}{6\,EI} - \frac{wl^4}{24\,EI} = \frac{wl^4}{8\,EI}.$$

$$\Delta = -\frac{wl^4}{8\,EI} = -\frac{Wl^3}{8\,EI}$$

which is the maximum deflection and occurs at the free end. W is the total load.

$$y = -\frac{wl^4}{8\,EI} + \frac{wl^3x}{6\,EI} - \frac{wx^4}{24\,EI},$$

which is the equation of the elastic curve for a cantilever beam with a uniform load.

100. Cantilever Beam, Various Loading. If the end projects beyond the uniform load, the load, shear, and moment diagrams will be similar to those of Fig. 74. The slope will be constant from the free end to the load as indicated by the dotted lines $OJKO'$, Fig. 74 (*d*). The additional deflection will be equal to the area $O'OJK$. If the beam has a concentrated load at the end and a uniform load the equations for the two cases may be combined. Any other combination of uniform and concentrated loads may be made, and corresponding equations derived similarly to the foregoing deductions.

101. Simple Beam, Concentrated Load at the Center. The curves in Fig. 76 are drawn for a simple beam with a load W at the center. The vertical shear to the left of the load is

$$V = \frac{W}{2}.$$

The bending moment to the left of the load is

$$M = \frac{Wx}{2}.$$

The slope curve must pass through the axis at the center of the beam, since the elastic curve is horizontal there, making the slope at that point equal to zero. The change in the slope from the end to the middle equals the area to the left of the center in the moment diagram divided by EI, which is $\frac{Wl^2}{16\,EI}$; therefore, the slope at the left end is $-\frac{Wl^2}{16\,EI}$, and the slope at the section AB to the left of the center is

$$\alpha = \frac{Wx^2}{4\,EI} - \frac{Wl^2}{16\,EI}.$$

The deflection at the end is zero. The change in the deflection from the end to the section AB to the left of the center can be obtained by calculating the area under the slope curve (d).

Area $OABO'$ = area $OACO'$ − area $O'BC$

$$= -\left(\frac{Wl^2x}{16\,EI} - \frac{wx^3}{12\,EI}\right) = -\frac{Wl^2x}{16\,EI} + \frac{wx^3}{12\,EI}$$

(see Art. 97). Since the area is below the axis it is negative.

$$y = -\frac{Wl^2x}{16\,EI} + \frac{Wx^3}{12\,EI}.$$

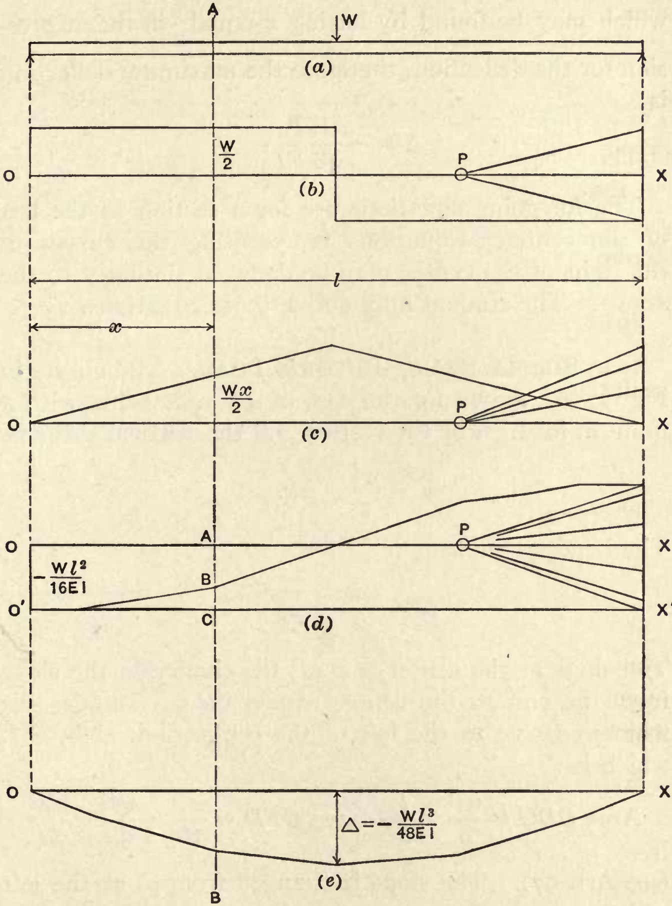


FIG. 76. SIMPLE BEAM, CONCENTRATED LOAD AT THE CENTER.

The maximum deflection occurs at the center and equals the area below the axis in the slope diagram, which may be found by letting x equal $\frac{l}{2}$ in the expression for the deflection; therefore the maximum deflection is

$$\Delta = -\frac{Wl^3}{48\,EI}.$$

The foregoing equations are for a section to the left of the center. Equations representing the curves to the right of the center may be deduced similarly to the above. The student may derive those equations.

102. SIMPLE BEAM, UNIFORM LOAD. The curves in Fig. 77 are drawn for the case of a simple beam with a uniform load. For the section AB the vertical shear is

$$V = \frac{wl}{2} - wx.$$

The bending moment is

$$M = \frac{wlx}{2} - \frac{wx^2}{2}.$$

The slope at the center is zero; the change in the slope from the end to the center equals the area under the moment curve to the left of the center divided by EI, which is

$$\text{Area } ODG = \frac{wl^2}{8} \times \frac{l}{2} - \text{area } OFD = \frac{wl^3}{16} - \frac{wl^3}{48} = \frac{wl^3}{24}$$

(see Art. 97). The slope is changed from α_1 at the left end to zero at the center; therefore,

$$\alpha_1 + \frac{wl^3}{24\,EI} = 0 \text{ and } \alpha_1 = -\frac{wl^3}{24\,EI}.$$

The change in the slope to the section AB equals the

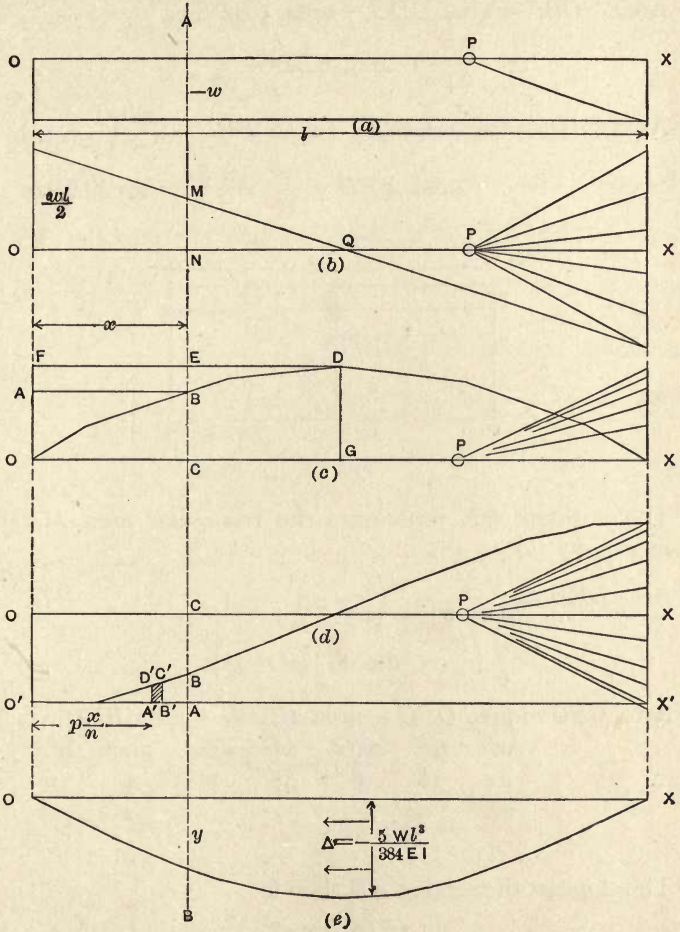


FIG. 77. SIMPLE BEAM, UNIFORM LOAD.

area OBC divided by EI. By referring to Fig. 78 we see that

$$\text{Area } OBC = \text{area } ODG - \text{area } CBDG$$
$$= \frac{wl^3}{24} - \text{area } CBDG.$$

$$\text{Area } CBDG = \text{area } CEDG - \text{area } BED = \frac{wl^2}{8}\left(\frac{l}{2} - x\right)$$
$$- \text{area } BED = \frac{wl^3}{16} - \frac{wl^2x}{8} - \text{area } BED.$$

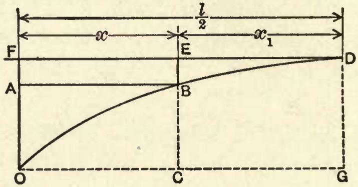


FIG. 78.

The ordinate BE represents the triangular area MNQ in Fig. 77 (b).

$$\text{Area } BED = \frac{wx_1^3}{2 \times 3} = \frac{w}{6}\left(\frac{l}{2} - x\right)^3$$
$$= \frac{w}{6}\left(\frac{l^3}{8} - \frac{3}{4}l^2x + \frac{3}{2}lx^2 - x^3\right).$$

$$\text{Area } OBC = \text{area } ODG - \text{area } CEDG + \text{area } BED,$$
$$= \frac{wl^3}{24} - \frac{wl^3}{16} + \frac{wl^2x}{8} + \frac{wl^3}{48} - \frac{wl^2x}{8} + \frac{wlx^2}{4} - \frac{wx^3}{6}$$
$$= \frac{wlx^2}{4} - \frac{wx^3}{6}.$$

The slope at the section AB then is

$$\alpha = -\frac{wl^3}{24\,EI} + \frac{wlx^2}{4\,EI} - \frac{wx^3}{6\,EI}.$$

Since the deflection at the end is zero, that at the section AB equals the area under the slope curve between the origin and that section. In Fig. 77 (*d*) the deflection is shown by

Area $OO'BC = -$(area $OO'AC -$ area $O'AB$)

$$= -\left(\frac{wl^3x}{24\,EI} - \text{area } O'AB\right).$$

Divide the area $O'AB$ into a great number of strips n. The width of each strip is $\frac{x}{n}\cdot$ If the strip shown is the pth strip its distance from O' is $\frac{px}{n}$, and its depth is $\frac{wl}{4\,EI}\left(\frac{p^2x^2}{n^2}\right) - \frac{w}{6\,EI}\left(\frac{p^3x^3}{n^3}\right)\cdot$ The small area then is $\frac{wl}{4\,EI}\left(\frac{p^2x^3}{n^3}\right) - \frac{w}{6\,EI}\left(\frac{p^3x^4}{n^4}\right)$, in which p represents all numbers to n, and to obtain the total area $O'BA$ all such areas must be added; therefore

$$\text{Area } O'BA = \frac{wlx^3}{4\,EIn^3}\Sigma(1^2 + 2^2 + 3^2 + \cdots + p^2 + \cdots + n^2)$$

$$- \frac{w}{6\,EI}\frac{x^4}{n^4}\Sigma(1^3 + 2^3 + 3^3 + \cdots + p^3 + \cdots + n^3)$$

$$= \frac{wlx^3}{4\,EIn^3}\Sigma(n)^2 - \frac{wx^4}{6\,EIn^4}\Sigma(n)^3$$

$$= \frac{wlx^3}{4\,EIn^3}\left(\frac{2\,n^3 + 3\,n^2 + n}{6}\right)$$

$$- \frac{wx^4}{6\,EIn^4}\left(\frac{n^4 + 2\,n^3 + n^2}{4}\right)$$

$$= \frac{wlx^3}{4\,EI}\left(\frac{1}{3} + \frac{1}{2\,n} + \frac{1}{6\,n^2}\right)$$

$$- \frac{wx^4}{6\,EI}\left(\frac{1}{4} + \frac{1}{2\,n} + \frac{1}{4\,n^2}\right)$$

$$= \frac{wlx^3}{12\,EI} - \frac{wx^4}{24\,EI}.$$

When n is infinitely large $\frac{1}{2\,n}$, $\frac{1}{6\,n^2}$, and $\frac{1}{4\,n^2}$ reduce to zero.

$$\therefore \quad \text{Area } O'OCB = -\left(\frac{wl^3x}{24\,EI} - \frac{wlx^3}{12\,EI} + \frac{wx^4}{24\,EI}\right);$$

$$\therefore \quad y = -\frac{wl^3x}{24\,EI} + \frac{wlx^3}{12\,EI} - \frac{wx^4}{24\,EI}.$$

The maximum deflection occurs at the center or when $x = \frac{l}{2}$ in the expression for y. Substituting this value for x the maximum deflection is found to be

$$\Delta = -\frac{5\,wl^4}{384\,EI} = -\frac{5\,Wl^3}{384\,EI}.$$

103. **BEAM FIXED AT BOTH ENDS, CONCENTRATED LOAD *W* AT THE CENTER.** A beam with fixed ends has restraining moments at the walls which keep the beam horizontal at those points. These moments must be determined in the solution of the problem. From the symmetry of the beam the reactions at the walls are equal, and the restraining moments are equal. In Fig. 79 the shear and moment curves are drawn in the usual manner; then by using the pole P' in Fig. 79 (c) a slope curve (d') is drawn. Since the beam is horizontal at the ends and at the center the true slope curve must pass through the axis at those three points. Therefore connect the ends by $O'X'$. Now in (c) draw $P'X$ parallel to $O'X'$ in (d'), and through X draw the horizontal axis OX which is the true moment axis giving the moment OO' at the left support, which equals XX', the moment at the right support. Then with a new pole P on the true axis in the moment diagram the true slope curve (d) is drawn. To prove that the slope in (d) will be reduced to zero at the right end, draw the closing line OX in (d) and draw OZ parallel to $O'X'$ in (d'), then the angle ZOX must be equal to $\alpha = \alpha'$ in (d') because $Z''X'$ and $O'X''$ are both parallel to $P'X'$ in (c); i.e., hori-

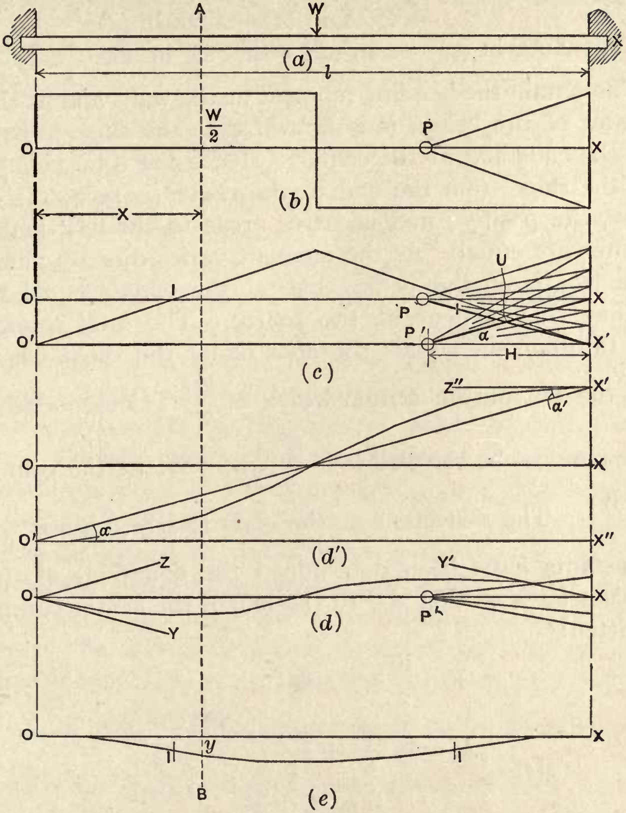


FIG. 79. BEAM FIXED AT BOTH ENDS, CONCENTRATED LOAD AT THE CENTER

zontal. $Y'XO$ and YOX in (d) are equal because $Y'X$ and YO are drawn parallel to PX in (c).

$X'UX$ in (c) = YOZ in (d),

YOX in (d) = $X'PX$ in (c),

$UX'P' + UP'X'$ in (c) = $X'PX + X'P'X$ in (c)

= YOX in (d) + α in (d'),

$$\therefore \quad YOZ \text{ in } (d) = YOX \text{ in } (d) + \alpha$$
$$= YOX \text{ in } (d) + XOZ \text{ in } (d),$$
$$\therefore \quad ZOX \text{ in } (d) = \alpha \text{ in } (d') = X'P'X \text{ in } (c).$$

To obtain the bending moment at the walls and at the center of the beam, it is known that the slope is zero at the ends and at the center. Hence the total change in the slope from the end to the center is zero; therefore, the positive and negative areas to the left of the center are equal. As the moment varies directly along the length of the beam that at the end is equal to minus the moment at the center. The total change in the moment equals the area under the shear curve to the left of the center which is $\frac{Wl}{4}\cdot$ Consequently, the moment at the wall is $-\frac{Wl}{8}$ and that at the center is $\frac{Wl}{8}\cdot$ The deflection at the ends is zero. Since the constants have been determined the equations of the curves for a section *AB* to the left of the center can be written:

$$V = \frac{W}{2},$$

$$M = -\frac{Wl}{8} + \frac{Wx}{2},$$

$$\alpha = -\frac{Wlx}{8\,EI} + \frac{Wx^2}{4\,EI},$$

$$y = -\frac{Wlx^2}{16\,EI} + \frac{Wx^3}{12\,EI}.$$

In order to obtain the deflection y the area under the slope curve is determined by the same method as that by which the area under the moment curve for a simple beam uniformly loaded was determined (Art. 102).

The maximum deflection is at the center and equals the area under the slope curve to the left of the center, which is found by letting $x = \frac{l}{2}$ in the expression for that area:

$$\Delta = -\frac{Wl^3}{64\,EI} + \frac{Wl^3}{96\,EI} = -\frac{Wl^3}{192\,EI}.$$

104. POINTS OF INFLECTION. In Fig. 79 (*c*) are shown two points marked *I* where the moment is zero. At these points the moment changes from negative to positive in going toward the center of the beam. The stresses also change from tension to compression in the top fibers and from compression to tension in the bottom fibers. These points where the fiber stresses change are called the **points of inflection** or **points of contraflexure.** Outside these points the beam curves downward, and inside them it curves upward. Since there are no flexural stresses at the points of inflection, the beam could be hinged at those points without affecting the stresses at the other sections of the beam.* For a beam fixed at both ends with a concentrated load at the center the inflection points occur at the two outside quarter points, hence it may be considered as a simple beam of length $\frac{l}{2}$ with the load W at the center and two cantilevers each of length $\frac{l}{4}$ with the load of $\frac{W}{2}$ at the ends. The simple beam may be considered as resting on the two cantilevers. Wherever the tensile stresses in a beam are to be taken by steel, as in reinforced concrete beams, part of the steel is bent down somewhere near the inflection points. The inflection points are located where the greatest positive and negative slopes occur.

* On account of secondary stresses and horizontal shear which have not yet been considered, the behavior of the beam may be somewhat different if hinged at the points of inflection.

105. Beam Fixed at Both Ends, Uniform Load. The reactions equal one-half the total load for a beam fixed at both ends and carrying a uniform load. The restraining moments at the ends keep the beam horizontal at these points. It is also horizontal at the center. In Fig. 80 curves (*a*), (*b*), and (*c*) are drawn in the usual manner. The curve (d') is then drawn by using the pole P' in (*c*). Connect $O'X'$ in (d'), then, in (*c*), $P'X$ is drawn parallel to $O'X'$ in (d'), and OX, which is the true moment axis, is drawn, giving the bending moments OO' and XX' at the walls. By selecting a new pole P on the moment axis the true slope curve (*d*) is drawn, from which in turn the elastic curve (*e*) is drawn.

In order to determine the bending moment at the wall and at the center it is known that the slope is zero at both sections, and, therefore, that the positive moment area ABC equals the negative area OAO'. The total change in the moment from the end to the center equals the area under the shear curve between those two sections, which is $\frac{wl^2}{8}\cdot$ By methods similar to those already given,

$$\text{Area } OAO' = \frac{wlx_1^2}{4} - \frac{wx_1^3}{3}$$

and

$$\text{Area } ABC = \text{area } ABCD - \text{area } ACD,$$

$$(BC) = (AD) = \frac{w\,(DC)^2}{2}, \text{ (area } HIF \text{ in } (b)).$$

$$\text{Area } ABC = \frac{w\,(DC)^2(DC)}{2} - \frac{w\,(DC)^3}{6} = \frac{w\,(DC)^3}{3},$$

$$(DC) = \frac{l}{2} - x_1,$$

$$\text{Area } OAO' = \text{area } ABC,$$

$$\frac{wlx_1^2}{4} - \frac{wx_1^3}{3} = \frac{w}{6}\left[\frac{2\,l^3}{8} - \frac{6\,l^2x_1}{4} + \frac{6\,lx_1^2}{2} - 2\,x_1^3\right].$$

Collecting and reducing,

$x_1 = \frac{l}{2} \pm \frac{l}{12}\sqrt{12}$, which is the distance from the end to the inflection point.

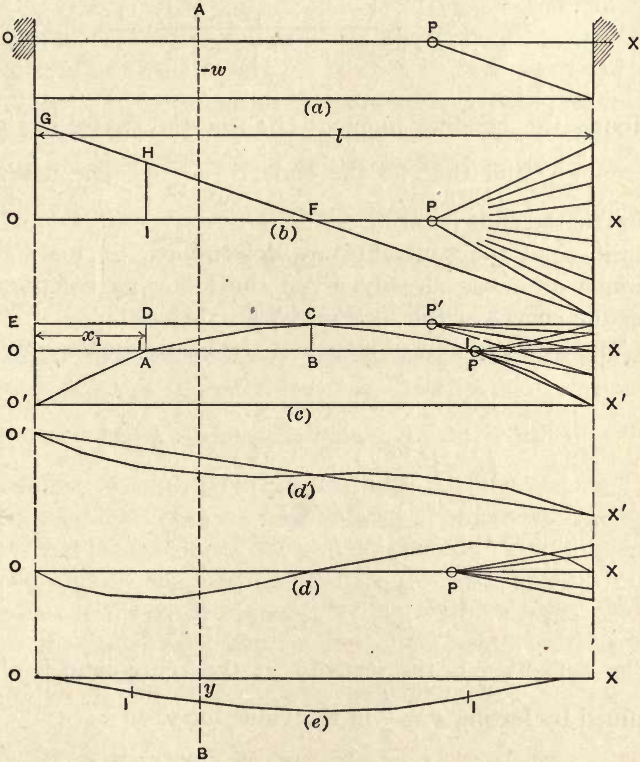


FIG. 80. BEAM FIXED AT BOTH ENDS, UNIFORM LOAD.

(AD) in (c) represents the area FHI in (b), and $(O'E)$ in (c) represents the area FGO in (b);

therefore

$$\frac{(AD)}{(O'E)} = \frac{(DC)^2}{(EC)^2} = \frac{\left(\frac{l}{2} - x_1\right)^2}{\left(\frac{l}{2}\right)^2},$$

$$\frac{(AD)}{(O'E)} = \frac{\left(\frac{l}{2} - \frac{l}{2} \mp \frac{l}{12}\sqrt{12}\right)^2}{\left(\frac{l}{2}\right)^2} = \frac{1}{3}.$$

Hence, the bending moment (BC) at the center is $\frac{1}{3}$ of $\frac{wl^2}{8} = \frac{wl^2}{24}$, and that at the ends is $-\frac{wl^2}{12}$. The deflection at the ends is zero.

Since all the constants are determined, by methods similar to those already given the following equations for the curves at the section AB are deduced.

The load per unit of length of the beam equals $-w$,

$$V = \frac{wl}{2} - wx,$$

$$M = -\frac{wl^2}{12} + \frac{wlx}{2} - \frac{wx^2}{2},$$

$$\alpha = -\frac{wl^2x}{12\,EI} + \frac{wlx^2}{4\,EI} - \frac{wx^3}{6\,EI},$$

$$y = -\frac{wl^2x^2}{24\,EI} + \frac{wlx^3}{12\,EI} - \frac{wx^4}{24\,EI}.$$

The deflection is the greatest at the center and is obtained by letting $x = \frac{l}{2}$ in the value for y, or

$$\Delta = -\frac{wl^4}{384\,EI} = -\frac{Wl^3}{384\,EI}.$$

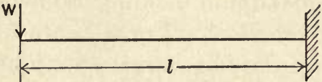
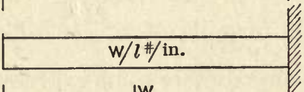
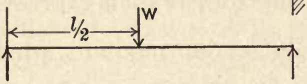
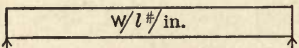
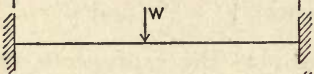
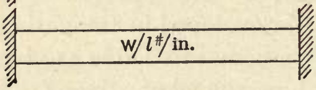
106. RELATIVE STRENGTH AND STIFFNESS OF BEAMS. The strength of a beam is proportional to the load it will carry with an assigned value of the maximum stress. For a beam of given section the allowable resisting shear and resisting moment are fixed by the allowable stresses. If the shearing stress or the deflection is not the controlling factor in the design of a beam, the strength depends upon the allowable resisting moment $\frac{fI}{c}$. The allowable bending moment is equal to this resisting moment. The strength of a certain type of beam is inversely proportional to the maximum bending moment produced in the beam by a given load. For a beam of length l, the load W to develop the fiber stress f may be obtained by use of the moment formula $M = \frac{fI}{c}$ in which M is the maximum bending moment, here to be expressed in terms of W and l, and $\frac{I}{c}$ is the section modulus, which is fixed for a given section. For example, in a cantilever beam with a concentrated load at the end $M = Wl$ (the sign being neglected); hence, $Wl = \frac{fI}{c}$, and $W = \frac{fI}{cl}$. Column two in Table 14 contains the expressions for the maximum bending moment developed in the various types of beams given, and column two in Table 15 gives the value of the load to produce the fiber stress f. If beams of the same material for the various types given are of equal length and section, their relative strengths will be proportional to the coefficients of $\frac{fI}{cl}$ given in Table 15, as $\frac{fI}{cl}$ will be the same for all the beams.

The stiffness of a beam is proportional to the load necessary to produce a given maximum deflection. The load W to cause a maximum deflection may be obtained by solving for W in terms of the maximum deflection Δ.

For example, the maximum deflection of a cantilever beam with a concentrated load at the end is $\Delta = \frac{Wl^3}{3EI}$ (the sign being neglected); hence, $W = 3\frac{EI}{l^3}\Delta$.

TABLE 14

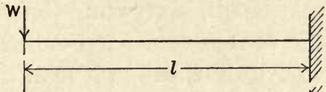
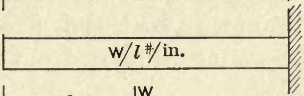
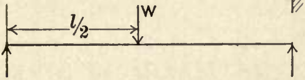
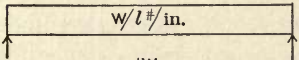
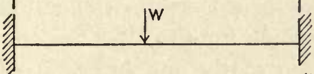
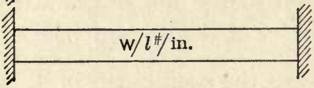
MAXIMUM MOMENTS AND MAXIMUM DEFLECTIONS

Kind of beam.	Maximum moment *M*.	Maximum deflection Δ.
	$-Wl$	$-\frac{Wl^3}{3EI}$
	$-\frac{Wl}{2}$	$-\frac{Wl^3}{8EI}$
	$\frac{Wl}{4}$	$-\frac{Wl^3}{48EI}$
	$\frac{Wl}{8}$	$-\frac{5Wl^3}{384EI}$
	$\pm\frac{Wl}{8}$	$-\frac{Wl^3}{192EI}$
	$-\frac{Wl}{12}, +\frac{Wl}{24}$	$-\frac{Wl^3}{384EI}$

Column three in Table 14 gives the expression for the maximum deflection of each type of beam there shown, and column three of Table 15 gives the value of the load to produce the deflection Δ. If beams of the same material for the types given are of equal length and section their relative stiffnesses will be proportional to the coefficients of $\frac{EI}{l^3}\Delta$, since $\frac{EI}{l^3}\Delta$ is the same for all types, assuming equal deflections.

TABLE 15

LOAD TO CAUSE A GIVEN MAXIMUM STRESS AND A GIVEN MAXIMUM DEFLECTION

Kind of beam.	Load W to cause stress f.	Load W to cause deflection Δ
	1. $\frac{fI}{cl}$	3. $\frac{EI}{l^3}\Delta$
	2. $\frac{fI}{cl}$	8. $\frac{EI}{l^3}\Delta$
	4. $\frac{fI}{cl}$	48. $\frac{EI}{l^3}\Delta$
	8. $\frac{fI}{cl}$	76 4/5. $\frac{EI}{l^3}\Delta$
	8. $\frac{fI}{cl}$	192. $\frac{EI}{l^3}\Delta$
	12. $\frac{fI}{cl}$	384. $\frac{EI}{l^3}\Delta$
General type.	$\alpha\frac{fI}{cl}$	$\beta\frac{EI}{l^3}\Delta$

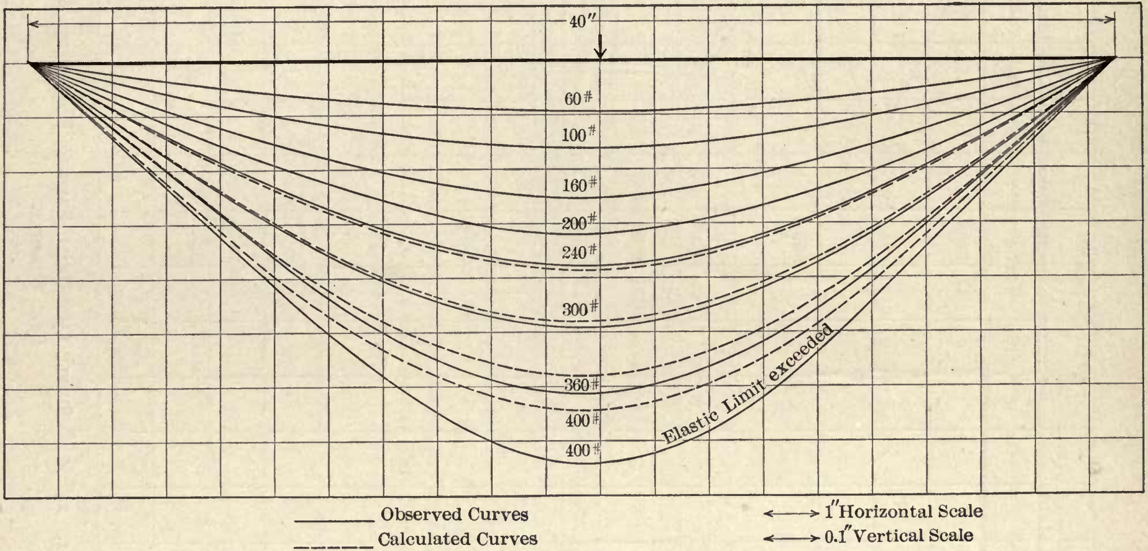
107. Maximum Stress and Deflection. From Art. 106 it is seen that if a given beam used as a cantilever will safely carry a given load at the end, it would carry twice that load uniformly distributed on the cantilever, four times that load if used as a simple beam with the load concentrated at the center, eight times that load if used as a simple beam with a uniform load, eight times that load if both ends are fixed and the load is concentrated at the center, and twelve times that load if both ends are fixed and the load is uniformly distributed. It is also seen that if a given load at the end of a cantilever beam will cause a given maximum deflection, to cause the same maximum deflection it will take two and two-thirds times that load uniformly distributed over the cantilever, sixteen times that load if the beam is used as a simple beam with the load concentrated at the center, twenty-five and one-fifth times that load if uniformly distributed over a simple beam, sixty-four times that load if concentrated at the center of a fixed-ended beam, and one hundred and twenty-six times that load if uniformly distributed over a beam with both ends fixed. In all the above cases the stresses are supposed not to exceed the elastic limit.

108. Relation Between the Maximum Stress and the Maximum Deflection. In column 2 of Table 15 appears the maximum stress f developed under the load W, and in column 3 the maximum deflection. By equating these two expressions for the load the relation of the maximum deflection to the maximum stress for a given load W is obtained. Let α represent the coefficients of $\frac{fI}{cl}$, and let β represent the coefficients of $\frac{EI}{l^3}\Delta$. Then by equating the two expressions for W there results

$$\alpha\frac{fI}{cl} = \beta\frac{EI}{l^3}\Delta,$$

$$\Delta = \frac{\alpha f l^2}{\beta E c}.$$

The last equation gives the maximum deflection in terms of the maximum stress.



COMPARISON OF OBSERVED AND CALCULATED CURVES OF A TIMBER BEAM.

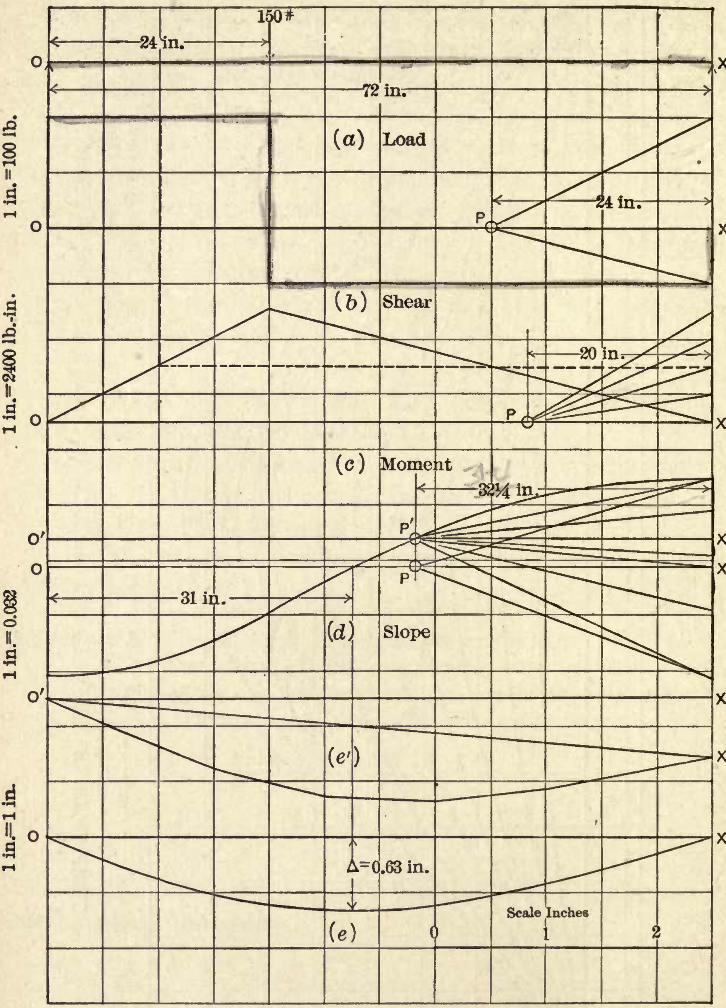


FIG. 81. $1\frac{1}{2}'' \times 2''$ SIMPLE OAK BEAM, SPAN 6′, LOAD 150# 2′ FROM LEFT SUPPORT.

EXAMPLE

Draw the elastic curve for a $1\frac{1}{2}$-inch by 2-inch oak beam of 6-ft. span carrying a load of 150 pounds 2 feet from the left support.

Solution: $I = \frac{bd^3}{12} = \frac{1.5 \times 2 \times 2 \times 2}{12} = 1 \text{ (inch)}^4,$

$$E = 1,500,000 \text{ pounds per square inch.}$$

In Fig. 81 the horizontal scale is 1 inch equals 12 inches. (On the diagram the length representing 1 inch is indicated at the bottom.) (*a*) represents the beam. (*b*) is the shear diagram in which the vertical scale is 1 inch equals 100 pounds. To draw the moment curve the pole distance is taken equal to 24 inches; the vertical scale of the moment curve then is 1 inch equals $24 \times 100 = 2400$ pound-inches. In drawing the slope curve the pole distance was taken equal to 20 inches; the vertical scale of the slope curve then is 1 inch equals $20 \times 2400 \div EI = 20 \times 2400 \div 1,500,000 \times 1 = 0.032$. As yet the slope is not known for any point along the beam; consequently an arbitrary axis $O'X'$ is assumed in curve (*d*) and the pole P' taken with a pole distance equal to $31\frac{1}{4}$ inches giving values of the deflection scale to be 1 inch equal to $0.032 \times 31\frac{1}{4} = 1.0$ inch. With the pole P' the curve (e') is drawn. This gives a deflection at the right support equal to 0.53 inch. It should be zero. The closing line $O'X'$ in (e') is drawn, and parallel to this line the ray $P'X$ in (*d*) is drawn, then the true axis OX in (*d*) is drawn, and with the pole P on this axis, with a pole distance of $32\frac{1}{4}$ inches, draw the true elastic curve (*e*). The deflections can be measured directly from this curve. The maximum deflection occurs at the point of zero slope which is 31 inches from the left support. The deflection at that point is $\Delta = 0.63$ inch.

If it is desired to find the maximum deflection for a 50-pound load divide the value for 150 pounds by 3; this gives 0.21 inch. For any beam with any concentrated load at the one-third point this set of curves can be used simply by changing the scale to agree with the data of the given beam.

PROBLEMS

1. Draw the load, shear, moment, slope, and deflection curves and determine the maximum deflection and the maximum fiber stress for the following beams:

(a) A 4-inch by 8-inch timber beam used as a cantilever of 8-ft. span with a concentrated load of 500 pounds at the end.

(b) A 15-inch, 42-pound, cantilever I-beam of 10-ft. span carrying a uniform load of 15,700 pounds.

(c) A simple timber beam of 14-ft. span, 8 inches wide and 14 inches deep carrying a load of 1550 pounds concentrated at the center.

(d) Same as (c) with an additional uniform load of 200 pounds per foot.

(e) A 12-inch, 31.5-pound I-beam of 16-ft. span when fixed at both ends and carrying a concentrated load of 2400 pounds at the center.

(f) An 18-inch, 55-pound I-beam used as a simple beam of 20-ft. span carrying a uniform load of 60,000 pounds.

(g) A simple timber beam of 10-ft. span, 10 inches wide and 12 inches deep carrying a uniform load of 8000 pounds and a concentrated load of 2000 pounds at the center.

2. In a test of a $1\frac{1}{2}$-inch by 2-inch yellow pine beam of 6-ft. span the following maximum deflections for the corresponding loads at the center were observed:

Load, pounds.	Deflection, inches.
50	.218
100	.374
150	.562
200	.718
250	.905

What is the modulus of elasticity of the yellow pine?

Ans. 2,040,000 lb. per sq. in.

3. What is the bending moment at the walls and at the center of a beam fixed at both ends of 16-ft. span, and carrying a concentrated load of 8300 pounds at the center? What is the maximum fiber stress developed if a 10-inch, 25-pound I-beam is

used? What is the fiber stress 4 feet from the walls? What is the shearing stress at the section 4 feet from the walls?

4. Design a longleaf pine beam with both ends fixed to carry a uniform load of 6000 pounds on a span of 12 feet. What will be the maximum fiber stress developed at the center of the span? Locate the inflection points.

5. What steel I-beam with fixed ends is required for a span of 20 feet to support a uniform load of 20,000 pounds, with a maximum unit-stress of 15,000 pounds per square inch? Find also the maximum deflection.

6. For a simple beam with a load concentrated at the distance *kl* from the left support, *k* being a fraction, show that the equation of the elastic curve to the left of the load is

$$y = \frac{W}{6\,EI}(1-k)\,x^3 - \frac{W}{6\,EI}(2\,k - 3\,k^2 + k^3)\,l^2x.$$

7. Design a beam of 20-ft. span to carry 18,000 pounds, fixed ends.

8. Calculate the maximum deflection of a steel bar, supported at its ends, 1 in. sq., 6 ft. long, with a load of 100 pounds at its center.

9. A floor is to support a uniform load of 100 pounds per square foot. The 10-inch, 25-pound I-beams have a span of 20 feet and are spaced 6 feet apart between centers. Does the maximum deflection of the beams exceed $\frac{1}{360}$ of the span?

10. Deduce the equation of the elastic curve and the expression for the maximum deflection for a beam on which the load varies uniformly from zero at the ends to w pounds per lineal unit at the center. Given $\Sigma\,(n^4 = [n\,(n+1)\,(6\,n^3 + 9\,n^2 + n - 1)] \div 30$.

Ans. To the left of the center,

$$y = \frac{wlx^3}{24\,EI} - \frac{wx^5}{60\,EIl} - \frac{5\,wl^3x}{192\,EI}. \qquad \Delta = -\frac{Wl^3}{60\,EI}.$$

In the following problems write the special equations of the elastic curve and obtain the maximum deflections.

11. A 12-in., 31½-pound I-beam used as a cantilever beam of 20-ft. span and carrying a concentrated load of 1000 pounds at free end.

12. A 10-inch, 25-pound I-beam used as a cantilever beam of 15-ft. span and carrying a uniform load of 500 pounds per foot.

13. A 20-inch, 65-pound I-beam used as a simple beam of 24-ft. span carrying a concentrated load of 20,000 pounds at the center.

14. An 18-inch, 55-pound I-beam used as a simple beam of 15-ft. span carrying a uniform load of 4000 pounds per foot.

CHAPTER XI

OVERHANGING, FIXED AND SUPPORTED, AND CONTINUOUS BEAMS

109. Overhanging Beam, Concentrated Loads. In Fig. 82 are drawn the shear, moment, slope, and deflection diagrams for two concentrated loads on an overhanging beam, W_1 at the left end which overhangs the support, and W_2 between the supports. After drawing curve (*b*), curve (*c*) is obtained by use of the pole P in (*b*). The bending moment is zero at the ends and also at the point of inflection I. By use of the pole P in (*c*) the slope curve (*d*) is drawn. Since the value of the slope is not known at any point, the curve (*e'*) is drawn by using the pole P' in (*d*), thus assuming the slope at the left end to be $O'A$. The supports A' and B' should be on a horizontal line. Therefore, to obtain the true elastic curve and the correct value of the slope at the left end, connect $A'B'$ in (*e'*), then draw $P'X$ in (*d*) parallel to $A'B'$; through X draw the horizontal axis OX, which is the true axis of reference for the slope curve. This gives the slope at the left end to be OA. Then by use of any pole P on the axis OX in (*d*) the true elastic curve (*e*) is drawn. This method is general and may be employed for any system of loading for cases in which the beam rests on two supports. If desired, the equations for the different parts of the elastic curve can be obtained by methods similar to those in Chapter X and the expressions for the maximum moment, the maximum deflection, and the location of the inflection point may be obtained.

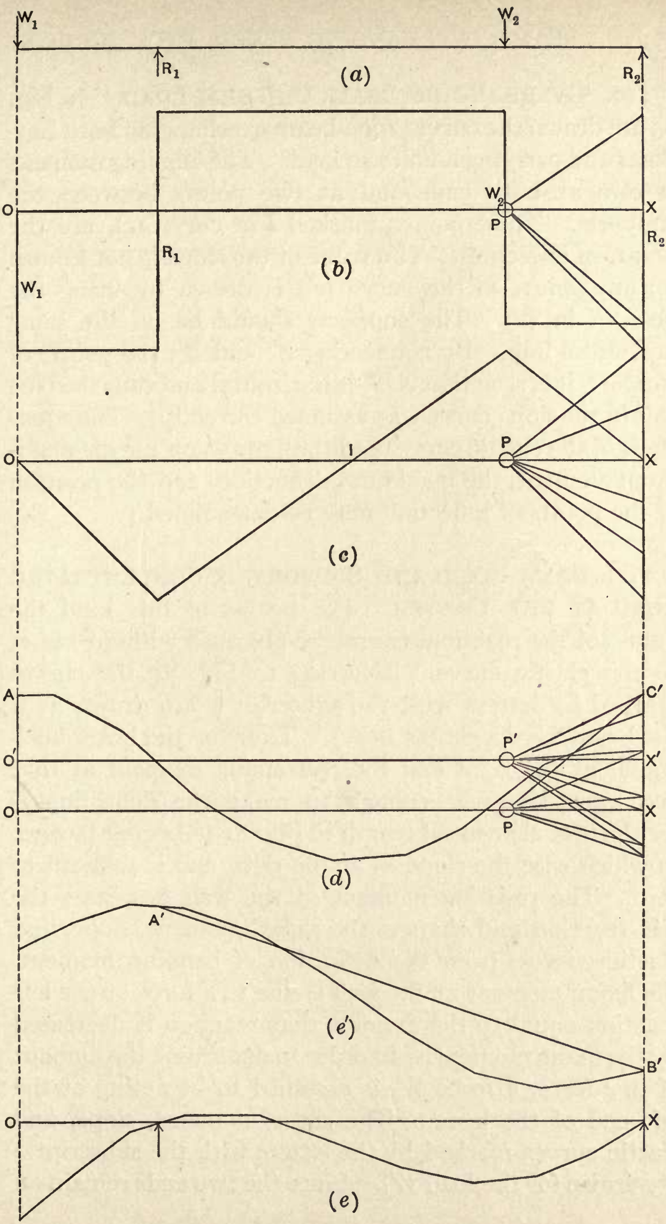


FIG. 82. OVERHANGING BEAM, CONCENTRATED LOADS.

110. Overhanging Beam, Uniform Load. In Fig. 83 are drawn the curves for a beam overhanging both supports and carrying a uniform load. The bending moment is zero at both ends and at two points between the supports. These points, marked I in curve (c), are the points of inflection. The value of the slope is not known for any point, so the curve (e') is drawn by using the pole P' in (d). The supports should be on the same horizontal line. By connecting A' and B', the points of support, it is seen that $A'B'$ is horizontal and thus the true axis in the slope curve was assumed correctly. The equations of the elastic curve, and the expression for the maximum moment, the maximum deflection, and the position of the points of inflection may be determined.

111. Beam Fixed and Supported, Concentrated Load at the Center. For beams of this kind the values of the reactions cannot be obtained without resort to the elastic curve. Referring to Fig. 84 the curves marked by letters with the subscript 1 are drawn as if the beam were a simple beam. To make the beam horizontal at the right end the restraining moment at that end must be great enough to make the deflection f_1 for the first element of length in Fig. 84 (e_1) equal to zero, in which case the slope α_1 at the right end is reduced to zero. The resisting moment at the wall decreases the left reaction and changes the shear, moment, slope, and elastic curves; from the definition of bending moment, the fixing moment at the wall is due to a force at the left reaction equal to the amount that reaction is decreased by the fixing moment. In order to determine the amount of this force, a force W_1 is assumed to be acting at the left end of the beam. The shear, moment, slope, and elastic curves marked by the letters with the subscript 2 are drawn for the load W_1. Since the two ends remain on

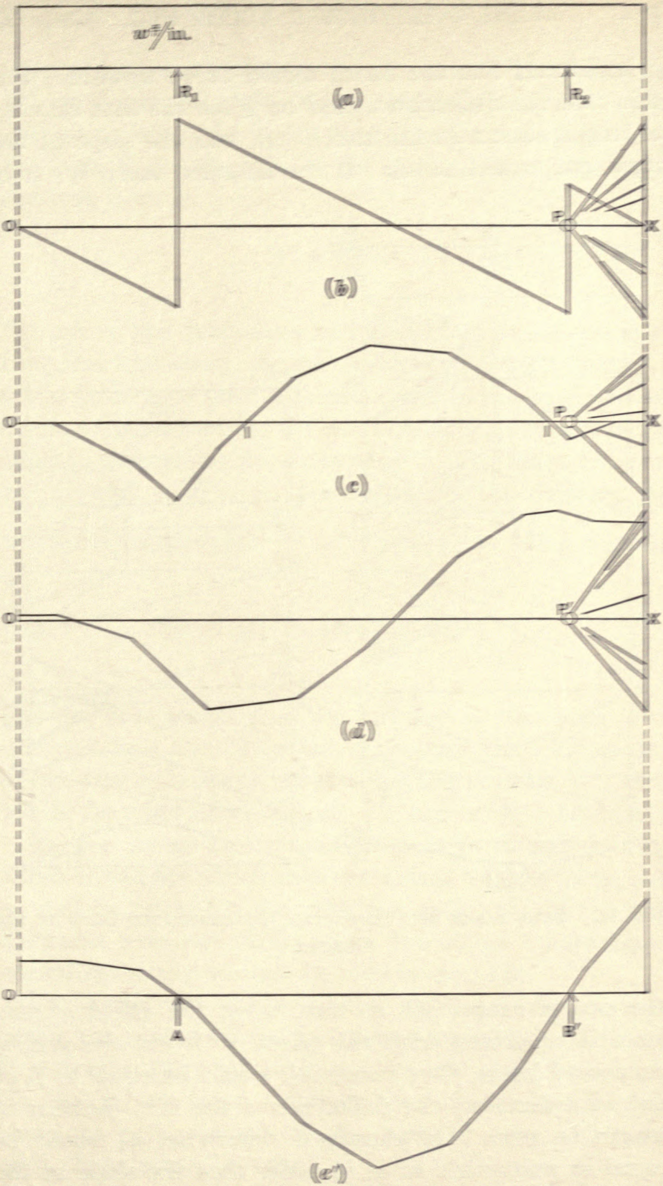


FIG. 83. OVERHANGING BEAM, UNIFORM LOAD.

a horizontal line the beam would curve upward. For this case the deflection would be f_2 for the first element of length shown in the curve (e_2), and the slope at the right end would be α_2. If the assumed force W_1 is of

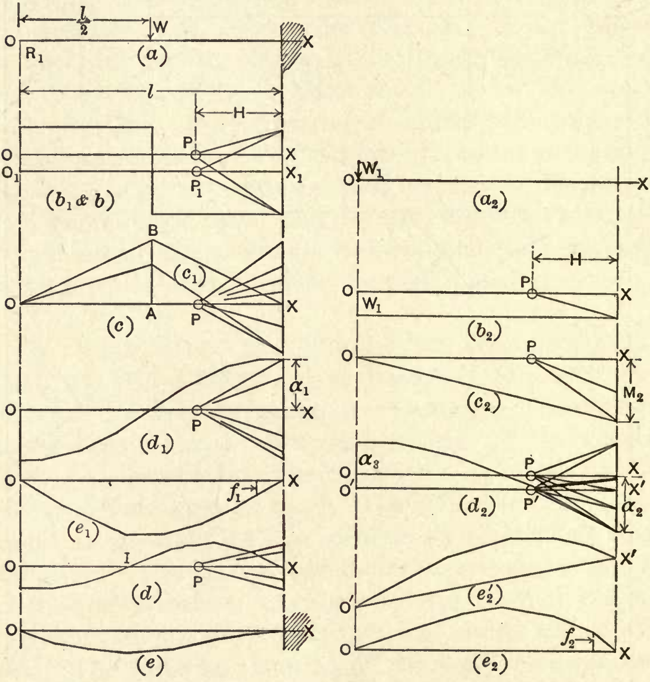


FIG. 84. BEAM FIXED AND SUPPORTED, CONCENTRATED LOAD AT THE CENTER.

the proper magnitude so that when the effect of this force is combined with the effect of W on the simply supported beam, f_2 of curve (e_2) would be equal to f_1 of (e_1) in order that the deflection of the first element of length be zero. Or, expressed otherwise, α_2 would be equal in magnitude to α_1 in order that the slope at the right end be zero under the combination. The true force W' necessary to make the moment at the wall great enough to bring the beam horizontal at that point is to the force W_1 as the ratio of the slopes α_1 to α_2; therefore the true force is

$$W' = \frac{\alpha_1}{\alpha_2} W_1.$$

By use of the definitions and equations in the previous chapters, the shear, moment, slope, and deflection of a given beam are directly proportional to the load causing them. Consequently, the reaction at the left end of the beam is lessened by the amount W'. The fixing moment at the wall is $W'l$, or it is equal to the moment M_2 multiplied by the ratio $\frac{\alpha_1}{\alpha_2}\cdot$ Therefore, the fixing moment is

$$M' = M_2 \frac{\alpha_1}{\alpha_2} = W_1 \frac{\alpha_1}{\alpha_2} l = W'l.$$

For the true reaction at the left end of the beam with one end fixed and the other supported, the left reaction of the simple beam is reduced by the amount W', after which the true curves (*b*), (*c*), (*d*), and (*e*) may be drawn.

In the foregoing solution the pole distances were taken equal, for which case the actual lengths for α_1 and α_2 may be taken for the reduction ratio. If all the pole distances were not taken equal, the actual values represented by α_1 and α_2 must be used in the ratio.

To obtain the values of the reactions, the moment under the load, and the restraining moment at the wall, it is known that the positive area and negative area in Fig. 84 (d_2) are equal, since the total change in deflection over the entire length of the beam is zero. In order that these areas be equal:

$\alpha_2 = 2\,\alpha_3$ (Art. 105).

$\alpha_2 + \alpha_3 = \dfrac{M_2}{EI} \times \dfrac{l}{2} = \dfrac{W_1 l^2}{2\,EI}$, (area in (c_2) divided by EI).

$\alpha_2 = \dfrac{W_1 l^2}{3\,EI}$.

$\alpha_1 = \dfrac{W l^2}{16\,EI}$, (area ABX in (c_1) divided by EI).

$$\therefore \quad W' = W_1 \frac{\alpha_1}{\alpha_2} = W_1 \times \frac{\dfrac{Wl^2}{16\,EI}}{\dfrac{W_1 l^2}{3\,EI}} = \frac{3}{16} W.$$

$$R_1 = \frac{W}{2} - \frac{3}{16} W = \frac{5}{16} W.$$

$$R_2 = W - \frac{5}{16} W = \frac{11}{16} W.$$

$$M' = -\frac{3}{16} Wl \text{ (at the wall).}$$

$$M = \frac{5}{32} Wl \text{ (under the load).}$$

The inflection point occurs at the point of zero moment which is $\frac{3}{11} l$ from the fixed end.

112. Beam, Both Ends Fixed, Concentrated Load at Any Point. The method of the last article may be followed for a beam fixed at one end and supported at the other with any system of loading. When both ends are fixed and the loading is not symmetrical a method quite similar to that of the last article may be followed, but instead of finding the fixing moment at one end only, it is necessary to determine the fixing moment at both ends. In Fig. 85 draw the curves (b_1), (c_1), (d_1), and (e_1) as for a simple beam. The resisting moments at the walls are proportional to α_1 and α_2 if the slopes at these points are reduced to zero by the resisting moments. Assume a force W_1 acting at the left end, producing a moment at the right wall, and draw the curves (b_2), (c_2), and (d_2) for this load. If the assumed force were

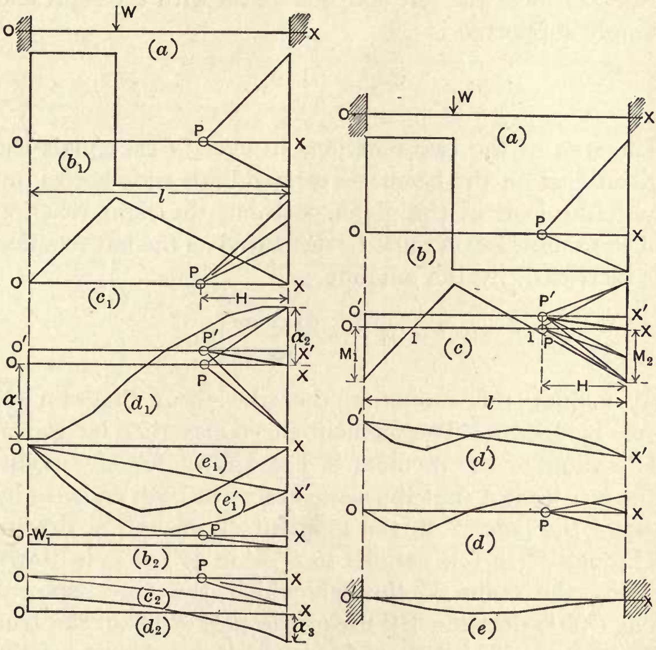


FIG. 85. BEAM FIXED AT BOTH ENDS, CONCENTRATED LOAD AT ANY POINT.

enough to make the beam horizontal at the right end α_3 would have been equal to α_2. If the force were taken at the other end and had been of the right magnitude to make the beam horizontal at the left end, α_3 would have been equal to α_1. Therefore, the amount the force or reaction acting on the left end is lessened to make the beam horizontal at the right end with the left one simply supported is

$$W_1' = \frac{\alpha_2}{\alpha_3} W_1,$$

and the amount the right reaction would be decreased in order to have the left end horizontal with the right end simply supported is

$$W_2' = \frac{\alpha_1}{\alpha_3} W_1.$$

The sum of the two reactions in every case equals the given load on the beam. To have both ends horizontal with the load in the given position, the right reaction of the simple beam must be decreased as the left reaction is increased, by the amount

$$W_2' - W_1' = \frac{\alpha_1 - \alpha_2}{\alpha_3} W_1.$$

By making this reduction the true shear diagram (*b*) may be drawn. The moment curve may then be drawn. The value of the moment is not known for any point, but it is known that the slope is zero at both ends, so by use of the pole P' in the moment curve, (d') is drawn. Through P' in (*c*), parallel to $O'X'$ in (d') $P'X$ is drawn giving the point X through which the true moment axis OX is drawn. By use of the pole P in (*b*) the true slope curve (*d*) is drawn from which, by the use of the pole P, the deflection curve (*e*) is drawn.

113. CONTINUOUS BEAMS. The definitions and general equations given in the foregoing chapters are applicable to continuous beams as well as the general method of determining the elastic curves. The reactions of continuous beams are determined by the use of the principles involved in determining the elastic curve.

For a given beam the reactions may be determined graphically, but that is left for a more advanced treatment of the subject. In the analytical treatment of continuous beams two spans are usually considered. Let Fig. 86 (*a*) represent two spans of a continuous

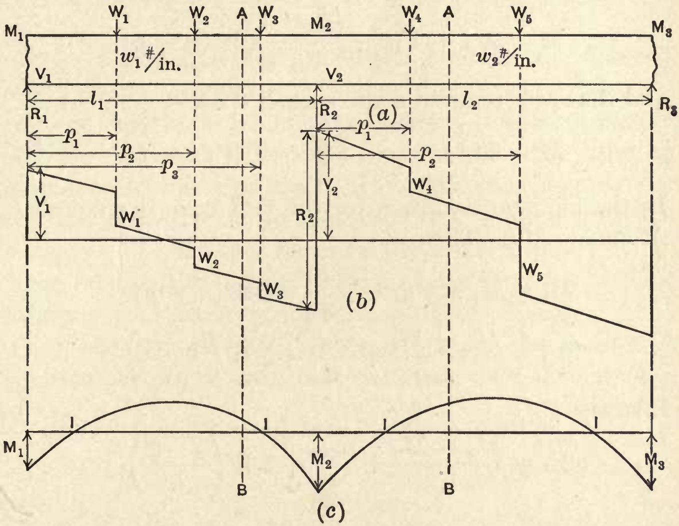


FIG. 86.

beam, and (*b*) and (*c*) represent the shear and moment curves for the same two spans. The spans are taken equal to l_1 and l_2; the uniform loads are w_1 and w_2 per unit of length, with several concentrated loads. Let the vertical shear just to the right of the left support be V_1 and that just to the right of the second support be V_2. Let the bending moment at the supports be M_1, M_2, and M_3.

From the definition of the vertical shear for a section, the vertical shear for the section AB in the first span is

$$V = V_1 - w_1x - \Sigma W,$$

and that for a section in the second span is

$$V = V_2 - w_2x - \Sigma W.$$

From the definition of bending moment the bending moment at the section AB in the first span is

$$M = M_1 + V_1x - \frac{w_1x^2}{2} - \Sigma W(x - p),$$

and that at a section in the second span is

$$M = M_2 + V_2x - \frac{w_2x^2}{2} - \Sigma W(x - p).$$

In the moment equation for the first span if x equals l_1, M equals M_2. Then

$$M_2 = M_1 + V_1l_1 - \frac{w_1l_1^2}{2} - \Sigma W(l_1 - p),$$

$$V_1 = \frac{M_2 - M_1}{l_1} + \frac{w_1l_1}{2} + \Sigma W\left(\frac{l_1 - p}{l_1}\right).$$

Likewise

$$V_2 = \frac{M_3 - M_2}{l_2} + \frac{w_2l_2}{2} + \Sigma W\left(\frac{l_2 - p}{l_2}\right).$$

By comparing the value of V_1 with the value of the vertical shear just at the right of the left support of a simple beam it is seen that when there are bending moments over the supports the value of the vertical shear is increased by the amount $\frac{M_2 - M_1}{l}$. The value of V_1 obtained shows that if the bending moments at the supports are known the vertical shear to the right of each support may be obtained, and the value of the vertical shear and the bending moment for any section then may be found. If the dimensions of the beam and the load are given, by use of the shear and moment formulas the stresses developed in the beam may be calculated, or, if the load is given, a beam may be designed to carry the load. The points of inflection may also be obtained by finding where the bending moment equals zero. Consequently, the subject of the investigation and design of continuous beams consists primarily of determining the bending moments at the supports.

114. THE THEOREM OF THREE MOMENTS. Instead of giving the graphical method of determining the reactions of continuous beams, the theorem of three moments will be discussed. In order to determine the bending moments at the supports of a continuous beam, or the relation between them, the system of loading must be known. The relation between the bending moments at three consecutive supports may be deduced for various systems of loading. (See Example 7, Chapter XII.) The "theorem of three moments" which expresses this relation for beams with uniform loads over each span is

$$M_1l_1 + 2\,M_2(l_1 + l_2) + M_3l_3 = -\tfrac{1}{4}(w_1l_1^3 + w_2l_2^3),$$
$$= -\tfrac{1}{4}(W_1l_1^2 + W_2l_2^2)$$

where M_1, M_2, and M_3 are the moments over the supports, w_1 and w_2 are the values of the uniform loads, and

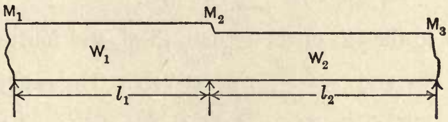


FIG. 87.

l_1 and l_2 are the spans. (See Fig. 87.) In applying the theorem to a given beam, unless restrained, it is known that the bending moment at the end support is zero. As many equations as there are bending moments may be written and those equations solved simultaneously to determine the bending moments at the supports.

For beams of equal spans and a uniform load over the entire length of the beam the theorem reduces to

$$M_1 + 4\,M_2 + M_3 = -\frac{wl^2}{2} = -\frac{Wl}{2}.$$

ILLUSTRATIVE EXAMPLE

Given a beam carrying a uniform load over three equal spans, to determine the bending moment at the supports, the vertical shear at the supports, the reactions, the maximum positive bending moment, and the inflection points. See Fig. 88.

From symmetry $M_1 = M_4 = 0$, and $M_2 = M_3$. Making these substitutions in the theorem of three moments:

$$0 + 4\,M_2 + M_3 = -\frac{wl^2}{2}$$

$$M_2 + 4\,M_3 + 0 = -\frac{wl^2}{2}$$

$$\therefore\ M_2 = M_3 = -\frac{wl^2}{10}$$

$$V_1 = -\frac{wl}{10} + \frac{wl}{2} = \frac{4\,wl}{10}$$

$$V_2 = \frac{wl}{2}$$

$$V_3 = \frac{wl}{10} + \frac{wl}{2} = \frac{6\,wl}{10}$$

The shear to the left of the second, third, and fourth supports is $-\frac{6\,wl}{10}$, $-\frac{wl}{2}$, and $-\frac{4\,wl}{10}$, respectively. The reaction equals the algebraic difference of the shear to the right and to the left of the support. Therefore,

$$R_1 = R_4 = \frac{4\,wl}{10},$$

$$R_2 = R_3 = \frac{11\,wl}{10}.$$

The maximum positive bending moment occurs between the supports where the vertical shear is zero. To obtain the position of zero shear let V equal zero.

$$V = \frac{4\,wl}{10} - wx_1 = 0. \qquad \therefore\; x_1 = \frac{4}{10}l,$$

$$M_m = 0 + \frac{4\,wl}{10} \times \frac{4\,l}{10} - \frac{w}{2}\left(\frac{4\,l}{10}\right)^2 = \frac{16\,wl^2}{200}.$$

Since the moment is zero at the left support it will be zero again where the shear areas above and below the axis are equal, which is at the distance $2\,x_1$ or $\frac{8}{10}l$ from the left support. This gives the inflection point. In the middle span the shear is

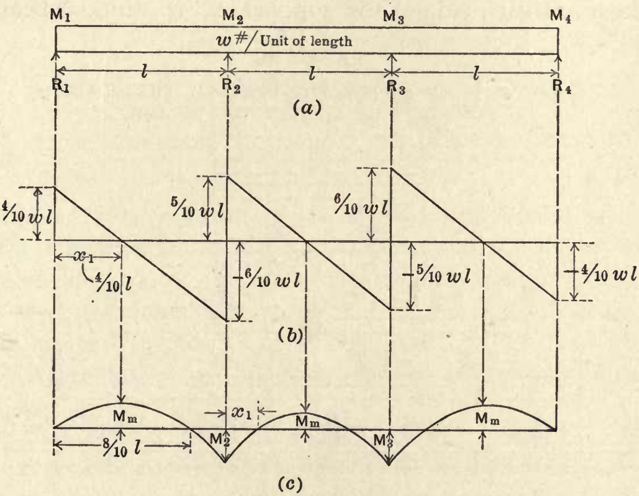


FIG. 88.

zero at the center. The area under the shear curve to the center is $\frac{wl}{2} \times \frac{l}{2} = \frac{wl^2}{4}$; then the bending moment at the center is $-\frac{wl^2}{10} + \frac{wl^2}{4} = \frac{3\,wl^2}{20}$. The inflection points occur where the bending moment passes through zero. Therefore, the distance from the second support to the inflection point is found from:

$$-\frac{wl^2}{10} + \frac{wl}{2}x - \frac{wx_1^2}{2} = 0.$$

$$x_1 = \frac{l}{2}\left(1 \pm \frac{1}{\sqrt{5}}\right).$$

The slope and the deflection may be obtained by graphical methods or by the calculation of areas as is done in the previous chapters.

In Fig. 88 (*b*) is drawn the shear diagram and in Fig. 88 (*c*) is drawn the moment diagram for this beam.

By a method similar to that employed in the foregoing example the coefficients for wl for the vertical shears at each side of the supports for continuous beams

TABLE 16

COEFFICIENTS OF wl FOR THE VERTICAL SHEAR AT THE SUPPORTS OF CONTINUOUS BEAMS.

No. Spans							
2	$\frac{3}{8}$	$-\frac{5}{8}$ ¦ $\frac{5}{8}$	$-\frac{3}{8}$				
3	$\frac{4}{10}$	$-\frac{6}{10}$ ¦ $\frac{5}{10}$	$-\frac{5}{10}$ ¦ $\frac{6}{10}$	$-\frac{4}{10}$			
4	$\frac{11}{28}$	$-\frac{17}{28}$ ¦ $\frac{15}{28}$	$-\frac{13}{28}$ ¦ $\frac{13}{28}$	$-\frac{15}{28}$ ¦ $\frac{17}{28}$	$-\frac{11}{28}$		
5	$\frac{15}{38}$	$-\frac{23}{38}$ ¦ $\frac{20}{38}$	$-\frac{18}{38}$ ¦ $\frac{19}{38}$	$-\frac{19}{38}$ ¦ $\frac{18}{38}$	$-\frac{20}{38}$ ¦ $\frac{23}{38}$	$-\frac{15}{38}$	
6	$\frac{41}{104}$	$-\frac{63}{104}$ ¦ $\frac{55}{104}$	$-\frac{49}{104}$ ¦ $\frac{51}{104}$	$-\frac{53}{104}$ ¦ $\frac{53}{104}$	$-\frac{51}{104}$ ¦ $\frac{49}{104}$	$-\frac{55}{104}$ ¦ $\frac{63}{104}$	$-\frac{41}{104}$

(Each span = 1.)

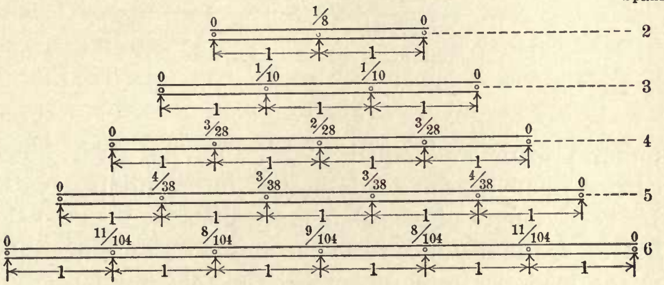
carrying uniform loads over the entire length were obtained as given in Table 16. The negative coefficients of wl^2 for the bending moments at the supports were also obtained as given in Table 17.

Tables 16 and 17 may be extended in the following manner: By following down to the right or to the left a line of similar supports for the different spans, to obtain the coefficients for a beam having an odd number of spans, as five, for the second support, the moment coefficient is $\frac{4}{38}$. The 4 is obtained by adding the 3 of $\frac{3}{28}$ to the 1 of $\frac{1}{10}$. The 38 is obtained by adding 28 of $\frac{3}{28}$ to the 10 of $\frac{1}{10}$. This method may be employed for any

TABLE 17

COEFFICIENTS OF $-wl^2$ FOR THE BENDING MOMENT AT THE SUPPORTS OF CONTINUOUS BEAMS.

No. Spans



support of any beam with an odd number of spans. For a beam with an even number of spans, as four, the coefficient is $\frac{3}{28}$. The 3 is obtained by multiplying the 1 of $\frac{1}{10}$ by 2 and adding the 1 of $\frac{1}{8}$. The 28 is obtained by multiplying the 10 of $\frac{1}{10}$ by 2 and adding the 8 of $\frac{1}{8}$. This method can be followed for extending either Table 16 or 17.

115. Hinging Points for Continuous Beams. If a continuous beam is to be made of several parts, it is necessary to know at what points the various parts should be hinged, in order that the "continuous" effect may be secured, as a continuous beam is stronger than several simple beams over the various spans. Any given continuous beam may be hinged at the inflection points, and the bending moment would be unchanged along the beam.

An economical method is to hinge the beam at such points as to make the maximum negative bending moments at the supports and the maximum positive bending moments in the spans equal in magnitude. The portion of the beam between the hinges in a span acts as a simple beam and the portions from the support to the hinges act as cantilever beams. For the case of

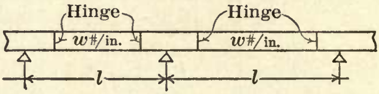


FIG. 89.

uniform loads and equal spans, Fig. 89, each hinge carries one-half the load on the intermediate length. If l is the length of one span, l_1 the distance between the hinges, l_2 the distance from the support to a hinge, and w the load per unit of length, the maximum bending moment in the center is $\frac{wl_1^2}{8}$ and the maximum bending moment at the support is $-\left(\frac{wl_2^2}{2}+\frac{wl_1l_2}{2}\right)$. For equal maximum bending moments

$$\frac{wl_1^2}{8}=\frac{wl_2^2}{2}+\frac{wl_1l_2}{2}.$$

$$l_1^2-4\,l_2^2-4\,l_1l_2=0.$$

$$\frac{l_1^2}{l_2^2}-4\frac{l_1}{l_2}-4=0.$$

$$\frac{l_1}{l_2}=(2+\sqrt{8})=4.284.$$

$$l_1+2\,l_2=l.$$

$$l_2=\frac{l}{4+\sqrt{8}}=\frac{l}{2}-\frac{l}{\sqrt{8}}=.14644\,l.$$

$$l_1=\frac{2+\sqrt{8}}{4+\sqrt{8}}l=\frac{l}{\sqrt{2}}=.70712\,l.$$

From this relation the maximum bending moment is found to be

$$M_m = \frac{wl_1^2}{8} = \frac{Wl}{16}.$$

Thus it is seen that if the beams are hinged at the proper points the efficiency is increased from thirty to sixty per cent. To use beams hinged in this way they should be fixed at the end supports. With uniform load the beams would remain horizontal at the supports, but if the load is not uniform at any time, the beam should be fixed at all the supports. Two lengths of beams could be used, one length about three-tenths the length of one span, to be used over the supports, and the other length about seven-tenths the length of one span, to be used between the hinges. If the loads are concentrated at the middle of the spans the lengths should be made equal.

PROBLEMS

1. Draw the shear, moment, slope, and elastic curves for a 9-inch, 21-pound I-beam of length 20 feet, overhanging each support 4 feet, carrying concentrated loads of 10,000 pounds at the left end, 12,000 pounds 8 feet from the left support, and 15,000 pounds at the right end. From the curves determine the deflection at each load and the maximum deflection.

2. What are the maximum shearing and fiber stresses developed in the beam of Problem No. 1?

3. Design a rectangular Washington fir beam 18 feet long, overhanging one support 4 feet, to carry a total uniform load of 9000 pounds. The shearing unit-stress is not to exceed 100 pounds

per square inch, the maximum fiber stress is not to exceed 1200 pounds per square inch, and the maximum deflection is not to exceed $\frac{1}{360}$ of the span between the supports.

4. Draw the shear, moment, slope, and elastic curves of a beam fixed at one end and supported at the other, of length l, carrying a uniform load of w pounds per lineal inch, and determine the value of the reactions, the restraining moment at the wall, the maximum positive moment, and the elastic curve.

$$\textit{Ans.}\quad \frac{3}{8}\,wl;\quad -\frac{wl^2}{8};\quad \frac{9}{128}\,wl^2;$$

$$y = \frac{wx^4}{24\,EI} + \frac{wlx^3}{16\,EI} - \frac{wl^3x}{48\,EI}.$$

5. Draw the shear, moment, slope, and elastic curves for an 8-inch by 10-inch beam of 12-ft. span fixed at one end and supported at the other, carrying a concentrated load of 8000 pounds 7 feet from the restrained end. What are the maximum shearing and fiber stresses developed in the beam?

6. Solve Problem No. 5 if both ends are fixed.

7. A continuous beam of two spans carries a load of 100 pounds per foot over one span of 12 feet and 200 pounds per foot over the other span of 8 feet. Determine the moment at the middle support and the reactions.

Ans. $M_2 = 20{,}640$ lb.-in., $R_1 = 457$ lb., $R_2 = 1758$ lb., $R_3 = 585$ lb.

8. Determine the bending moment at the middle support and the maximum positive bending moments in each span of a beam 24 feet long, one span being 10 feet and the uniform load for that span 24,000 pounds, the other span being 14 feet and the uniform load for that span 28,000 pounds. Select the proper **I**-beam for this loading.

9. Select the proper continuous **I**-beam to carry a uniform load of 144,000 pounds uniformly distributed over six spans of 12 feet each.

10. If the beam of Problem No. 9 were fixed at the supports and hinged so as to make the bending moment at the supports equal to that at the middle of the span, what **I**-beam would be required? What would be the length of each section?

CHAPTER XII

ELASTIC CURVE OF BEAMS DETERMINED BY THE ALGEBRAIC METHOD *

116. The Algebraic Relations between the Five Curves. As deduced in Art. 89 the expression for the radius of curvature of a beam is

$$r = \frac{EI}{M} \tag{1}$$

where E is the modulus of elasticity, I is the moment of inertia of the cross section about the neutral axis, and M is the bending moment. The algebraic expression for the radius of curvature for a curve as deduced in the calculus is

$$r = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \tag{2}$$

where x and y are the coördinates of the point of the given curve, for which r is the radius of curvature and $\frac{dy}{dx}$ is the slope of the tangent to the curve at the given point. For beams the X-axis is horizontal and the Y-axis is vertical, and since the slope of the elastic curve is small at all points of the beam the value of $\left(\frac{dy}{dx}\right)^2$

* This chapter introduces the calculus method for the only time and is intended only for students who have had courses in differential and integral calculus.

generally is very small, and in comparison with 1 may be neglected, with which approximation

$$r = \frac{1}{\frac{d^2y}{dx^2}}. \tag{3}$$

By substituting this value of r in equation (1) there results

$$EI\frac{d^2y}{dx^2} = M. \tag{4}$$

By combining this relation with those given in Chapters V and IX the following values for the section distant x from the origin result:

The ordinate to the elastic curve is

$$y = f(x). \tag{a}$$

The slope of the elastic curve is

$$\alpha = \frac{dy}{dx}. \tag{b}$$

The bending moment is

$$M = EI\frac{d\alpha}{dx} = EI\frac{d^2y}{dx^2}. \tag{c}$$

The vertical shear is

$$V = \frac{dM}{dx} = EI\frac{d^2\alpha}{dx^2} = EI\frac{d^3y}{dx^2}. \tag{d}$$

The load per unit of length is

$$w = \frac{dV}{dx} = \frac{d^2M}{dx^2} = EI\frac{d^3\alpha}{dx^3} = EI\frac{d^4y}{dx^4}. \tag{e}$$

If the value of any one of the variables is known for the above equations, the values of those lower in the scale may be determined by differentiation as indicated, but usually it is necessary to start with the lower equations and derive the higher ones by integration. In problems concerning beams the operation of integration between definite limits is not generally applied, consequently each operation introduces a constant of integration which must be determined from the known conditions governing the case. For deriving higher curves the equations may be written in the following form:

The load per unit of length is

$$w = w. \tag{1}$$

The vertical shear is

$$V = \int w dx + V_1. \tag{2}$$

The bending moment is

$$M = \int V dx + M_1 = \int\int w dx^2 + \int V_1 dx + M_1. \tag{3}$$

The slope is

$$\alpha = \int \frac{M dx}{EI} + \alpha_1$$

$$= \frac{1}{EI}\left(\int\int\int w dx^3 + \int\int V_1 dx^2 + \int M_1 dx\right) + \alpha_1. \tag{4}$$

The deflection is

$$y = \int \alpha dx + y_1$$

$$= \frac{1}{EI}\left(\int\int\int\int w dx^4 + \int\int\int V_1 dx^3 + \int\int M_1 dx^2\right)$$

$$+ \int \alpha_1 dx + y_1. \tag{5}$$

The method of evaluating these expressions will be given later.

The latter set of equations is the one to be employed in determining the elastic deflections. Any one of the equations may be used to start with, if the variables can be expressed in terms of x. The load, shear, and moment equations can usually be written by applying the definitions. If the moment equation is used to start with, one integration and the determination of one constant of integration are avoided, but since these operations are of the simplest in calculus there is no advantage in starting with any other than the load or shear equation. The constants should be determined as they appear if convenient. In the case of concentrated loads the equation of the load curve is zero, and the shear curve probably would be the best with which to start.

117. THE CHOICE OF COÖRDINATE AXES. In the deduction of the formula for the radius of curvature $r = \frac{EI}{M}$, the X-axis was taken parallel to the axis of the beam before bending, and the Y-axis at right angles to the X-axis. The origin may be chosen arbitrarily, and for some particular cases it is more convenient to take the origin at the center of the beam, but in this book the X-axis will be taken to coincide with the axis of the beam before the beam is bent, and the Y-axis will be taken at right angles to the X-axis at the left end of the span under consideration. In the solutions the proper algebraic signs should be observed.

118. THE CONSTANTS OF INTEGRATION. In all cases an approximate diagram of the deflected beam will be of value in determining the constants of integration. For problems in the determination of the deflection of beams, the constant of integration for any curve is the value of the variable at the origin, as here treated. Thus, V_1, introduced in equation (2), Art. 116, is the value of the vertical shear at the origin. See Fig. 90. M_1 introduced in equation (3) is the value of the bending moment at the origin; α_1 introduced in equation (4) is the value of the slope at the origin; y_1 introduced in equation (5) is the value of the deflection at the origin.

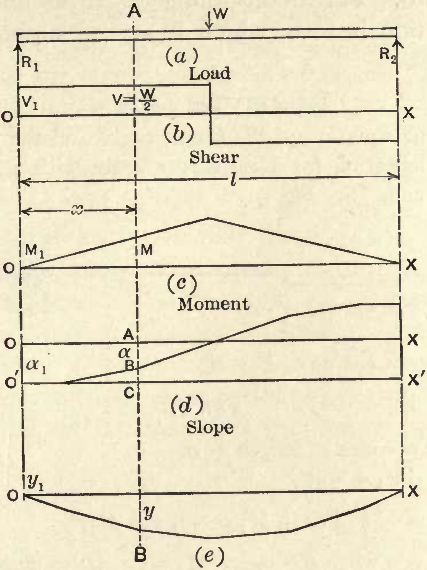


FIG. 90.

119. DETERMINATION OF THE CONSTANTS OF INTEGRATION. If the values of the constants can be determined, they may be inserted into the equations at once; thus, V_1 and M_1 can be determined in many cases at first. For other cases it may be known where the shear is zero, and then the value of zero, for V and the corresponding value of x may be substituted in equation (2) to give the value of V_1. If the position of zero bending moment is known, the value of M and the corresponding value of x substituted in equation (3) will give M_1. Likewise to determine α_1 it may be known where the slope is zero, i.e. where the beam is horizontal, and that value of α and the corresponding value of x substituted in equation (4) will give the value of α_1. For determining y_1 it is known where the deflection is zero, which is at the supports for the usual cases.

ILLUSTRATIVE EXAMPLE

Deduce the equation of the elastic curve and the value of the maximum deflection for a cantilever beam with a concentrated load at the end. See Fig. 91.

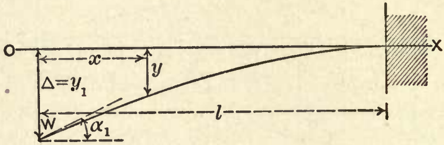


FIG. 91.

The load per unit of length = 0.

$$V = V_1 = - W,$$

$$M = - \int W dx + M_1 = - Wx + M_1,$$

$$M = 0 \text{ when } x = 0, \quad \therefore \quad M_1 = 0. \quad \text{(Zero moment)},$$

$$\alpha = - \frac{1}{EI} \int Wx dx + \alpha_1 = - \frac{1}{2\,EI} Wx^2 + \alpha_1.$$

The slope α equals zero when x equals l, as the beam is horizontal at the wall, therefore

$$0 = - \frac{Wl^2}{2\,EI} + \alpha_1; \quad \alpha_1 = \frac{Wl^2}{2\,EI}. \quad \text{(Zero slope)},$$

$$y = - \frac{1}{2\,EI} \int Wx^2 dx + \frac{1}{2\,EI} \int Wl^2 dx + y_1,$$

$$y = - \frac{Wx^3}{6\,EI} + \frac{Wl^2 x}{2\,EI} + y_1,$$

$$y = 0, \text{ for } x = l,$$

$$\therefore \quad 0 = - \frac{Wl^3}{6\,EI} + \frac{Wl^3}{2\,EI} + y_1; \quad y_1 = - \frac{Wl^3}{3\,EI}. \quad \text{(Zero deflection)},$$

$$y = - \frac{Wx^3}{6\,EI} + \frac{Wl^2 x}{2\,EI} - \frac{Wl^3}{3\,EI}.$$

The maximum deflection occurs where $x = 0$, and is

$$\Delta = - \frac{Wl^3}{3\,EI}.$$

When there are concentrated loads on the beam the shear equation changes at every concentrated load, consequently the equations for the moment, slope, and elastic curves have different expressions on each side of the load, and for each of the curves there is one more equation than there are concentrated loads on the beam. Care should be taken in substituting values of x in these equations to see that the equation is true for the particular value of x used. With concentrated loads the two sections of the beam on each side of the load have a common slope at the load, and also a common deflection. For continuous beams and overhanging beams the two sections on each side of a support have a common slope and a common deflection at the support.

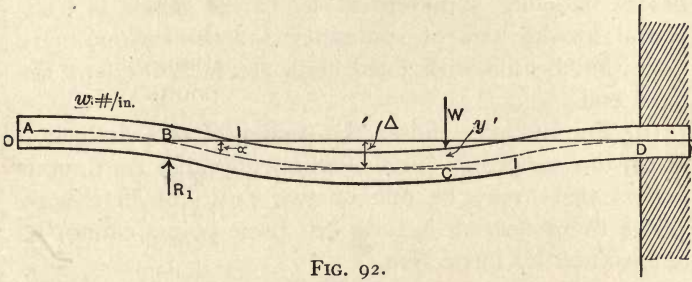


FIG. 92.

Thus, in Fig. 92 the portions of the elastic curve AB and BC have a common tangent (i.e. a common slope) and a common deflection at the point B. Also the portions BC and CD have a common tangent and a common deflection at the point C. The beam is fixed at the point D, hence the slope of the portion CD is zero at the wall.

The following principles, then, may be used in the determination of the constants of integration:

(a) The **section of zero vertical shear** can be obtained by drawing the shear diagram, and if it occurs at a point where there is no concentrated load or reaction the corresponding value of x may be used in the shear equation, with V equal to zero and the value of the constant V_1 determined. However, this substitution will seldom be necessary, as the value of V_1 will usually be determined by other methods.

(b) The **section of zero bending moment** will be at the free ends of beams, as at A, Fig. 92, and at the ends supported without restraint; also at the points of inflection for overhanging, continuous, and restrained beams, as at I, Fig. 92, but the inflection points in such beams cannot be obtained by inspection.

(c) The **section of zero slope** is at the horizontal portion of the beam, as at D, Fig. 92. For symmetrical beams carrying symmetrical loads the beam is horizontal at the axis of symmetry (at the center). By definition beams with fixed ends are horizontal at the fixed ends.

(d) For the axes chosen the **section of zero deflection** is at the supports. For overhanging and continuous beams there may be one or two positions in a span where the deflection is zero, but these points cannot be determined by inspection.

120. ESSENTIAL QUANTITIES TO BE KNOWN ABOUT BEAMS. In all kinds of beams the important things to be obtained are the position and magnitude of the maximum stresses and the maximum deflection. For overhanging, continuous, and fixed beams the inflection points need to be found. When the maximum vertical shear is determined, the maximum shearing stress is then obtained by use of the shear formula $s = \frac{V}{kA}$. When the maximum bending moment is found the maximum fiber stress developed may be obtained by use of the moment formula $f = \frac{Mc}{I}$. These formulas may be used to determine the safe load for a given beam, and also to design beams. Building specifications usually state that the maximum deflection shall not exceed a given amount, therefore it is necessary to be able to determine the maximum deflection for beams.

EXAMPLES

1. Deduce the equation of the elastic curve and the maximum deflection for a cantilever beam with a uniform load of w pounds per lineal unit. See Fig. 93.

The load per lineal unit is $-w$.

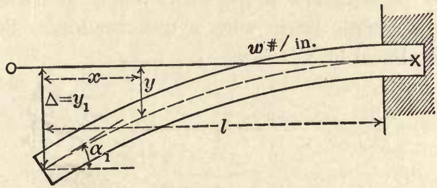


FIG. 93.

$$V = -\int wdx + V_1 = -wx + V_1.$$

When $x = 0$, $V = 0$, $\therefore$ $V_1 = 0$. (Zero shear),

$$M = \int Vdx + M_1 = -\int wxdx + M_1 = -\frac{wx^2}{2} + M_1.$$

When $x = 0$, $M = 0$, $\therefore$ $M_1 = 0$. (Zero moment),

$$\alpha = \frac{1}{EI}\int Mdx + \alpha_1 = -\frac{1}{EI}\int \frac{wx^2}{2}dx + \alpha_1 = -\frac{wx^3}{6EI} + \alpha_1.$$

When $x = l$, $\alpha = 0$,

$$\therefore \quad 0 = -\frac{wl^3}{6EI} + \alpha_1 \text{ and } \alpha_1 = \frac{wl^3}{6EI}. \quad \text{(Zero slope)},$$

$$\therefore \quad \alpha = -\frac{wx^3}{6EI} + \frac{wl^3}{6EI},$$

$$y = \int \alpha dx + y_1 = -\int \frac{wx^3 dx}{6EI} + \int \frac{wl^3 dx}{6EI} + y_1,$$

$$y = -\frac{wx^4}{24EI} + \frac{wl^3 x}{6EI} + y_1.$$

When $x = l$, $y = 0$,

$$\therefore \quad 0 = -\frac{wl^4}{24EI} + \frac{wl^4}{6EI} + y_1,$$

and $y_1 = -\frac{wl^4}{8EI}$. (Zero deflection),

$$\therefore \quad y = -\frac{wx^4}{24EI} + \frac{wl^3 x}{6EI} - \frac{wl^4}{8EI}. \quad \text{(Elastic curve).}$$

The maximum deflection is at the end, where $x = 0$, and is

$$\Delta = -\frac{wl^4}{8EI} = -\frac{Wl^3}{8EI}.$$

2. Deduce the equation of the elastic curve and the maximum deflection for a simple beam with a uniform load. See Fig. 94. The load per lineal unit is $-w$.

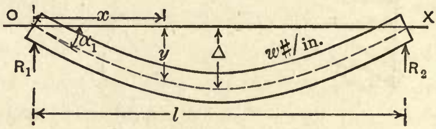


FIG. 94.

$$V = -\int wdx + V_1 = -wx + V_1.$$

When $x = 0$ the vertical shear is equal to the left reaction, which is $\frac{wl}{2}$.

$$\therefore \quad V_1 = \frac{wl}{2},$$

$$M = -\int wxdx + \int \frac{wl}{2} dx + M_1 = -\frac{wx^2}{2} + \frac{wlx}{2} + M_1.$$

When $x = 0$, $M = 0$, $\therefore$ $M_1 = 0$. (Zero moment).

$$\alpha = -\frac{1}{EI}\int \frac{wx^2}{2} dx + \frac{1}{EI}\int \frac{wlx}{2} dx + \alpha_1 = -\frac{wx^3}{6EI} + \frac{wlx^2}{4EI} + \alpha_1.$$

The beam is horizontal at the center, hence

when $x = \frac{l}{2}$, $\alpha = 0$.

$$\therefore \quad 0 = -\frac{wl^3}{48\,EI} + \frac{wl^3}{16\,EI} + \alpha_1, \quad \alpha_1 = -\frac{wl^3}{24\,EI}. \quad \text{(Zero slope)},$$

$$y = -\frac{1}{6\,EI}\int wx^3\,dx + \frac{1}{4\,EI}\int wx^2\,dx - \frac{1}{24\,EI}\int wl^3 dx + y_1,$$

$$y = -\frac{wx^4}{24\,EI} + \frac{wlx^3}{12\,EI} - \frac{wl^3x}{24\,EI} + y_1.$$

When $x = 0$ the deflection is zero, or $y = 0$, $\therefore y_1 = 0$. (Zero deflection),

$$y = -\frac{wx^4}{24\,EI} + \frac{wlx^3}{12\,EI} - \frac{wl^3x}{24\,EI}. \quad \text{(Elastic curve).}$$

The maximum deflection occurs at the center and is obtained by letting $x = \frac{l}{2}$ in the equation for y, and it is found to be

$$\Delta = -\frac{5}{384}\frac{wl^4}{EI} = -\frac{5}{384}\frac{Wl^3}{EI}.$$

3. Deduce the equation of the elastic curve, and the value of the maximum deflection for a simple beam with a concentrated

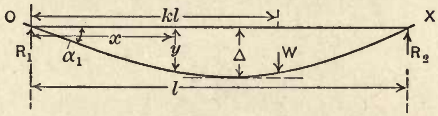


FIG. 95.

load at the distance kl from the left support in which k is a fraction. See Fig. 95.

$$R_1 = W(1 - k).$$

The load curve is at zero. To the left of the load,

$$V = W(1 - k).$$

$$M = \int W(1 - k)\,dx + M_1 = W(1 - k)\,x + M_1;$$

when $x = 0$, $M = 0$, $\therefore M_1 = 0$,

$$\therefore \quad M = W(1 - k)\,x. \qquad \text{(b)}$$

$$\alpha = \frac{W}{EI}\int (1 - k)\,x dx + \alpha_1 = \frac{W}{EI}(1 - k)\frac{x^2}{2} + \alpha_1. \qquad \text{(c)}$$

The beam is horizontal between the load and the center but for what value of x it is not known, hence we must let α_1 remain in the equations till enough conditions are obtained to determine its value.

$$y = \frac{W}{EI}\int (1-k)\frac{x^2}{2}\,dx + \int \alpha_1 dx + y_1,$$

$$y = \frac{W}{EI}(1-k)\frac{x^3}{6} + \alpha_1 x + y_1.$$

When $\quad x = 0,\ y = 0,\ \therefore\ y_1 = 0,$

$$y = \frac{W}{EI}(1-k)\frac{x^3}{6} + \alpha_1 x. \qquad \text{(d)}$$

This is all that is known about the equations to the left of the load, hence we must use those to the right of the load.

To the right of the load,

$$V = -Wk. \qquad (1)$$

$$M = -\int Wkdx + M_1' = -Wkx + M_1'.$$

When $x = l$, $M = 0$, $\therefore\ M_1' = Wkl$, and $M = Wk(l-x)$ (2)

$$\alpha = \frac{Wk}{EI}\int (l-x)\,dx + \alpha_1',$$

$$\alpha = \frac{Wk}{EI}\left(lx - \frac{x^2}{2}\right) + \alpha_1'. \qquad (3)$$

The value of α for this curve is not known for any value of x. Hence α_1' must be kept in the equation till its value can be determined.

$$y = \frac{Wk}{EI}\int \left(lx - \frac{x^2}{2}\right) dx + \int \alpha_1'\,dx + y_1,$$

$$y = \frac{Wk}{EI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + \alpha_1' x + y_1'.$$

When $\quad x = l,\ y = 0,\ \therefore\ y_1' = -\dfrac{Wkl^3}{3\,EI} - \alpha_1' l,$

$$y = \frac{Wk}{EI}\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + \alpha_1' x - \frac{Wkl^3}{3\,EI} - \alpha_1' l. \qquad (4)$$

To determine α_1 and α_1' it is known that the slope of both portions of the elastic curve at the load are the same. Therefore (c) and (3) are equal when x equals kl.

$$\frac{W}{EI}(1-k)\frac{k^2l^2}{2}+\alpha_1=\frac{Wk}{EI}\left(kl^2-\frac{k^2l^2}{2}\right)+\alpha_1'. \qquad \text{(A)}$$

This gives one relation between α_1 and α_1'. To obtain another relation between them, it is known that the deflection of both portions of the elastic curve are the same at the load. Therefore (d) and (4) are equal when x equals kl.

$$\frac{W}{EI}(1-k)\frac{k^3l^3}{6}+\alpha_1 kl=\frac{Wk}{EI}\left(\frac{k^2l^3}{2}-\frac{k^3l^3}{6}\right)+\alpha_1'kl-\frac{Wkl^3}{3\,EI}-\alpha_1'l. \qquad \text{(B)}$$

Solving equations (A) and (B) for α_1 and α_1',

$$\alpha_1=-\frac{Wl^2}{6\,EI}(2\,k+k^3-3\,k^2),$$

$$\alpha_1'=-\frac{Wl^2}{6\,EI}(2\,k+k^3).$$

To the left of the load, then,

$$y=\frac{Wx^3}{6\,EI}(1-k)-\frac{Wl^2x}{6\,EI}(2\,k+k^3-3\,k^2).$$

To the right of the load,

$$y=-\frac{Wkx^3}{6\,EI}+\frac{Wklx^2}{2\,EI}-\frac{Wl^2x}{6\,EI}(2\,k+k^3)+\frac{Wk^3l^3}{6\,EI}.$$

The value of x for the maximum deflection is obtained by equating α to zero and solving for x. If k is greater than $\frac{l}{2}$, the value of x is found to be

$$x_1=l\sqrt{\frac{2\,k+k^3-3\,k^2}{3\,(1-k)}}=l\sqrt{\frac{2\,k-k^2}{3}}.$$

This value of x substituted in the expression for y gives the maximum deflection to be

$$\Delta=\frac{Wl^3}{3\,EI}(1-k)\left(\frac{2}{3}\,k-\frac{1}{3}\,k^2\right)^{\frac{3}{2}}.$$

4. Deduce the equation of the elastic curve and determine the value of the maximum deflection and the maximum moment, and locate the inflection points for a beam fixed at both ends with a concentrated load at the center. See Fig. 96.

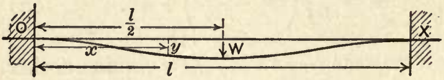


FIG. 96.

From symmetry both reactions are equal. The load curve is zero. To the left of the load,

$$V = \frac{W}{2},$$

$$M = \frac{Wx}{2} + M_1.$$

The bending moment at the wall, which is the restraining moment and keeps the beam horizontal at that point, is not known at the start. The position of zero moment is not known either, so M_1 must be retained at the present.

$$\alpha = \frac{Wx^2}{4\,EI} + \frac{M_1 x}{EI} + \alpha_1.$$

By definition of restrained beam the slope at the wall is zero, therefore when $x = 0$, $\alpha = 0$, and $\alpha_1 = 0$. From symmetry it is seen that the beam is also horizontal at the center, therefore when $x = \frac{l}{2}$, $\alpha = 0$.

$$\therefore \quad 0 = \frac{Wl^2}{16\,EI} + \frac{M_1 l}{2\,EI}, \quad \text{and} \quad M_1 = -\frac{Wl}{8}.$$

$$M = \frac{Wx}{2} - \frac{Wl}{8},$$

$$\alpha = \frac{Wx^2}{4\,EI} - \frac{Wlx}{8\,EI},$$

$$y = \frac{Wx^3}{12\,EI} - \frac{Wlx^2}{16\,EI} + y_1.$$

When $\qquad x = 0, \ y = 0, \ \therefore \ y_1 = 0,$

$$\therefore \quad y = \frac{Wx^3}{12\,EI} - \frac{Wlx^2}{16\,EI}.$$

The maximum deflection occurs at the center and is found to be

$$\Delta = \frac{Wl^3}{96\,EI} - \frac{Wl^3}{64\,EI} = -\frac{Wl^3}{192\,EI}.$$

Similar equations can be deduced for the right half of the beam.

The bending moment at the wall is $-\frac{Wl}{8}$, and at the middle of the beam it is $+\frac{Wl}{8}$.

The inflection point is the point at which the bending moment is zero, and is found by equating M to zero and solving for the corresponding value of x, which may be called x_1.

$$\therefore \quad 0 = \frac{Wx_1}{2} - \frac{Wl}{8}; \quad x_1 = \frac{l}{4}.$$

5. Determine the left reaction, the maximum bending moment, and the equation of the elastic curve, and locate the inflection

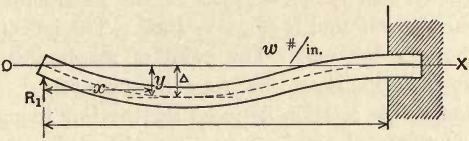


FIG. 97.

point for a beam fixed at one end and supported at the other end, carrying a uniform load. See Fig. 97.

The load per lineal unit is $-w$,

$$V = -wx + V_1.$$

Since the reactions cannot be determined at the start, the value of V_1 cannot be determined at first. The left reaction is less than it would be if the beam were not restrained at the right end.

$$M = -\frac{Wx^2}{2} + V_1 x + M_1.$$

When $x = 0, \quad M = 0, \quad \therefore \quad M_1 = 0,$

$$\alpha = -\frac{Wx^3}{6\,EI} + \frac{V_1 x^2}{2\,EI} + \alpha_1.$$

When $x = l, \ \alpha = 0.$

$$\therefore \quad 0 = -\frac{wl^3}{6EI} + \frac{V_1x^2}{2EI} + \alpha_1, \quad \alpha_1 = \frac{wl^3}{6EI} - \frac{V_1l^2}{2EI},$$

$$\alpha = -\frac{wx^3}{6EI} + \frac{V_1x^2}{2EI} + \frac{wl^3}{6EI} - \frac{V_1l^2}{2EI},$$

$$y = -\frac{wx^4}{24EI} + \frac{V_1x^3}{6EI} + \frac{wl^3x}{6EI} - \frac{V_1l^2x}{2EI} + y_1.$$

When $x = 0, \ y = 0, \ \therefore \ y_1 = 0.$

Also when $x = l, \ y = 0,$

$$\therefore \quad 0 = -\frac{wl^4}{24EI} + \frac{V_1l^3}{6EI} + \frac{wl^4}{6EI} - \frac{V_1l^3}{2EI},$$

$$V_1 = \tfrac{3}{8}wl = R_1,$$

$$\therefore \quad y = -\frac{wx^4}{24EI} + \frac{wlx^3}{16EI} - \frac{wl^3x}{48EI}.$$

The maximum positive moment is at the distance $\frac{3}{8}l$ from the left support, and is $M_m = \frac{9}{128}wl^2$. The maximum negative moment is at the wall and is $M_m = -\frac{1}{8}wl^2$. The inflection point is at $\frac{3}{4}l$ from the left end. The point of maximum deflection occurs where the slope is zero between the supports.

6. Determine the relation between the bending moments over the three supports of two consecutive spans of a continuous beam carrying uniform loads on both spans. See Fig. 98.

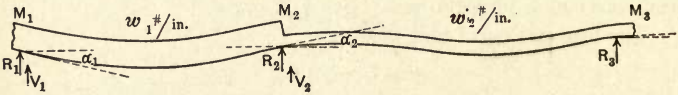


FIG. 98.

Using the relations deduced in Art. 116 we have for the first span:

The load per unit of length is $-w_1$.

$V = -w_1x + V_1$. (No relations yet to determine V_1). (a)

$M = -\dfrac{wlx^2}{6EI} + V_1x + M_1$. (No relations yet to determine M_1). (b)

$$\alpha = \frac{w_1x^3}{6\,EI} + \frac{V_1x^2}{2\,EI} + \frac{M_1x}{EI} + \alpha_1. \quad \text{(No relations yet to determine } \alpha_1\text{).} \quad \text{(c)}$$

$$y = -\frac{w_1x^4}{24\,EI} + \frac{V_1x^3}{6\,EI} + \frac{M_1x^2}{2\,EI} + \alpha_1 x + y_1. \quad \text{(d)}$$

When $x = 0,\ y = 0. \quad \therefore\ y_1 = 0.$

When $x = l_1,\ y = 0,$

$$\therefore \quad \alpha_1 = \frac{w_1l_1^3}{24\,EI} - \frac{V_1l_1^2}{6\,EI} - \frac{M_1l_1}{2\,EI}.$$

By the same process we obtain equations for the second span.

For the second span,

The load per unit of length is $-\,w_2$.

$$V = -\,w_2x + V_2. \quad \text{(No relations yet to determine } V_2\text{).} \quad (1)$$

$$M = -\,\frac{w_2x^2}{2} + V_2x + M_2. \quad \text{(No relations to determine } M_2\text{).} \quad (2)$$

$$\alpha = -\frac{w_2x^3}{6\,EI} + \frac{V_2x^2}{2\,EI} + \frac{M_2x}{EI} + \alpha_2. \quad \text{(No relations to determine } \alpha_2\text{).} \quad (3)$$

$$y = -\,\frac{w_2x^4}{24\,EI} + \frac{V_2x^3}{6\,EI} + \frac{M_2x^2}{2\,EI} + \alpha_2x + y_2. \quad (4)$$

When $x = 0,\ y = 0,\ \therefore\ y_2 = 0.$

Also when $x = l_2,\ y = 0,$ and

$$\alpha_2 = \frac{w_2l_2^3}{24\,EI} - \frac{V_2l_2^2}{6\,EI} - \frac{M_2l_2}{2\,EI}.$$

It is known that the slope of each portion of the elastic curve is the same at the middle support. By letting $x = l_1$ in equation (c), and letting $x = 0$ in equation (3), and equating (c) to (3), and remembering the value of V_1 and V_2 from Art 113, there results the *theorem of three moments,*

$$M_1l_1 + 2\,M_2\,(l_1 + l_2) + M_3l_2 = -\,\frac{w_1l_1^3}{4} - \frac{w_2l_2^3}{4}.$$

7. What is the maximum deflection of an 8-inch, 18-pound cantilever I-beam 10 feet long carrying a load of 1800 pounds concentrated at the end?

$$\Delta = \frac{Wl^3}{3\,EI} = \frac{1800 \times 120 \times 120 \times 120}{3 \times 30{,}000{,}000 \times 56.9} = 0.608 \text{ inch.}$$

PROBLEMS

1. For a simple beam with a concentrated load at the center show that the equation of the elastic curve to the left of the center is

$$y = \frac{W}{48\,EI}(4\,x^3 - 3\,l^2x),$$

and show that the maximum deflection is

$$\Delta = \frac{Wl^3}{48\,EI}.$$

2. Deduce the equation of the elastic curve for a cantilever beam carrying a load W concentrated at the end and a uniform load w per lineal unit. Also determine the maximum deflection.

Ans. $\Delta = -\frac{Wl^3}{3\,EI} - \frac{wl^4}{8\,EI}.$

3. Deduce the value of the maximum deflection for a simple beam carrying a load W concentrated at the center and a uniform load w per lineal unit.

Ans. $\Delta = -\frac{Wl^3}{48\,EI} - \frac{5\,wl^4}{384\,EI}.$

4. (a) Deduce the equation of the elastic curve for a beam fixed at both ends carrying a uniform load. (b) Determine the value of the maximum bending moment. (c) Determine the value of the maximum deflection. (d) Locate the inflection points.

Ans. (a) $y = \frac{w}{2\,EI}(-\,l^2x^2 + 2\,lx^3 - x^4),$

(b) $M_m = -\frac{wl^2}{12},$

(c) $\Delta = -\frac{wl^4}{384\,EI},$

(d) $x_1 = \frac{l}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right).$

5. For a beam fixed at both ends and carrying a concentrated load at the distance kl from the left support show that:

The left reaction is $R_1 = W(1 - 3\,k^2 + 2\,k^3)$.

The moment at the left support is $M_1 = -\,Wlk\,(1 - 2\,k + k^2)$.

The moment under the load is $M_3 = Wlk^2\,(2 - 4\,k + 2\,k^2)$.

Also determine the value of the deflection y.

6. For a beam fixed at one end and supported at the other, and carrying a concentrated load at the center, show that:

The left reaction is $\frac{5}{16} W$.

The moment at the wall is $-\frac{3}{16} Wl$.

The moment at the load is $\frac{5}{32} Wl$.

The values of y are:

To the left of the load, $y = \frac{5 Wx^3}{96 EI} + \frac{Wl^2x}{8 EI} - \frac{5 Wl^2x}{32 EI}$.

To the right of the load, $y = -\frac{11 Wx^3}{96 EI} + \frac{Wlx^2}{4 EI} - \frac{5 Wl^2x}{32 EI} + \frac{Wl^3}{48 EI}$.

The inflection point is $\frac{8}{11} l$ from the free support.

7. For a beam fixed at the right end and supported at the left carrying a concentrated load of W at the distance kl from the left support show that:

The reaction at the left end is $\frac{W}{2}(2 - 3k + k^3)$.

The moment at the wall is $-\frac{Wl}{2}(k - k^3)$.

The moment at the load is $\frac{Wlk}{2}(2 - 3k + k^3)$.

8. For equal spans and equal uniform loads on all spans show that the theorem of three moments reduces to

$$M_1 + 4 M_2 + M_3 = -\frac{Wl}{2}.$$

9. Solve the problems at the end of Chapters IX and X.

CHAPTER XIII

SECONDARY STRESSES

121. Horizontal Shear in Beams. When one board is placed on top of another one and the two are then used as a beam the upper board will slip over the lower one in one direction at one end and in the opposite direction at the other end. To prevent this motion and to make the beam stronger the boards may be nailed together, the nails taking shear. In all beams there is the tendency

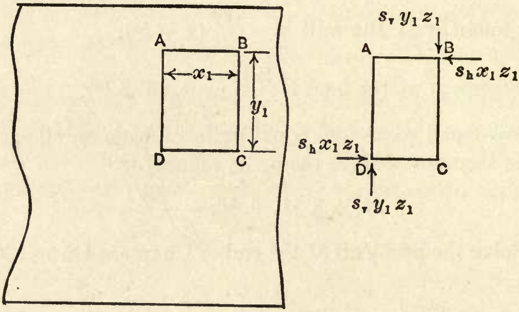


Fig. 99.

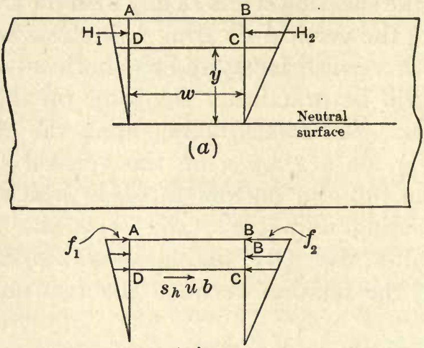
of the upper part to slip past the lower part along any horizontal plane, a horizontal shearing stress thus being produced.

It is the object of this article to show that **the vertical shearing unit-stress and the horizontal shearing unit-stress at any point in a beam are the same.** Proof: In a beam let a small parallelopiped $ABDC$, of dimensions x_1, y_1, z_1, be imagined cut from the beam. Neglecting the effect upon the shearing stress of any load for the element of length x_1, the vertical shearing unit-stress will be the same on both vertical faces, and the horizontal shearing unit-stress will be practically the same on the top and bottom faces. The forces acting upon the element as shown in Fig. 99 are $s_v y_1 z_1$ on the vertical faces, and $s_h x_1 z_1$ on the top and bottom faces, in which s_v is the vertical shearing unit-stress, and s_h is the horizontal shearing unit-stress. By taking moments about any point, as A, the relation between the two unit-stresses is deduced.

$$\Sigma M_A = s_h x_1 z_1 y_1 - s_v y_1 z_1 x_1 = 0,$$

$$\therefore \quad s_h = s_v.$$

122. THE MAGNITUDE OF THE HORIZONTAL AND VERTICAL SHEARING UNIT-STRESSES AT A POINT. In Art. 63 it was assumed that the maximum vertical shearing unit-stress for a section is greater than the average, and values of the ratio between the two were given for several standard sections. To determine those ratios the value of the horizontal shearing unit-stress must be deduced. The expression for the horizontal shearing unit-stress will now be deduced. Let Fig. 100 (a) represent a beam with a portion $ABCD$ imagined cut from the beam. The stresses on the fibers of the section AD in general will not be equal to the stresses on the fibers of the section BC, because the bending moments at the two sections are usually different. If the bending moment at the section AD is less than that at the section BC, the resultant H_1 of the stresses acting upon the face AD is less than H_2, which is the resultant of the stresses acting upon the face BC. To maintain equilibrium a horizontal shearing force $s_h ub$ must act upon the face CD. s_h is the horizontal shearing unit-stress, u is an element of length, AB, and b is the thickness of the beam.



(b)

FIG. 100.

By Art. 71 H_1 is equal to the area of the section above the plane multiplied by the unit-stress on the centroid of that area.

$$H_1 = f_1 \frac{\bar{y}'}{c} A'.$$

Likewise $$H_2 = f_2 \frac{\bar{y}'}{c} A'.$$

f_1 is the unit-stress developed on the outside fiber at the section AD, and f_2 is the unit-stress on the outside fiber at the section BC. From the moment formula, $\frac{f_1}{c} = \frac{M_1}{I}$ and $\frac{f_2}{c} = \frac{M_2}{I}$. For equilibrium of the element:

$$H_2 - H_1 = s_h u b,$$

$$\frac{f_2 - f_1}{c} A'\bar{y}' = s_h u b,$$

$$\frac{M_2 - M_1}{I} A'\bar{y}' = s_h u b,$$

$$s_h = \left(\frac{M_2 - M_1}{u}\right)\left(\frac{A'\bar{y}'}{Ib}\right),$$

$$s_h = \frac{V}{Ib} A'\bar{y}'.$$

$\frac{M_2 - M_1}{u}$ is the rate of change of the bending moment, and this equals V when u is made indefinitely small. $A'\bar{y}'$ is zero for y equal to c and is a maximum for y equal to zero. Therefore the shearing unit-stress is zero at the outside fiber of beams, and is greatest at the neutral surface.

The maximum shearing stress, both horizontal and vertical, developed in a rectangular beam of breadth b and depth d where V is the vertical shearing force, is

$$s_m = \frac{V}{Ib} A'\bar{y}' = \frac{V}{\frac{bd^3b}{12}} \times \frac{bd}{2} \times \frac{d}{4} = \frac{3V}{2\,bd} = \frac{3V}{2A}.$$

This is one-half greater than the average vertical shearing stress.

For circular sections the maximum shearing stress is

$$s_m = \frac{V}{\frac{\pi d^4}{64} \times d} \times \frac{\pi d^2}{8} \times \frac{2\,d}{3\,\pi} = \frac{4V}{3A}.$$

This is one-third greater than the average vertical shearing stress.

For built-up and I-sections the maximum shearing stress is approximately equal to that obtained by dividing the vertical shear by the area of the web A_1.

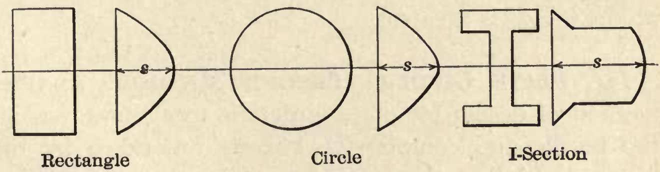


FIG. 101.

The variation in the intensity of the shearing unit-stress for various sections is shown by the diagrams of Fig. 101.

Because of their small strength in shear parallel to the grain, timber beams frequently fail by shearing along the neutral surface. Beams should always be investigated for the maximum shearing stress developed.

123. PLATE GIRDERS, FIRST METHOD. One method of the design of plate girders is to consider all plates and angles acting as a homogeneous beam. The girder may then be designed by the use of the moment and shear formulas, and the pitch of the rivets can be determined by the use of the formula for horizontal shear. In $s_h = \frac{V}{Ib} A'\bar{y}'$, s_h is the stress developed upon a unit of area of the horizontal plane. Multiplying this by b gives the total force which would be transmitted from the upper section to the lower one in one unit of length of the girder. For built-up sections the stress must be transmitted from the upper plates to those next below, through the rivets connecting the plates. If p is the pitch of the rivets, and there are n rivets in the distance p, the force that each rivet must carry will be

$$P = \frac{pbs_h}{n} = \frac{Vp}{In} A'\bar{y}'.$$

The greatest number of rivets will be required where the product VA' is greatest.

124. PLATE GIRDERS, SECOND METHOD. Another method of design for plate girders is to assume that all the tensile and compressive stresses are taken by the flanges, and that the stress is uniform over the section of the flanges, and that the shear is taken by the web. The stresses calculated in this way are probably a little in excess of those actually developed, but the error is on the side of safety.

Let Fig. 102 be the cross section of a girder, A the effective area of one flange, d the distance between the centroids of the flanges, and f the unit-stress developed in the flanges, then the compressive and tensile forces equal Af and the resultants act at the centroids of the flanges. The moment of these internal stresses resists the bending moment due to the external forces. For equilibrium, then,

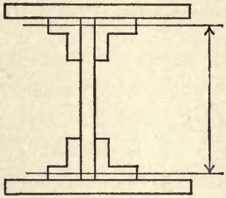


FIG. 102.

$$M = Afd,$$

where M is the bending moment and Afd is the resisting moment.

The bending moment increases toward the center of the span, and to increase the resisting moment with the same unit-stress the area A is increased by adding cover plates in the center. The pitch p of the rivets connecting the flanges to the web may now be found. (See Fig. 103.) The change in the stress in the flanges between any two sections must be transmitted through the rivets to the web. This difference in a unit of length is

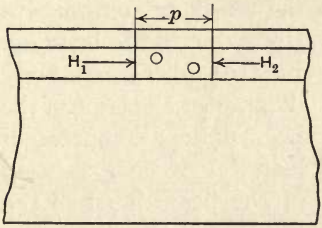


FIG. 103.

$$H_2 - H_1 = f_2A - f_1A$$
$$= \frac{M_2 - M_1}{d} = \frac{V}{d},$$

since $M_2 - M_1$ is the rate of change of the bending moment, as the distance between the sections was taken as unity. If the rate of change of the bending moment,

or V, is constant, the change for the distance p is $p\,\frac{V}{d}$.
If there are n rivets in the pitch p, and R is the allowable force one rivet will transmit,

$$nR = p\frac{V}{d}$$

$$\therefore\ p = \frac{nRd}{V}.$$

125. COMBINED FLEXURE AND TENSION OR COMPRESSION. When a beam is subject to axial loads in connection with the flexural loads the maximum stress developed may be considered as made up of two parts — that due to bending and that due to the axial load. The axial load increases the tensile or compressive stress due to the bending. Fig. 104 (*b*) is a free-body diagram of a portion of the beam under a compression load P. If the deflection of the beam is small the moment due to P may be neglected. If M is the bending moment due to the flexural loads, A the sectional area, and $\frac{I}{c}$ the section modulus, the maximum flexural stress developed is $f_1 = \frac{Mc}{I}$ (indicated by AB in compression). The compressive unit-stress due to the axial load is $f_2 = \frac{P}{A}$ (indicated by FA). It is seen that the maximum compressive stress is developed in the most remote fiber in compression

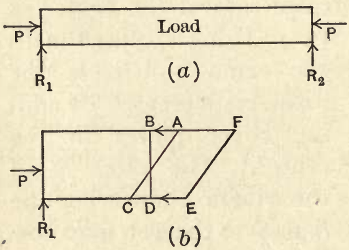


FIG. 104.

and equals the sum of the flexural and direct stress and is

$$f = f_1 + f_2 = \frac{Mc}{I} + \frac{P}{A}.$$

If P is a tension load the maximum tensile stress is of the same form as that given for compression.

In longer beams, where the deflection is appreciable, there is an additional moment M' due to the load P. The moment is decreased if P is tension and increased if P is compression. (See Fig. 105.) If Δ_1 is the maximum deflection due to both the transverse and longitudinal loads the moment of P is $P\Delta_1$, and the total moment is

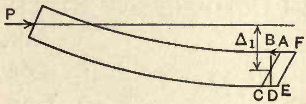


FIG. 105.

$$M \pm M' = M \pm P\Delta_1.$$

$$f_1 = \frac{M_1 c}{I} = \frac{(M \pm P\Delta_1)c}{I}.$$

The plus sign is for a compression load and the minus is for a tension load. In order to find the stress AB, or f_1 due to the moment of both loads, Δ_1 must be expressed in terms of f_1, the maximum stress for the deflection Δ_1.

From Art. 107, $\Delta_1 = \dfrac{\alpha f_1 l^2}{\beta E c}.$

$$\therefore \quad f_1 = \frac{Mc}{I} \pm \frac{\alpha P f_1 l^2 c}{\beta E c I},$$

from which $$f_1 = \frac{Mc}{I}\left(\frac{1}{1 \mp \dfrac{\alpha P l^2}{\beta E I}}\right).$$

To this add the direct stress $\dfrac{P}{A}$ due to the axial load.

Then the maximum stress developed is found to be

$$f = f_1 + \frac{P}{A} = \frac{Mc}{I}\left(\frac{1}{1 \mp \dfrac{\alpha P l^2}{\beta E I}}\right) + \frac{P}{A}\,.$$

The minus sign is used for P in compression and the plus for P in tension.

126. COMBINED SHEARING STRESSES AND TENSILE OR COMPRESSIVE STRESSES. Let Fig. 106 represent a small portion of a beam where the known unit-stresses are s in shear and f in tension or compression, s and f being at right angles to each other. Along all diagonal planes, as AB, there are normal and tangential components of the stresses.

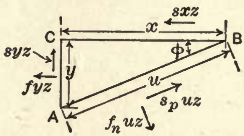


FIG. 106.

s_p is the shearing unit-stress along the plane and f_n is the tensile or compressive unit-stress normal to the plane. In more advanced texts it is shown that the value of ϕ to give the maximum s_p is such that $\tan 2\,\phi = \dfrac{f}{2\,s}$, and the corresponding maximum shearing stress is

$$s_p = \sqrt{s^2 + \left(\frac{f}{2}\right)^2}\,.$$

The value of ϕ to give the maximum f_n is such that $\cot 2\,\phi = -\dfrac{f}{2\,s}$, and the maximum tensile stress is

$$f_n = \frac{f}{2} \pm \sqrt{s^2 + \left(\frac{f}{2}\right)^2}\,.$$

In the latter equation the maximum stress will be obtained with the plus sign, and if f_n is a tensile stress, the maximum f_n will be a tensile stress. If f is a compressive stress the maximum f_n will be a compressive stress. If the minus sign is used before the radical the resulting f_n will be negative, which indicates that it is of opposite sign from f, i.e., f_n is compression for f tension, and f_n is tension for f compression.

EXAMPLES.

1. Determine the maximum horizontal shearing unit-stress in a timber beam 8 inches by 14 inches under a load of 24,000 pounds applied at the third points.

$$R_1 = R_2 = 12{,}000 \text{ pounds.}$$

$$s_h = \frac{V}{Ib} A'\bar{y}' = \frac{3}{2}\frac{V}{A} = \frac{3 \times 12{,}000}{2 \times 112} = 160.7 \text{ lb. per sq. in.}$$

2. Compute the pitch of $\frac{7}{8}$-inch rivets for a plate girder of 72-ft. span and 7 feet 2 inches deep, the cover plates being 14 inches wide with a total thickness of $\frac{7}{8}$ inch at the center, being connected to angles $\frac{3}{8}$ inch thick. (NOTE: This girder was designed to carry a live train load together with the weight of the tracks and the girder.)

The maximum vertical shear at the ends was found to be 137,000 pounds; at 5 feet from the ends, 121,000 pounds; at 10 feet from the ends, 102,000 pounds, etc.

Each rivet will carry .601 × 8000 = 4810 pounds in shear, and $\frac{3}{8} \times \frac{7}{8} \times 18{,}000 = 5900$ pounds in bearing. The shear governs in this case.

Taking 2 rivets in the pitch p,

$$p = \frac{2 \times 4810 \times 86}{137{,}000} = 6 \text{ inches at the ends.}$$

$$p = 6.8 \text{ inches at 5 feet,}$$

$$p = 8.05 \text{ inches at 10 feet, etc.}$$

For concentration of the loads on the girder the maximum allowable pitch would be about 6 inches.

3. A timber 8 inches by 10 inches is used as a simple beam of 12-ft. span to carry a uniform load of 4000 pounds and end compression loads of 40,000 pounds. What is the maximum stress developed?

By assuming the deflection negligible,

$$f = \frac{P}{A} + \frac{Mc}{I} = \frac{40,000}{80} + \frac{4000 \times 144 \times 5 \times 12}{8 \times 8 \times 10 \times 10 \times 10},$$

$$= 500 + 540 = 1040 \text{ pounds per square inch.}$$

By use of the formula assuming the deflection not negligible,

$$f = \frac{P}{A} + \frac{Mc}{I}\left(\frac{1}{1 - \frac{\alpha P l^2}{\beta E I}}\right)$$

$$= 500 + 540\left(\frac{1}{1 - \frac{8 \times 40,000 \times 144^2 \times 3}{76.8 \times 1,500,000 \times 2000}}\right)$$

$$= 500 + 540\left(\frac{1}{1 - .0864}\right) = 500 + \frac{540}{.9136}$$

$$= 500 + 590 = 1090 \text{ pounds per square inch.}$$

Thus it is seen that the moment of the axial load about the central section increases the stress about 5 per cent.

4. A bolt 1 inch in diameter is subjected to a tension of 3000 pounds and at the same time to a cross shear of 5000 pounds. Determine the maximum tensile and shearing unit-stresses.

$$s = 5000 \div .7854 = 6370 \text{ pounds per square inch.}$$

$$f = 3000 \div .7854 = 3820 \text{ pounds per square inch.}$$

By substitution in the formulas for the maximum tensile and shearing stresses,

$$f_n = \frac{3820}{2} + \sqrt{6370^2 + \frac{3820^2}{4}} = 8560 \text{ lb. per sq. in., tension.}$$

$$s_p = \sqrt{6370^2 + \frac{3820^2}{4}} = 6650 \text{ lb. per sq. in., shear.}$$

PROBLEMS

1. A simple rectangular timber beam 8 inches by 12 inches and of 10-ft. span carries a uniform load of 2000 pounds per foot. Determine the horizontal shearing unit-stress at the following points: (a) At the neutral surface over a support. (b) 3 inches from the neutral surface over a support. (c) 4 inches from the neutral surface at a quarter point.

2. What is the maximum shearing unit-stress developed in an I-beam of the largest standard section which carries a uniform load over a span of 20 feet if the maximum fiber stress does not exceed 16,000 pounds per square inch?

3. A girder of $55\frac{1}{3}$-ft. span is built up of $\frac{1}{2}$-inch by 4 feet 10-inch web plate, four 5-inch by 6-inch by $\frac{5}{8}$-inch angles, with the 5-inch leg riveted to the web, and four cover plates at the quarter points 14 inches by $\frac{1}{2}$ inch. The rivets are $\frac{7}{8}$ inch in diameter and spaced 3 inches apart, and there are two rows in each 5-inch leg. Determine the maximum shearing and bearing unit-stresses that would probably come on the rivets. (Section is similar to that shown in Fig. 102 with cover plates added. Girder is to carry a train load.)

4. Determine the maximum stress developed in a 6-inch, 15-pound I-beam of 6-ft. span with both ends fixed, carrying a uniform load of 8 tons and tension loads at the ends of 6 tons.

5. What will be the maximum fiber stress developed in a simple timber beam 6 inches by 8 inches of 8-ft. span, with a concentrated load of 1500 pounds at the center and end compression loads of 10,000 pounds?

6. A 12-inch, 40-pound I-beam of 6-ft. span carries a uniform load of 1200 pounds per foot, and is subjected to an axial compression of 60,000 pounds. Find the maximum stress developed.

7. Find the size of a square maple simple beam for a simple span of 12 feet to carry a load of 500 pounds at the middle, when it is also subjected to an axial compression of 2000 pounds.

8. A bar of iron is under a direct tensile stress of 4000 pounds per square inch and a shearing stress of 3500 pounds per square inch. Find the maximum tensile and shearing unit-stresses.

9. Design a white oak beam with both ends fixed, for a span of

12 feet, which is to carry a concentrated load of 4 tons at the center and a tension load of 5 tons.

10. What I-beam would be required for the loading given in Problem No. 6 if the unit-stress is not to exceed 16,000 pounds per square inch?

11. What will be the maximum shearing and tensile unit-stresses developed in a $\frac{7}{8}$-inch bolt if it is subjected to a tension load of 5000 pounds and a cross shearing load of 5000 pounds?

CHAPTER XIV

COLUMNS AND STRUTS

127. Discussion. The terms **columns** and **struts** are usually applied to prismatic members designed to carry compression loads when the length has an effect on the strength of the member. For short compression members, lateral deflection is inappreciable under load, while for longer ones (i.e. **columns**) it may be of consequence. Long columns will not carry so great loads as shorter ones of the same material and section, since the lateral bending of the column causes the stress to be distributed unevenly over the cross section of the column and makes it greater on the concave side than the value obtained by dividing the load by the sectional area (see Fig. 107). The formulas used in designing columns, and in calculating the stress developed in them, are to a large extent empirical. A large number of formulas have been developed by different investigators, and those in most common use will be given.

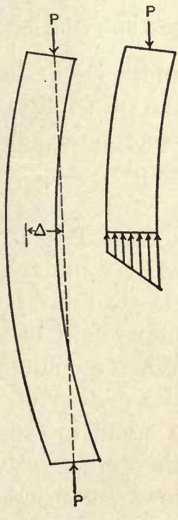


Fig. 107.

128. Stiffness of Columns. If a flat board is used as a column, bending will occur about an axis parallel to the longer side of a section. In all columns free to bend in any direction bending will occur in the direction in which the column is least stiff. In other words, the bending will occur about the axis for which the moment of inertia and the radius of gyration are the least. The most economical column section, therefore, would be one for which the tendency to bend would be the same for all axes.

The **slenderness ratio** is the ratio of the length of a column to the least radius of gyration of the cross section, and equals $\frac{l}{r}$, where l is the length of the column and r is the least radius of gyration as determined by the principles of Appendix A. l and r should be in the same units, and the inch is the unit most commonly employed.

129. THE STRENGTH OF COLUMNS. The yield point of the material, which is somewhat higher than the elastic limit, is practically the ultimate strength for columns built of structural steel or other similar material. When a column is sensibly bent, the bending moment at the section of greatest deflection increases rapidly with a small increase of load. The moment of the load at the danger section will cause the column to fail under a load somewhat greater than that load which will develop a stress equal to the elastic limit of the material.

Fig. 108 shows characteristic failures for compression specimens of timber. The short one shows oblique shear failure, the intermediate ones show failure in compression, and the longest one shows failure due to bending of the column.

The condition of the ends also has an effect on the strength of a column. Fig. 109 shows the position assumed by long homogeneous columns under load, with different end conditions: (a) with both ends round

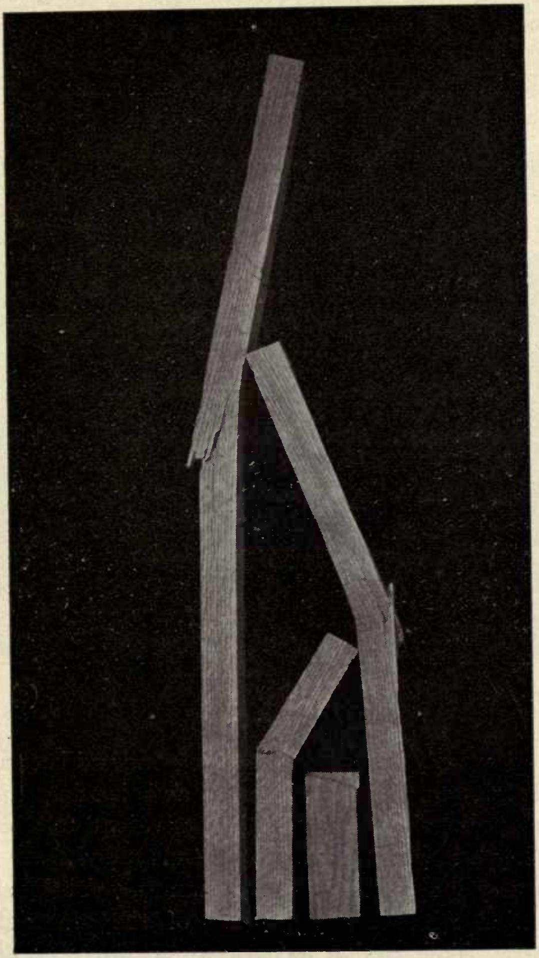
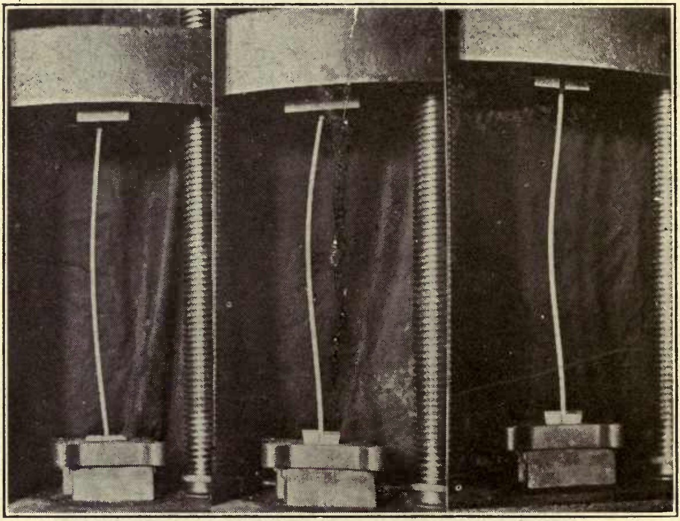


FIG. 108.

or pivoted, (b) with one end round and the other end fixed, and (c) with both ends fixed. Fixing the ends increases the strength of a column. In Fig. 109 it may be seen that about two-thirds of the column in (*b*) is in a condition similar to that in (*a*), and that half of the column in (*c*) is in a condition similar to that in (*a*).



(*a*) (*b*) (*c*)

FIG. 109.

It is commonly assumed that fixing one end is equivalent to decreasing the length to $\frac{2}{3}$ and leaving the ends round, and fixing both ends is equivalent to decreasing the length to $\frac{1}{2}$ and leaving both ends round.

In very long columns the column may fail by sidewise deflection without any portion of the material being injured. This action occurs at a lower slenderness ratio in a material like timber than in a material like steel. The phenomenon of sidewise failure can be illustrated by the blade of a tee-square.

130. THE STRAIGHT-LINE FORMULA. By examining the data of tests of columns it is found by plotting points to represent the average stress $\frac{P}{A}$ at rupture, for various values of the slenderness ratio $\frac{l}{r}$, that a straight line can be drawn which will fairly represent average ultimate values of $\frac{P}{A}$ for the different slenderness ratios. Thus, in Fig. 110 values of $\frac{P}{A}$ are given along the vertical axis and values of $\frac{l}{r}$ are given along the horizontal axis. An equation representing this straight line is of the form $\frac{P_1}{A} = f_1 - \frac{C_1 l}{r}$. In this formula P_1 is the load at rupture. A similar formula may be used to determine the safe load for a given column or to determine the proper sectional area to carry a given load. Of course, the safe load P will be considerably lower than the rupturing load P_1. The straight-line formula for the average stress over the section of the

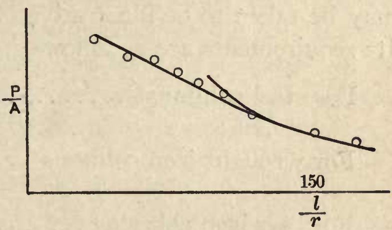


FIG. 110.

column is $\frac{P}{A} = f - \frac{Cl}{r}$ where P is the safe load and the values of f and C are to be specified. In the straight-line formula $f - C\frac{l}{r}$ is considered as the allowable safe unit-stress. This formula shows that the strength of a column becomes less as the length increases and as the radius of gyration decreases. It is purely empirical, as it is based entirely upon experimental data, but it is considered to be as reliable as any and is quite generally employed.

The values of f_1 and C_1 for various materials can be derived from experimental data, and they may be used as a guide to determine values of f and C to be used in design.

The greatest value of the slenderness ratio to be used in the application of the straight-line formula is usually given as 100 to 125. In any case it should not be greater than 150.

Cooper gives the formula $\frac{P}{A} = 17{,}000 - 90\frac{l}{r}$ as safe for soft steel columns of a through railroad bridge. Ketchum gives the formula $\frac{P}{A} = 16{,}000 - 70\frac{l}{r}$ for steel columns in building frames. The limit of $\frac{l}{r}$ is 125.

The Chicago building ordinance as revised in 1910 may be taken to be illustrative of architectural practice. Its requirements are as follows:

For steel columns . . . . $\frac{P}{A} = 16{,}000 - 70\frac{l}{r}$.

For wrought iron columns . . . $\frac{P}{A} = 12{,}000 - 60\frac{l}{r}$,

For cast iron columns . . . . $\frac{P}{A} = 10{,}000 - 40\frac{l}{r}$.

The maximum allowable compressive stress shall not exceed the values given in Table 18. $\frac{l}{r}$ shall not exceed 120.

For timber columns the following is a modification of the formula used by Ricker and of that given in the Chicago building ordinance:

$$\text{For timber columns} \ldots\; \frac{P}{A} = f - .0036f\frac{l}{r},$$

in which f is the value of the allowable compressive stress parallel to the grain given in Table 8. The ordinance provides that the slenderness ratio $\frac{l}{r}$ shall not exceed 120. The original form of this formula was for columns of rectangular section.

TABLE 18

MAXIMUM ALLOWABLE COMPRESSIVE STRESS IN POUNDS PER SQUARE INCH, CHICAGO BUILDING ORDINANCE, 1910.

Material.	Compressive stress, pounds per square inch.
Steel..................	14,000
Wrought iron..........	10,000
Cast iron..............	10,000

Two problems can be solved by the use of the straight-line formula,—(1) the safe load a given column will carry, and (2) the design of columns. For these problems the above formulas may be used unless the specifications state otherwise. Of these problems the design of columns is the one most commonly met by the engineer and the architect, and it admits of many solutions.

ILLUSTRATIVE EXAMPLES

1. What load would a 15-inch, 42-pound I-beam 9 feet long safely carry if used as a column in a bridge?

From Table No. 21 giving properties of I-sections, $A = 12.48$ square inches, the least $r = 1.08$ inches.

$$\frac{l}{r} = \frac{9 \times 12}{1.08} = \frac{108}{1.08} = 100.$$

By the use of Cooper's formula

$$P = A\left(17{,}000 - 90\frac{l}{r}\right),$$

$$P = 12.48 \times (17{,}000 - 9000) = 12.48 \times 8000 = 99{,}840 \text{ pounds.}$$

2. Design a square shortleaf pine column 11 feet long to carry a load of 10,000 pounds.

Let d be the dimension of a side and r the radius of gyration.

$$A = d^2.$$

The least value of r may be such that

$$\frac{l}{r} = 120, \quad \text{least } r = \frac{12 \times 11}{120} = \frac{132}{120} = 1.1 \text{ inch.}$$

$$r = \sqrt{I \div A} = \sqrt{\frac{d^4}{12 \div d^2}} = \frac{d}{\sqrt{12}}.$$

$$\text{Least } d = r\sqrt{12} = 1.1 \times 3.46 = 3.81 \text{ inches.}$$

For the formula $\dfrac{P}{A} = f - .0036 f \dfrac{l}{r}$,

$$f = 1200 \text{ from Table 8.}$$

$$\frac{10{,}000}{d^2} = 1200 - .0036 \times 1200 \times \left(\frac{\dfrac{11 \times 12}{d}}{3.46}\right),$$

$$10{,}000 = 1200\,d^2 - 1970\,d,$$

$$d^2 - 1.64\,d = 8.33,$$

$$d = 3.82 \text{ inches.}$$

The 3.82-inch by 3.82-inch timber would do, but a 4-inch by 4-inch would generally be used.

In this example the size of the column is the same, determined both by the slenderness ratio and by the allowable stress. This is seldom the case.

131. Eccentric Loads on Columns. The foregoing formulas are to be used only for axial loads. As shown in Art. 77, a load when eccentric produces a greater unit-stress than when axial, and when the load on a column is eccentric the formulas used must take account of the effect of the eccentricity. If the load P has the eccentricity e, the stress due to the eccentricity alone, as derived in Art. 77, is $\dfrac{P}{A}\dfrac{ec}{r^2}$. Consequently the allowable unit-stress for design of a column must be equal to

$\frac{P}{A} + \frac{P}{A}\frac{ec}{r^2}$, and by equating this to the allowable stress, the straight-line formula for columns carrying eccentric loads becomes

$$\frac{P}{A}\left(1 + \frac{ec}{r^2}\right) = f - C\frac{l}{r}.$$

The second member of this equation is the allowable stress as given in the formulas of Art. 130.

ILLUSTRATIVE EXAMPLE

What load would a solid round cast iron column 6 inches in diameter and 10 feet long safely carry if the load has the eccentricity of 1.5 inches?

For this column the formula is $\frac{P}{A}\left(1 + \frac{ec}{r^2}\right) = 10{,}000 - 40\frac{l}{r}.$

$A = \frac{\pi d^2}{4} = 28.3$ square inches, $e = 1.5$ inches, $c = 3$ inches,

$r^2 = I \div A = \frac{\pi d^4}{64} \div \frac{\pi d^2}{4} = \frac{d^2}{16} = \frac{36}{16} = 2.25$, $r = 1.5$ inches.

$\frac{l}{r} = \frac{10 \times 12}{1.5} = 80,$

$$\therefore \quad \frac{P}{28.3}\left(1 + \frac{1.5 \times 3}{2.25}\right) = 10{,}000 - 40 \times 80 = 6800.$$

$$P = \frac{28.3}{3} \times 6800 = 64{,}200 \text{ pounds.}$$

132. THE METHODS OF TRANSMITTING LOADS TO COLUMNS. In the columns of such structures as bridges the load is usually transmitted to the column through pins or rivets and plates, in such a manner that the load is axial or so that the eccentric stress will oppose any moment stress that may be developed by the weight of the member when not vertical. For buildings and many other structures, however, the load may be transmitted through angles and rivets on one, two, three, or four sides of the column, thus producing large eccentric stress which should be provided for in the design of the column. Beams and girders are usually supported on caps or brackets, and for purposes of design the load is considered as acting at the centroid of the area supporting the member. The resultant of all the loads may be found and dealt with, or the effect of each load may be determined separately and the resulting stresses combined. In making the combination of the loads it should be borne in mind that loads on opposite sides of the column will partly neutralize the eccentric effect.

OTHER COLUMN FORMULAS

133. COMPARATIVE STRENGTH AND STIFFNESS OF LONG, IDEAL COLUMNS. Condition of the Ends. From analogy with beams the maximum deflection for a given stress in the outer fiber is taken to vary directly with the square of the length of the column (Art. 108). Consequently the maximum moment and the maximum moment stress in columns are assumed to vary with the square of the length. For very long, ideal, homogeneous columns the assumptions are approximately true. Columns under load will deflect approximately, as shown in the curves of Fig. 111. Curve (*a*) is for both ends round or hinged; curve (*b*) is for one end round or hinged and the other end fixed; curve (*c*) is for both ends fixed; curve (*d*) is for one end fixed and the other end round

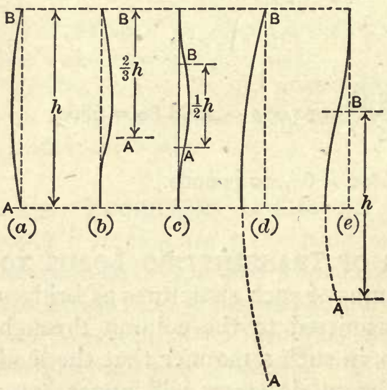


FIG. 111.

and free to move; curve (*e*) is for both ends fixed in direction but free to move laterally.

If f_1 is the moment stress developed in a column of length h with both ends round or hinged (represented by curve (*a*)) under a given load, the moment stress developed in the same column under the same load with its ends fixed in the various ways will now be found.

Fixing one end is equivalent to shortening the column to $\frac{2}{3}h$ and leaving the ends round. The portion AB of the column shown in curve (*b*) is in a condition similar to a round-ended column. For this case, one end fixed and one end round, therefore, the moment stress developed will be

$$f_2 = \frac{(\frac{2}{3}h)^2}{h^2} f_1 = \frac{4}{9} f_1.$$

Fixing both ends is equivalent to shortening the column to $\frac{1}{2}h$ and leaving both ends round. The portion AB may be considered as a round-ended column. The maximum moment stress developed for this case will be

$$f_3 = \frac{\left(\frac{h}{2}\right)^2}{h^2} f_1 = \frac{1}{4} f_1.$$

The column with one end fixed and the other end round or hinged is in the same condition as half of a column with both ends round, and for this case the moment stress developed will be

$$f_4 = \frac{(2h)^2}{h^2} f_1 = 4 f_1.$$

By keeping both ends restrained in direction but one end free to move laterally, as in (*e*), is equivalent to having two columns similar to the condition shown in (*d*) but one-half as long; therefore the moment stress developed in this case is

$$f_5 = \frac{h^2}{h^2} f_1 = f_1.$$

134. RANKINE'S FORMULA. Columns of Intermediate Length. Rankine derived an empirical formula for columns of intermediate lengths, such as are found most commonly in engineering practice. The following is the derivation (see Fig. 112): Let the column have the maximum deflection Δ, then the maximum stress, which is due to compression and flexure, will be at the point of maximum deflection and is $f = f_1 + f_2$, where f_1 is the direct compressive stress which equals $\frac{P}{A}$ and f_2 is the bending stress and is $\frac{Mc}{I}$ (Art. 69). Therefore

$$f = \frac{P}{A} + \frac{Mc}{I} = \frac{P}{A} + \frac{P\Delta c}{I}.$$

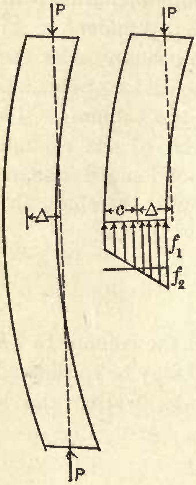


FIG. 112.

From analogy with the maximum deflection of beams (Art. 108) it is assumed that Δ varies with $\frac{l^2}{c}$. If ϕ is a factor depending upon the material of the column and the condition of the ends, $\Delta = \phi\frac{l^2}{c}$, then

$$f = \frac{P}{A} + \phi\frac{Pl^2}{I} = \frac{P}{A}\left(1 + \phi\left(\frac{l}{r}\right)^2\right) \quad \text{or} \quad P = \frac{Af}{\left(1 + \phi\left(\frac{l}{r}\right)^2\right)},$$

$I = Ar^2$, and $\frac{l}{r}$ is the slenderness ratio. The factor ϕ is a fraction

TABLE 19

VALUES OF ϕ USED IN RANKINE'S FORMULA

Conditions of the ends.	Timber.	Cast iron.	Wrought iron.	Steel.
Both ends round.........	$\frac{4}{3000}$	$\frac{4}{5000}$	$\frac{4}{36,000}$	$\frac{4}{25,000}$
Fixed end round.........	$\frac{1.78}{3000}$	$\frac{1.78}{5000}$	$\frac{1.78}{36,000}$	$\frac{1.78}{25,000}$
Both ends fixed..........	$\frac{1}{3000}$	$\frac{1}{5000}$	$\frac{1}{36,000}$	$\frac{1}{25,000}$

which is determined partly by experiment and partly from the theory of Art. 133. Having experimental data available and assuming the relative strengths as given in Art. 133 the values of ϕ were found to be as given in Table 19. It may be noted that the numerator of the fraction indicates the condition of the ends and the denominator is the characteristic for the material. f should be the allowable working stress for problems in design.

135. EULER'S FORMULA. Long Columns. Euler deduced a formula for long, ideal, homogeneous columns. For such columns it has been found that when the load reaches a certain limit, if a lateral deflection occurs the load will hold the column in equilibrium in that position. If the load is decreased the column will come back to a straight position, and if the load is increased the deflection increases until failure finally follows. From analogy with beams, the deflection of a column from the straight position varies inversely with the modulus of elasticity and the moment of inertia of the section, and directly as the square of the length (Art. 108); consequently the load a given long column will carry is directly proportional to the modulus of elasticity and to the moment of inertia and is inversely proportional to the square of the length. Therefore the formula based on these principles has the general form $P = \frac{nEI}{l^2}$.

For n Euler deduced the theoretical value π^2 for columns with both ends round, $2\frac{1}{4}\pi^2$ for columns with one end fixed and the other end round, $4\pi^2$ for columns with both ends fixed. Therefore, the values of the load P which will cause failure as determined by Euler's formula, are:

Both ends round,

$$P = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 EA}{\left(\frac{l}{r}\right)^2}.$$

One end fixed, the other end round,

$$P = 2\frac{1}{4}\frac{\pi^2 EI}{l^2} = 2\frac{1}{4}\frac{\pi^2 EA}{\left(\frac{l}{r}\right)^2}.$$

Both ends fixed,

$$P = 4\frac{\pi^2 EI}{l^2} = 4 \times \frac{\pi^2 EA}{\left(\frac{l}{r}\right)^2}.$$

One end fixed, the other round and free to move,

$$P = \frac{1}{4}\frac{\pi^2 EI}{l^2} = \frac{1}{4}\frac{\pi^2 EA}{\left(\frac{l}{r}\right)^2}.$$

In these formulas E is the modulus of elasticity, $I = Ar^2$ is the least moment of inertia of the cross section, r is the least radius of gyration of the cross section, and $\frac{l}{r}$ is the slenderness ratio.

Euler's formula as given is for the critical load, and it should be modified before being used for design, and it should be used only for columns of which the slenderness ratio is not less than about 200. For design the formula would be modified by introducing a "factor of safety." (See Example 4.)

136. **THE THREE PROBLEMS.** Three typical problems may be investigated by the use of the column formulas: (1) The investigation of columns, which consists of determining the maximum unit-stress developed in a given column under a given load. (This can be done only with Rankine's formula, for which case the stress is only nominal.) (2) The load which a given column will carry safely. (3) The design of a column to carry a given load.

137. **ECCENTRIC LOADS ON COLUMNS.** Rankine's and Euler's formulas as given above are to be used only when the load is axial. In Rankine's formula,

$$f = \frac{P}{A}\left(1 + \phi\left(\frac{l}{r}\right)^2\right)$$

for axial loads, the part of the stress $\frac{P}{A}$ is due to direct compression and the part $\frac{P}{A} \times \phi\left(\frac{l}{r}\right)^2$ is due to the bending moment in the column. If the load has the eccentricity e, the increase in the stress due to this eccentricity by Art. 77 is $\frac{P}{A}\frac{ec}{r^2}$; consequently the stress developed in a column under an eccentric load will be the sum of the three stresses and is

$$f = \frac{P}{A} + \frac{P}{A}\phi\left(\frac{l}{r}\right)^2 + \frac{P}{A}\frac{ec}{r^2} \text{ or } f = \frac{P}{A}\left(1 + \phi\left(\frac{l}{r}\right)^2 + \frac{ec}{r^2}\right),$$

which may be used for columns to which Rankine's formula would ordinarily be applied for an axial load.

For columns of large slenderness ratio a nearer approximation may be made as follows: If P is the load with the eccentricity e and the maximum deflection Δ, the total eccentricity of the load at the point of maximum deflection is $e_1 = e + \Delta$ (Fig. 113), and by considering the stress on the cross section at that point as the result of the eccentric load P, the maximum unit-stress from Art. 77 is

$$f = \frac{P}{A}\left(1 + \frac{ce_1}{r^2}\right) = \frac{P}{A}\left(1 + \frac{c(e + \Delta)}{r^2}\right),$$

in which e_1 must be calculated. Let P_0 be the load obtained by the use of Euler's formula for the column, and imagine it placed concentric with the column when the deflection is Δ, then the column will be in equilibrium under that load. As the column is in equilibrium under either the eccentric load P or the concentric load P_0, the moments at the danger section for both loads may be equated, hence

$$P_0\Delta = P(e + \Delta) \text{ or, } \Delta = \frac{Pe}{P_0 - P},$$

and $$e_1 = e + \Delta = e + \frac{Pe}{P_0 - P} = \frac{P_0 e}{P_0 - P}$$

and $$f = \frac{P}{A}\left(1 + \frac{cP_0 e}{(P_0 - P)r^2}\right).$$

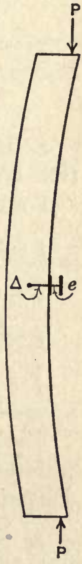


FIG. 113.

This formula for eccentric loads on columns may be used for long columns for which Euler's formula could be applied for axial loads.

EXAMPLES

1. If two 8-inch, 18-pound I-beams, latticed together so that the distance between their centroids is 6½ inches, are used as a

column 20 feet long with both ends round, what is the unit-stress developed by an axial load of 40 tons?

By Rankine's formula

$$f = \frac{P}{A}\left(1 + \phi\left(\frac{l}{r}\right)^2\right),$$

$$P = 80{,}000 \text{ pounds}, \quad A = 10.66 \text{ square inches},$$

$$\phi = \frac{4}{25{,}000}, \quad l = 20 \times 12 = 240 \text{ inches},$$

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{113.8}{10.66}} = 3.27 \text{ inches}.$$

$$f = \frac{80{,}000}{10.66}\left(1 + \frac{4}{25{,}000}\left(\frac{240}{3.27}\right)^2\right) = 7500 \times 1.86$$

$$= 14{,}000 \text{ pounds per square inch}.$$

2. If the load in Example No. 1 is applied 2 inches from the centroid of the section what would be the maximum stress developed?

For this case the formula is

$$f = \frac{P}{A}\left(1 + \phi\left(\frac{l}{r}\right)^2 + \frac{ec}{r^2}\right),$$

$$c = (x + \text{width of flange}) \div 2 = (6.32 + 4) \div 2 = 5.18 \text{ inches}.$$

$$f = \frac{80{,}000}{10.66}\left(1 + \frac{4}{25{,}000}\left(\frac{240}{3.27}\right)^2 + \frac{2 \times 5.18}{(3.27)^2}\right),$$

$f = 7500\,(1 + .86 + .96) = 7500 \times 2.82 = 21{,}200$ pounds per square inch.

These results show that when a load is axial the stress may be within the safe limit, while a slight shifting of the load may cause dangerously high stresses.

3. Design a square timber column 10 feet long with one end fixed and the other end round to carry a load of 5000 pounds safely.

Let d be one dimension of the section, then $A = d^2$

$$r = \sqrt{\frac{d^4}{12 \div d^2}} = \frac{d}{\sqrt{12}} \text{ inches}, \quad l = 10 \times 12 = 120 \text{ inches}.$$

For Rankine's formula

$$\phi = \frac{1.78}{3000} \text{ and } f = 800 \text{ pounds per square inch}.$$

$$\therefore \frac{5000}{d^2} = \frac{800}{1 + \dfrac{1.78 \times 120 \times 120 \times 12}{3000 \times d^2}} = \frac{800\,d^2}{d^2 + 102.5},$$

$$8\,d^4 - 50\,d^2 - 5125 = 0.$$

Solving this as a quadratic equation,

$$d^2 = 29.4,$$
$$d = 5.4 \text{ inches.}$$

For Euler's formula, by using a factor of safety of 10,

$$5000 \times 10 = \frac{2.25 \times 9.87 \times 1,500,000 \times d^2}{\dfrac{120 \times 120 \times 12}{d^2}} = 192\, d^4,$$

$$d^4 = 260,$$
$$d = 4+ \text{ inches.}$$

As the slenderness ratio of the column is in the neighborhood of 100, the result obtained by Euler's formula is not so reliable as the other.

4. What will be the maximum stress developed in a cast iron column 4 inches in diameter and 12 feet 6 inches long, with both ends round, carrying a load of 7000 pounds placed $1\frac{1}{2}$ inches from the center of the end?

By the use of Rankine's modified formula for eccentric loading, we find the stress.

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{\pi d^4}{64 \times \dfrac{\pi d^2}{4}}} = \frac{d}{4} = 1 \text{ inch},$$

$l = 150$ inches, $A = 12.57$ square inches,

$$f = \frac{7000}{12.57}\left(1 + \frac{4}{5000}\left(\frac{150}{1}\right) + \frac{1.5 \times 2}{1}\right) = \frac{7000}{12.57}(1 + 18 + 3)$$
$$= 12,300 \text{ pounds per square inch.}$$

138. Behavior of Columns under Load. In columns of ordinary length used in construction the stresses set up by eccentricity of loading due to non-straightness, unevenness of bearing at ends, and other causes due to shop and erection processes, often are so great that the effect of the length of the column is almost negligible. This is especially true of columns built up of several parts (e.g., a column built up of two channels connected by lattice work). Due to bends in the component parts of such built-up columns, slip of rivets and other causes, the extreme fiber stress, even in short columns, may be as much as 50 per cent greater than the average stress.* Furthermore, in designing columns great care should be taken that they are not built up of so thin metal that there is danger of failure by "wrinkling" of plates under load. So much uncertainty exists as to the action of built-up columns that low stresses should be used in designing them, and care should be taken to see that any new column is not built up of parts relatively thinner and more liable to "wrinkling" failure than are the parts of existing successful columns.

A formula depending upon experimental data for its constants should be used only in designing columns similar to those from which the data were derived. For example, if a series of experiments is made upon columns of one shape of cross section, the data should not be relied upon in designing columns of a different shape of cross section, although the material and slenderness ratio may be the same. Whether the results of tests of *small* columns can be used for determining the allowable stresses in similar *large* columns is a disputed question among engineers. Such a procedure is sometimes necessary, and in such a case working stresses in the large columns should be low.

* See Bulletin No. 44 of the Engineering Experiment Station of the University of Illinois.

EXAMPLES

1. What should be the distance from center to center of two 8-inch, 18-pound I-beams latticed together for a column section to make the radii of gyration equal for the two principal axes? (Fig. 114.)

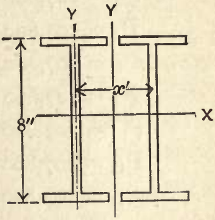


FIG. 114.

The moments of inertia about the X and Y axes must be made equal. From the table of properties of I-sections, the moment of inertia of one of the sections about the X-axis is 56.9 (in.)^4, and the moment of inertia about the Y'-axis is 3.78 (in.)^4, and the area of one section is 5.33 square inches. The moment of inertia of one section about the Y-axis is $\bar{I}+A\bar{d}^2$ and equals $3.78 + 5.33\left(\frac{x}{2}\right)^2$.

For both sections it is $2\left(3.78 + 5.33\left(\frac{x}{2}\right)^2\right)$. For equal moments of inertia of the built-up section about the X and Y axes,

$$2\,(56.9) = 2\left(3.78 + 5.33\left(\frac{x}{2}\right)^2\right),$$

$$\therefore x = 6.32 \text{ inches.}$$

2. A column built up of two 10-in., 15-lb. channels laced together with a distance of 6.33 inches between the backs, is 18 feet long. Determine the load the column will carry with eccentricities of 0 inch, 1 inch, 2 inches, 4 inches, 6 inches, 8 inches, 10 inches, 12 inches, and 14 inches, respectively, the point of application of the load being on the centroidal axis which is perpendicular to the web of the channel, and plot the curve showing the relation between the load and the eccentricity.

The moments of inertia about both principal axes are equal for the given spacing and $r = 3.87$ inches, $A = 8.92$ square inches, $l = 12 \times 18 = 216$ inches, $\frac{l}{r} = \frac{216}{3.87} = 55.8$, $c = \frac{6.33 + 2 \times 2.6}{2}$ $= 5.77$ inches, $\frac{c}{r^2} = \frac{5.77}{15} = .385$.

By the use of Ketchum's formula,

$$\frac{P}{A}\left(1 + \frac{ec}{r^2}\right) = 16{,}000 - 70\frac{l}{r} = 16{,}000 - 70 \times 55.8 = 12{,}100 \text{ pounds per square inch,}$$

$$P(1 + .385\,e) = 8.92 \times 12{,}100 = 107{,}800 \text{ pounds.}$$

$\therefore P_0 = 107{,}800$ pounds,

$P_1 = 107{,}800 \div 1.385 = 78{,}000$ pounds,

$P_2 = 107{,}800 \div 1.77 = 60{,}900$ pounds,

$P_4 = 107{,}800 \div 2.54 = 42{,}500$ pounds,

$P_6 = 107{,}800 \div 3.31 = 32{,}600$ pounds,

$P_8 = 107{,}800 \div 4.08 = 26{,}400$ pounds,

$P_{10} = 107{,}800 \div 4.85 = 22{,}200$ pounds,

$P_{12} = 107{,}800 \div 5.62 = 19{,}200$ pounds,

$P_{14} = 107{,}800 \div 6.39 = 16{,}900$ pounds.

These results are plotted in Fig. 115.

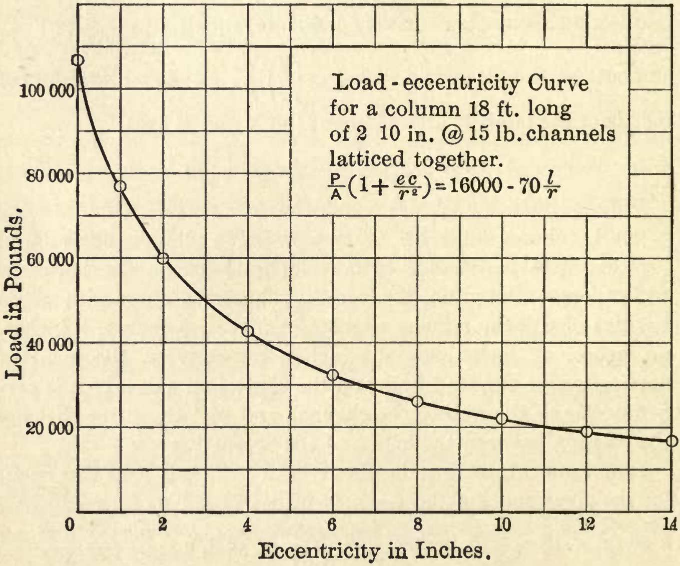


FIG. 115.

3. Design the upper chord of a roof truss in which the maximum stress is 65,100 pounds compression, and the length between supported points is 5 feet. Use two angles connected by $\frac{3}{8}$-in. gusset plates and $\frac{3}{4}$-in. rivets.

Since the member is in compression the rivet holes need not be deducted.

The least allowable r is $\frac{5 \times 12}{120} = \frac{1}{2}$ inch.

By use of a handbook we find the properties of the angles.

For direct compression alone the required area is 65,100 ÷ 16,000 = 4.07 square inches. For the column section the area must be somewhat greater. Try two 5-inch by 3-inch by $\frac{5}{16}$-inch angles placed with the short legs outstanding. The least r is 1.22 inches, $A = 4.82$ square inches. The allowable stress is $\frac{P}{A} = 16{,}000 - 70 \times \frac{60}{1.22} = 12{,}560$ pounds per square inch. The actual stress is $\frac{P}{A} = \frac{65{,}100}{4.28} = 13{,}500$ pounds per square inch.

This is not safe.

Try two 5-inch by $3\frac{1}{2}$-inch by $\frac{5}{16}$-inch angles.

The least r is 1.47 inches, $A = 5.12$ square inches.

The allowable stress is

$$\frac{P}{A} = 16{,}000 - 70 \times \frac{60}{1.47} = 13{,}140 \text{ pounds per square inch.}$$

The actual stress is

$$\frac{P}{A} = \frac{65{,}100}{5.12} = 12{,}750 \text{ pounds per square inch.}$$

This is safe.

The actual average unit-stress is nearly equal to the allowable, so use two 5-inch by $3\frac{1}{2}$-inch by $\frac{5}{16}$-inch angles with short legs outstanding.

4. Draw the diagrams representing the relation between the load and the length of columns of hemlock for the common rectangular sections.

By use of the formula $\frac{P}{A} = f - .0036 f \frac{l}{r}$ the values are obtained. From Table 8, $f = 1000$ pounds per square inch. For a 2-inch by 2-inch column, $A = 4$ square inches, $d = 3.46\ r$. The maximum length for which this section may be used is $l = 120\ r = 120\sqrt{\frac{bd^3}{12} \div bd} = 34.7\ d = 34.7 \times 2 = 69\frac{1}{2}$ inches $= 5$ feet, $9\frac{1}{2}$ inches.

When $\frac{l}{r} = 0$, $P = 4000$ pounds. When $l = 5$ feet $= 60$ inches,

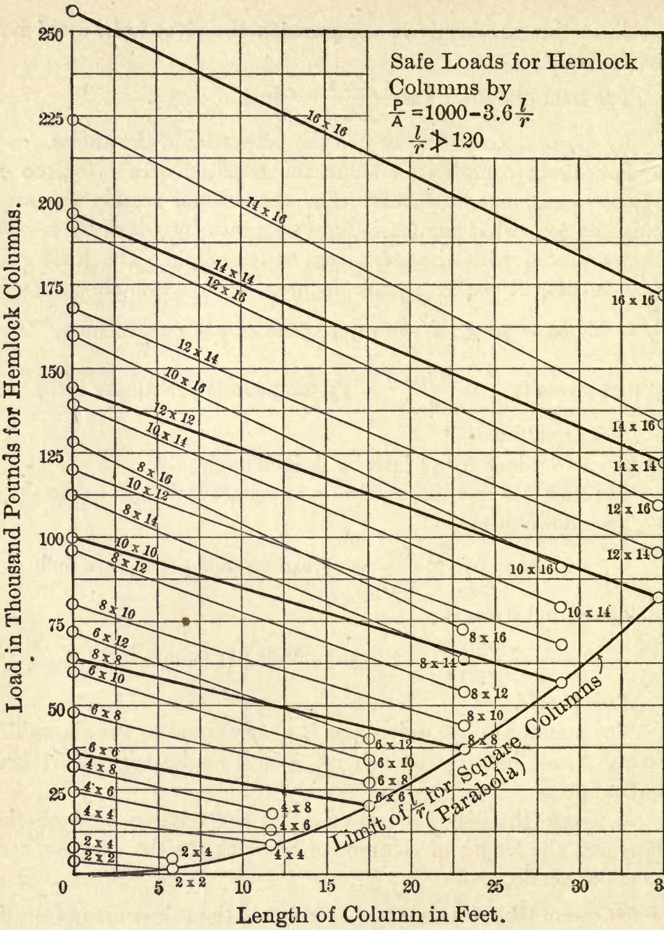


Fig. 116.

$\frac{l}{r} = \frac{60 \times 3.46}{2} = 103.8$, then $P = 4\,(1000 - .0036 \times 1000 \times 103.8)$
$= 2500$ pounds.

The other values given in the diagram, Fig. 116, were obtained in the same manner as those for a 2-inch by 2-inch column.

PROBLEMS

1. Determine the distance between the backs of two 9-in., 13.25-lb. channels latticed back to back, for equal radii of gyration.

Ans. 5.62 in.

2. What will be the radii of gyration with respect to the two principal axes of a column section built up of two 10-in., 25-lb. I-beams and two ¾-in. cover plates 12 inches wide?

3. In a compression test of specimens of different lengths of the same piece of red oak of cross section 1½ inches by 2 inches the following values were obtained:

Length, inches.	Maximum load, pounds.
6	22,000
12	18,000
24	14,200
36	8,000

Plot a curve showing the relation between the average unit-stress and the slenderness ratio, and determine the value of f_1 and C_1 at rupture, in the straight-line formula.

Ans. $f_1 =$ 8000 lb. per sq. in.
$C_1 =$ 63 lb. per sq. in.

4. Design a square longleaf pine column 14 feet long to carry a load of 6 tons.

5. Design a latticed column 18 feet long built up of two steel channels to carry a load of 20 tons.

6. What safe load will a hollow cast iron column 10 feet long carry if the outside dimensions are 6 inches by 7 inches and the inside dimensions are 4 inches by 5 inches?

7. Design a steel column 14 feet long to carry an eccentric load of 20 tons applied 2 inches from the outside of the column.

8. What should be the spacing of 2-inch by 4-inch timber posts 6 feet long to carry a platform on which the maximum load is to be 200 pounds per square foot?

9. Find the load by Rankine's formula that would probably rupture a cast iron column with fixed ends, 18 feet long and 6 inches in diameter.

10. By the use of Rankine's modified formula for eccentric loads on columns calculate the load that would develop a unit-stress of 1000 pounds per square inch in a 6-inch by 6-inch column 10 feet long with round ends for the following eccentricities: (a), 0; (b), 1 inch; (c), 2 inches; (d), 4 inches; and (e), 6 inches. Plot a curve showing the relation between the load and the eccentricity.

11. By use of the straight-line formula solve Problem No. 10 if the column is of Washington fir.

12. If a 12-in., 40-lb. I-beam 18 feet long is used as a column with round ends, what is the slenderness ratio? According to Euler's formula, what load would cause rupture?

13. What safe load will a column 27 feet long built up of two 9-in., 13.25-lb. channels latticed together and placed 6 inches back to back, safely carry if used in a bridge? *Ans.* 67,300 lb.

14. What should be the greatest length for which timber columns of the following sections may be used? 2 inches by 2 inches, 4 inches by 4 inches, 4 inches by 6 inches, 6 inches by 6 inches, 6 inches by 10 inches, 6 inches by 12 inches, 8 inches by 8 inches, 10 inches by 12 inches, 12 inches by 12 inches.

Ans. $l = 34.7$ where d is the least lateral dimension. 4 in. × 4 in., 11 ft., 7 in.; 4 in. × 6 in., 11 ft., 7 in.; 10 in. × 12 in., 28 ft., 11 in.

15. Determine the safe load for 4-inch by 4-inch red oak columns, which are 3 feet, 7 feet, and 11 feet, 7 inches long respectively. Plot a curve showing the relation between the load and the length of the column. Do the same for various other sections of oak columns carrying the curves to the maximum allowable length of column. (This set of curves may be made to include all commercial sizes of sections and put on one diagram. Then the red oak column necessary for any load and any length can be selected directly from the diagram.)

16. Determine the safe load for various lengths and various sections of columns of the different kinds of timber given in Table 8, and plot the curves as in Problem No. 15.

17. Design a strut 12 feet, 9 inches long in a roof truss to carry a compression load of 12,000 pounds. Use two angles with a $\frac{3}{8}$-in. gusset plate between them, and $\frac{3}{4}$-in. rivets.

Ans. 4 in. × 3 in. × $\frac{5}{16}$ in. ⌊s.

18. Design a strut 5 feet, 9 inches long in a roof truss to carry a load of 20,800 pounds.

Ans. Two $2\frac{1}{2}$ in. × 2 in. × $\frac{5}{16}$ in. ⌊s, short legs outstanding, $\frac{3}{8}$-in. gusset plate.

19. What four angles with the long legs outstanding would be required to be riveted to a $\frac{5}{16}$-in. plate for a column 18 feet long to carry a load of 27,370 pounds?

Ans. 4 in. × 3 in. × $\frac{7}{16}$ in. ⌊s, width of plate 8 in.

20. A wooden stick 3-inch by 4-inch in cross section and 10 feet long is used as a column with fixed ends. Find by Rankine's formula the unit-stress developed under a load of $\frac{1}{2}$ ton.

21. Find the safe load for a hollow cast iron column of outside dimensions 8 inches by 6 inches, inside dimensions 6 inches by 4 inches and 12 feet long.

22. A hollow yellow pine column of square section, 5 inches outside dimensions, and 4 inches inside dimensions, has a length of 16 feet. What load could the column safely carry?

23. A cylindrical steel column with round ends is 36 feet long and 6 inches in diameter. Calculate by Euler's formula the axial load that would probably produce rupture.

24. Determine the safe load for a hollow round cast iron column of external diameter 12 inches, thickness 1 inch, and length 12 feet.

25. A square white oak column 12 feet long is to support a load of 16 tons. What must be the size of the column?

26. Determine the size of a rectangular loblolly column 20 feet long to carry safely a load of 24 tons. *Ans.* 8 in. by 10 in.

27. A round solid cast iron strut 15 feet long carries a load of 10 tons. What should be its diameter?

CHAPTER XV

TORSION

139. Stress and Deformation. Round Shafts. When a couple, as indicated by *Pa* in Fig. 117, in a

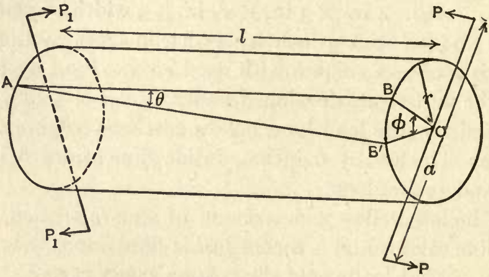


Fig. 117.

plane perpendicular to the axis of a shaft acts upon the shaft, it is twisted, and one cross section tends to slip by the section next to it. This tendency is resisted by the torsional stresses set up in the shaft. The stresses developed are shearing stresses. If *AB* in Fig. 115 is the original position and *AB′* the final position of an element of the surface of the shaft, the end of the shaft has twisted through the angle ϕ or *BOB′*, which is proportional to the couple acting on the shaft and to the length of the shaft, when the stresses developed are within the elastic limit. The element will have twisted through the angle θ or *BAB′*, which is proportional to the couple but independent of the length of the shaft.

140. THE TORSION FORMULA. ROUND SHAFTS. In Fig. 118 let the forces producing the couple be P and the arm between them be p, then the couple C equals Pp. Under the influence of this couple the radius OA will have swept through the angle AOA' to the position OA' while it still remains a straight line. The deformation

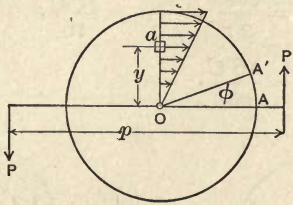


FIG. 118.

of a fiber of the section is proportional to the distance from the center O of the shaft to the fiber. Therefore the unit-stress developed on the fibers when the greatest stress is below the elastic limit is proportional to the distance of the fiber from the center. If s is the maximum unit-stress developed upon the outer fiber of the shaft, and r is the radius of the shaft, the unit-stress on the fiber a distance y from the axis is $s_y = s\frac{y}{r}$. The total stress on the elementary area a is $\frac{s}{r}\,ya$, and the moment of this stress about the axis of the shaft is $\frac{s}{r}\,yay = \frac{s}{r}\,ay^2$. The moment of the stresses acting on the entire cross section is the sum of all such expressions, and for equilibrium,

$$\sum M_0 = Pp - \sum \frac{s}{r} ay^2 = 0,$$

$$\therefore \quad C = \frac{sJ}{r}, \text{ or } \quad s = \frac{Cr}{J},$$

where $C = Pp$ and is the twisting moment, and $\Sigma ay^2 = J$ and is the polar moment of inertia of the section about the axis.

For solid circular shafts, Fig. 119,

$$J = \frac{\pi d^4}{32} = \frac{\pi r^4}{2} \text{ (Appendix A)},$$

$$\therefore \quad C = \frac{s\pi r^4}{2r} = \frac{s\pi r^3}{2} = \frac{s\pi d^3}{16},$$

$$s = \frac{16C}{\pi d^3}.$$

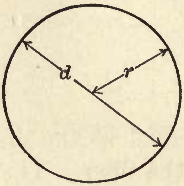


FIG. 119.

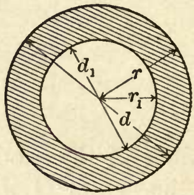


FIG. 120.

For hollow circular shafts of outer radius r and inner radius r_1, Fig. 120,

$$J = \frac{\pi (r^4 - r_1^4)}{2},$$

$$\therefore \quad C = \frac{s\pi (r^4 - r_1^4)}{2r} = \frac{s\pi (d^4 - d_1^4)}{16d},$$

$$s = \frac{16Cd}{\pi (d^4 - d_1^4)}.$$

Three typical problems may be investigated by the use of the torsion formula: (1) The investigation, (2) determining the allowable couple, and (3) the design of a shaft to transmit a given couple.

141. STIFFNESS OF SHAFTS. The relation between the angle of twist and the shearing modulus of elasticity may now be deduced. Since BB' in Fig. 121 is small, $BB' = r\phi = l\theta$, ϕ and θ being in radians. The detrusion of a fiber on the surface in the length l is BB', the

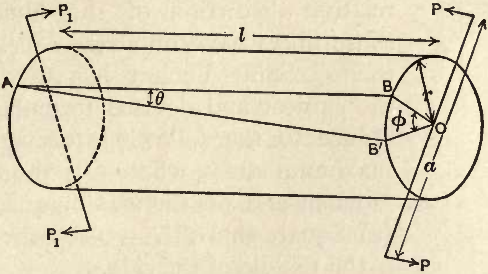


FIG. 121.

unit detrusion is $\frac{BB'}{l} = \frac{r\phi}{l}$. The shearing modulus of elasticity is

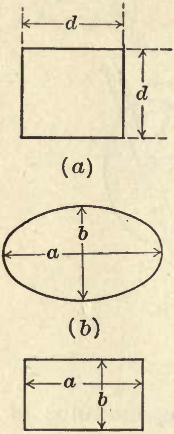
$$E_s = \frac{\text{Unit-stress}}{\text{Unit detrusion}} = \frac{s}{\epsilon} = \frac{\frac{Cr}{J}}{\frac{r\phi}{l}} = \frac{Cl}{J\phi},$$

$$E_s = \frac{Cl}{J\phi}, \text{ or } \phi = \frac{Cl}{JE_s}.$$

In the formulas, E_s is in pounds per square inch, C is in pound-inches, l is in inches, J is in (inches)^4, and ϕ is in radians. In tests, if the angle of twist is measured in degrees the value must be reduced to radians by the relation

One radian = 57.3 degrees.

142. OTHER SHAPES OF CROSS SECTION OF SHAFTS. For any other than circular sections the foregoing formulas cannot be applied. Experiment has shown that if the section has two axes of symmetry the fibers at the ends of the shorter axis have the greatest distortion, and consequently the greatest unit-stress will occur at those points. Along the corners of rectangular shafts there is no relative distortion of the fibers, and those fibers have no stress developed in them. Saint Venant has investigated the subject and devised formulas which reduce to the following forms for the maximum stress, where C is the twisting moment and s is the maximum stress:



(a) (b) (c)
Fig. 122.

(a) Square shaft,* $C = 0.208\, d^3 s$. s is at the middle of the side.

(b) Elliptical shaft,* $C = \frac{\pi}{16} ab^2 s$. s is at the end of the shorter axis.

(c) Rectangular shaft,* $C = \left(\frac{a^2 b^2}{3a + 1.8b}\right) s$. s is at the middle of the longer side.

Merriman† gives for rectangular cross sections the formula $C = \frac{2}{9} ab^2 s$.

143. Power Transmitted by Shafts. The primary purpose of shafting is the transmission of power. The pulleys are frequently fastened to the shaft by keys and keyways, in which case the formula for the relation between the maximum stress and the twisting moment is complex. However, the power a circular shaft without a keyway will transmit can easily be obtained if the allowable stress is known. If C is the couple acting on the shaft the work done by turning the shaft through an angle θ is $C\theta$. Proof: Let P be the force of the couple

* See "History of Elasticity," Vol. II, part 1, by Todhunter and Pearson.

† See Merriman's "Mechanics of Materials."

and p the arm. The distance through which P will move in turning through an angle θ is $p\theta$, and the work done is $Pp\theta$, or $C\theta$, as $Pp = C$. If the shaft makes N revolutions per minute the work done in one minute will be $C2\pi N$.

$$\text{H.P.} = \frac{C2\pi N}{33{,}000 \times 12} = \frac{sJ2\pi N}{396{,}000\, r} = \frac{sJN}{63{,}030\, r}.$$

144. **COMBINED TWISTING AND BENDING.** If a bending moment is developed in the shaft as well as a twisting moment, there is a combination of stresses. The maximum fiber stress developed by the bending moment may be obtained by the use of the moment formula $f = \frac{Mc}{I}$, and the maximum shearing stress may be obtained from the torsion formula $s = \frac{Cr}{J}$. These stresses may be combined to obtain the maximum shearing stress and the maximum tensile or compressive stress by the formulas

$$s_p = \sqrt{\left(s^2 + \left(\frac{f}{2}\right)^2\right)},$$

$$f_n = \frac{f}{2} \pm \sqrt{\left(s^2 + \left(\frac{f}{2}\right)^2\right)}.$$

EXAMPLES

1. A solid circular steel shaft 10 feet long and 2 inches in diameter has a couple of 126,000 pound-inches acting upon it. (a) What is the maximum unit-stress developed in the shaft? (b) What is the unit-stress ¾-inch from the axis of the shaft? (c) At 300 R.P.M. what is the horse power developed? (d) What is the angle through which one end would twist past the other? (e) Through what angle would a line on the surface twist?

(a) $$s = \frac{16C}{\pi d^3} = \frac{16 \times 12{,}600}{\pi\, 2^3} = 8000 \text{ lb. per sq. in.}$$

(b) $$s_y = \frac{8000}{1} \times \frac{3}{4} = 6000 \text{ lb. per sq. in.}$$

(c) $$\text{H.P.} = \frac{12{,}600 \times 2\pi \times 300}{33{,}000 \times 12} = 60 - \text{H.P.}$$

(d) $$\phi = 57.3 \times \frac{12{,}600 \times 10 \times 12}{12{,}000{,}000 \times 1.57} = 4^\circ\, 35',$$

(e) $$\theta = 1 \times (4^\circ\, 35') \div 120 = 2\tfrac{1}{4}'.$$

2. What should be the diameter of a solid shaft to transmit 500 horse power at 80 revolutions per minute if the maximum torsional stress is not to exceed 9000 pounds per square inch?

$$\text{H.P.} = \frac{sJN}{63{,}030\, r},$$

$$500 = \frac{9000\, \pi d^3 \times 80}{63{,}030 \times 16},$$

$$d = 6 \text{ inches.}$$

PROBLEMS

1. What maximum unit-stress will be developed in a hollow shaft of 3 inches outside and 2 inches inside diameter when twisted by a force of 3000 pounds at a distance of 1 foot from the axis? What is the minimum stress developed?

Ans. 8460 lb. per sq. in.

2. What horse power will be transmitted by the shaft in Problem No. 1 when making 90 revolutions per minute?

3. What must be the diameter of a solid steel shaft to transmit 120 horse power at 80 revolutions per minute if the allowable unit-stress is 10,000 pounds per square inch. *Ans.* 3.6 in.

4. If the shaft of Problem No. 2 is 20 feet long between the pulleys, what will be the angle of twist when transmitting the required power?

5. A wrought iron shaft 7 feet, 6 inches long and 2 inches in diameter twists through an angle of $10° 30'$ under the influence of a couple produced by a force of 2500 pounds at a distance of 1 foot from the axis. Compute the shearing modulus of elasticity.

6. What are the maximum shearing and tensile stresses developed in a shaft $2\frac{1}{2}$ inches in diameter under a twisting moment of 12,000 pound-inches and at the same time under a bending moment of 800 pound-feet?

7. What will be the maximum stress developed in a rectangular shaft of dimensions 1 inch by $1\frac{1}{2}$ inches if the twisting moment is 400 pound-feet?

8. Determine the maximum stress developed in a shaft 1 inch square if the twisting moment is produced by a force of 75 pounds at a distance of 14 inches from the axis.

9. What stress will be developed in an elliptical shaft of dimensions 1 inch by $1\frac{1}{2}$ inches if the twisting couple is 400 pound-feet?

10. What should be the diameter of a steel shaft to transmit safely 500 horse power at 150 revolutions per minute?

11. Calculate the horse power that a round, wrought iron shaft 8 inches in diameter and making 150 revolutions per minute will safely transmit.

12. A hollow steel shaft of outside diameter 6 inches safely transmits 450 horse power at 100 revolutions per minute. Find the inside diameter. *Ans.* $d_1 = 3.82$ in.

13. Find the shearing modulus of elasticity of a cast iron bar 10 inches long and 0.82 inch in diameter if twisted through an angle of $1.3°$ by a twisting moment of 50 pound-feet.

14. A structural steel shaft 120 feet long and 16 inches in diameter transmits 8000 horse power at 20 revolutions per minute. Find the angle of twist and the stress developed.

15. A solid shaft 6 inches in diameter is coupled by bolts 1 inch in diameter on a flange coupling. The centers of the bolts are 5 inches from the axis. Find the required number of bolts.

16. A wrought iron shaft is subjected simultaneously to a

bending moment of 10,000 pound-inches and a twisting moment of 12,000 pound-inches. Determine the least diameter of the shaft if the maximum tensile stress is not to exceed 10,000 pounds per square inch and the shearing stress is not to exceed 8000 pounds per square inch.

17. Find the horse power that can be transmitted safely by a cast iron shaft 3 inches in diameter and making 60 revolutions per minute.

18. A steel wire 0.18 inch in diameter and 10 inches long is twisted through an angle of 9.2° by a moment of 20 pound-inches. Determine the shearing modulus of elasticity of the wire.

CHAPTER XVI

REPEATED STRESSES, RESILIENCE, HYSTERESIS, IMPACT

145. **Repeated Stresses.** The behavior of materials under repeated stresses and impact is somewhat different from that for static or slowly applied stresses. The experiments of Wohler, Bauschinger, and others for repeated stresses show that a material will fail

TABLE 20

TESTS ON WROUGHT IRON

[*Wohler.*]

Number of applications.	Unit-stress producing rupture.
800	52,800
107,000	48,400
450,000	39,000
10,140,000	35,000

under stresses lower than the ultimate strength of the material. For an enormous number of applications of a stress about equal to the elastic limit, the material ruptured. When the stress was reversed and carried to about one-half to two-thirds the elastic limit for each reversal, an enormous number of applications of the stress caused rupture. These experiments were carried on in such a manner that the time between each application or reversal of stress was so short that the specimen had no time to rest. It is interesting to note in Table 20 the variation in the maximum applied stress with the number of applications for wrought iron. Fig. 123 shows graphically the number of applications of a given stress necessary to produce rupture in wrought iron.

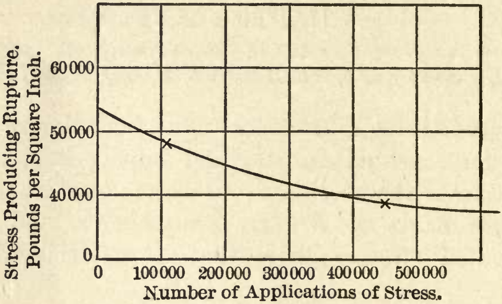


FIG. 123.

Instances in which repeated stress and reversed stresses would influence the design of the members would be shafting, car axles, piston rods, all rolling or vibrating members, etc.

146. RESILIENCE. When a load is applied to a member it will deform. On removing the load the member will resume its former size and shape for stresses below the elastic limit. And when the elastic limit has been exceeded the material will partly recover its original size and shape. The load applied to the material does work on it, and in turn when the load is being released the material gives out energy. **Resilience** is the amount of potential energy stored in a material when it is under stress. **Elastic resilience** is the amount of potential energy stored in a member when the stress is within the elastic limit. The **modulus of resilience** is the amount of energy stored in a unit of volume of a member when the stress is at the elastic limit. Resilience can be recovered to do work.

When the stress is carried beyond the elastic limit permanent set is developed. In such cases a larger amount of work has been done upon the specimen than it will give out upon releasing the load. The work that cannot be recovered is used in permanently distorting the material, and is converted into heat. Fig. 124 shows

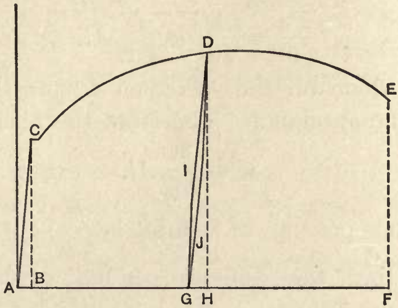


FIG. 124.

a typical soft steel stress-deformation diagram. In this diagram the ordinates represent the unit-stress and the abscissas represent the unit deformation. The work done on the material is the average force times the distance through which the force acts. Since the stress-deformation diagram shows the unit-stress developed in a specimen and the corresponding unit deformation under that stress, the area between the curve and the horizontal axis represents the work done on a unit of volume of the material. When the stress is not carried beyond the elastic limit all the work done can be recovered. The triangular area *ACB* represents the modulus of resilience. When the point *D* is reached the work done on a unit of volume of the material is represented by the area *ACDH*, and the work that can be recovered (the resilience) for that point is represented by the area *GJDH*. When the point of rupture *E* is reached, the total work done on a unit of volume of the material is represented by the area $ACDEF$.

147. RESILIENCE OF A BAR UNDER DIRECT STRESS. For tension or compression, let the load on the bar be P, the sectional area A, the length l, the deformation e. For stresses below the elastic limit the work done is

$$\frac{Pe}{2} = \frac{fAfl}{2E} = \frac{f^2}{2E}Al.$$

This work done on the specimen equals the resilience stored in the specimen. Therefore the resilience is

$$R = \frac{f^2}{2E}Al.$$

The resilience per unit of volume is $\frac{f^2}{2E}$. If f is equal to the elastic limit the resilience per unit of volume is the modulus of resilience.

148. RESILIENCE OF A BEAM. An expression for the resilience of a beam may be deduced similarly to the following method. Take the case of a cantilever beam of length l with a concentrated load W at the end, Fig. 125. The average force will be $\frac{W}{2}$ and the deflec-

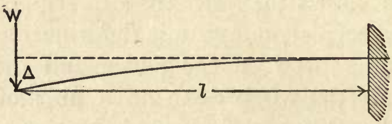


FIG. 125.

tion will be Δ. The work done on the beam which is equal to the resilience is

$$R = \frac{W}{2}\Delta = \frac{W}{2} \times \frac{Wl^3}{3EI}. \quad \text{(Art. 97.)}$$

$$\therefore \quad R = \frac{W^2l^3}{6EI}.$$

This can be expressed in terms of the maximum stress on the outer fiber from the formula

$$f = \frac{Mc}{I} = \frac{Wlc}{I}, \quad \text{or} \quad W = \frac{fI}{lc}.$$

$$\therefore \quad R = \frac{f^2 I^2 l^3}{6 l^2 c^2 EI} = \frac{f^2}{2E} \times \frac{Il}{3c^2}.$$

$\frac{f^2}{2E}$ is the same expression as obtained in Art. 147.

In the case of a uniform load each elementary load does an amount of work equal to one-half the load times the distance it deflects, and is $\frac{wu}{2}y$ where w is the load per unit of length, u is an element of length, and y is the deflection at the point (see Fig. 126). As the load is uniform the work done by each element is proportional to its deflection y. From Fig. 126 it is seen that uy is a small area between the X-axis and the elastic curve of the bent beam. The total work done is the summation of all such expressions as $\frac{wu}{2}y$ and equals

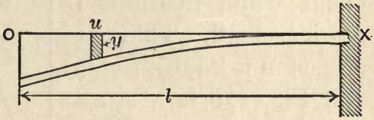


FIG. 126.

$$\sum \frac{wu}{2} y = \frac{w}{2} \sum uy = \frac{w}{2} \times \text{(area between the } X\text{-axis and the elastic curve)}$$

since $\Sigma\, uy$ is the area between the X-axis and the elastic curve. This area may be determined by the same method as is used in finding the deflection curves.

149. MECHANICAL HYSTERESIS. In Fig. 127 is shown the stress-deformation curve for the case where the elastic limit has been exceeded. After the point A had been reached the load was removed. The curve is convex downward, as ADC indicates. On reapplying the load the curve will be convex upward, as *CEA*. The resilience or work obtained from the material is *ABCD*, and that put into it is *ABCE*. The energy represented by the loop *ADCE* is lost as heat and is called **mechanical hysteresis.**

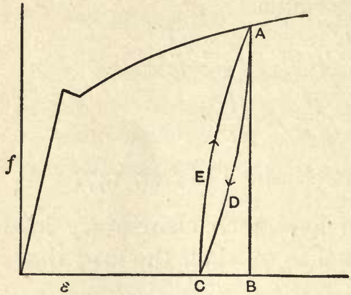


FIG. 127.

150. LAG. For some materials at stresses beyond the elastic limit when the load is stopped the specimen will continue to deform for some time. The metal yields while the load is not increased. This phenomenon is known as lag, and Fig. 128 is a stress-deformation diagram in which lag is shown.

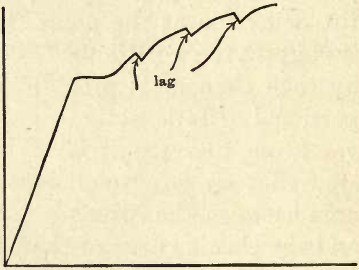


FIG. 128.

151. THE EFFECT OF REST. By allowing a specimen to rest after being stressed beyond the elastic limit, it will partly recover its elastic properties. Fig. 129 shows the stress-deformation curves for steel before and after resting. In the one marked "before resting" the stress had been carried beyond the elastic limit, and reversed several times, the specimen being heated by the work done on it.

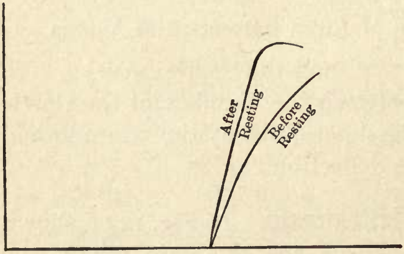


FIG. 129.

152. Suddenly Applied Loads. In the foregoing portion of the book the load was considered to be gradually applied to the specimen or member. If the load is suddenly applied the stresses are much higher than when the load is gradually applied. In order to get the relation between the stress produced by a gradually applied and a suddenly applied load let the deformation under the load gradually applied be e, and when suddenly applied be e_1. Having the deformation, the corresponding unit-stress developed can be determined, since the stress below the elastic limit is proportional to the deformation. The work done on the member when the load is gradually applied is equal to the product of the average force and the deformation and is $\frac{We}{2}$. The force varies from 0 to W. The area OAB in Fig. 130

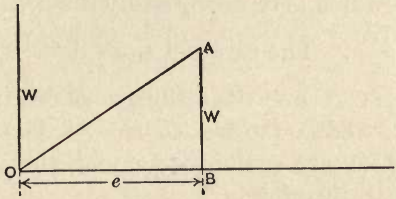


FIG. 130.

represents the work done. When the load is suddenly applied the total load acts through the entire deformation, as indicated by the line AB in Fig. 131, but the internal resisting stresses vary from zero to the value of FB along the line OB. When the point B is reached the external work done is We, while the work stored in the member or the resilience is $\frac{We}{2}$. According to the principle of the conservation of energy the load will not stop until the resilience equals the work done, consequently the deformation and the stress in the member will still increase. When the deformation is e the work done in excess to the resilience stored in the material is represented by the area OAB which equals $\frac{We}{2}$. Therefore the excess of resilience over the work done beyond

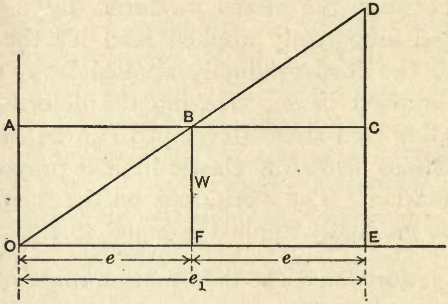


FIG. 131.

the point B which is represented by the area BCD, must be equal to $\frac{We}{2}$. The external work done is represented by the area $ACEO$ and the resilience stored in the material is represented by the area ODE. As the two triangles BOA and BDC are similar and equal, the similar sides must be equal, therefore,

$$BC = AB = e,$$
$$\therefore \quad e_1 = 2e,$$

and consequently $f_1 = 2f$ where f_1 is the stress due to the suddenly applied load and f is the stress developed when the load is gradually applied. This shows that **the deformation and the stress developed by a load when suddenly applied are twice what they would be if the load is gradually applied.**

153. **IMPACT LOADS.** A load W moving horizontally with a velocity v possesses the kinetic energy $\frac{Wv^2}{2g}$, which

is equal to Wh, where g is the acceleration due to gravity and h is the vertical distance the weight would fall to acquire the velocity v. This energy must be overcome by the resilience stored in the member. This energy may be equated to the resilience of the member for any given case to obtain the stress developed. For example, if the member is in direct tension or compression

$$Wh = \frac{f^2}{2E}Al,$$

$$f = \sqrt{\frac{2WhE}{Al}},$$

$$e_1 = \frac{fl}{E}.$$

A method more generally applied is to obtain the relation between the deformation the load would produce when gradually applied and the deformation produced under impact. The stress is proportional to the deformation. Let Q be the maximum total resisting force under the impact load and e_1 the deformation produced by the impact load. The work done by the resisting force is $\frac{Qe_1}{2}$ since the resisting force varies directly from zero to Q. This work is the resilience and equals the external work,

$$\therefore \quad Wh = \frac{Qe_1}{2}.$$

If the deformation under the static load W is e the following proportion results:

$$\frac{W}{e} = \frac{Q}{e_1}.$$

Solving these two equations for Q and e_1,

$$e_1 = \sqrt{2he},$$

$$Q = W\sqrt{\frac{2h}{e}}.$$

From these equations it is seen that **the deformation, and the resisting force, and the unit-stress developed, increase directly with the velocity of the load, or with the square root of the height *h*.**

154. **DROP LOADS.** If the impact load falls vertically onto a member through the height h before impinging upon it, the load also does work through the deformation of the member. Then, using the same nomenclature as given in the previous article we have,

$$\frac{Qe_1}{2} = W(h + e_1),$$

and

$$\frac{Q}{e_1} = \frac{W}{e}.$$

Solving these equations for e_1 and Q,

$$e_1 = e + \sqrt{2he + e^2},$$

and

$$Q = W + \frac{W\sqrt{2he + e^2}}{e}.$$

It is seen from these two equations that **a drop of a short distance develops a high stress compared with that developed under the static load *W*.**

EXAMPLE.

1. Find the amount of work necessary to stress a bar of wrought iron 5 feet long and 1 inch in diameter, from zero to the elastic limit 100 times.

$$P = 25{,}000 \times .7854 = 19{,}635 \text{ pounds.}$$

$$e = \frac{25{,}000 \times 5 \times 12}{25{,}000{,}000} = .06 \text{ inch} = .005 \text{ feet.}$$

$$\text{Work} = \frac{1}{2}PeN = \frac{19{,}635}{2} \times .005 \times 100 = 4909 \text{ ft.-lb.}$$

2. If a force of 50 pounds is suddenly applied at the center of a 2-inch by 2-inch simple timber beam of 6-ft. span what will be the deflection and what will be the maximum stress developed?

$$I = \frac{bd^3}{12} = \frac{2 \times 8}{12} = \frac{4}{3}, \qquad c = 1.$$

The deflection and stress developed are the same as those developed by twice the static load, or 100 pounds.

$$\Delta_1 = \frac{100 \times 72 \times 72 \times 72 \times 3}{48 \times 1{,}500{,}000 \times 4} = .389 \text{ inch.}$$

$$f_1 = \frac{Mc}{I} = \frac{100 \times 72 \times 1 \times 3}{4 \times 4} = 1350 \text{ lb. per sq. in.}$$

3. If the weight in Example No. 2 falls 1 inch before impinging on the beam what stress will be developed and what will be the maximum deflection?

$$\Delta_1 = .194 + \sqrt{2 \times .194 \times 1 + .04} = .194 + .65 = .844 \text{ inch.}$$

The stress developed is proportional to the deflection and is

$$f_1 = 1350 \times \frac{.844}{.389} = 2920 \text{ pounds per square inch.}$$

Or the stress is the same as that developed by a static load of

$$Q = 50 + \frac{50\sqrt{2 \times .194 \times 1 + .04}}{.194} = 50 + 168 = 218 \text{ lb.}$$

$$f_1 = \frac{218 \times 72 \times 3}{4 \times 4} = 2920 \text{ pounds per square inch.}$$

PROBLEMS

1. What is the resilience stored in a cubic inch of the following materials when the stress is at the elastic limit (modulus of resilience)? (a) Wrought iron. (b) Structural steel.

Ans. 12.5 in.-lb.; 20.4 in.-lb.

2. What horse power is required to stress a structural steel rod 2 inches in diameter and 6 feet long from zero to the elastic limit 120 times per minute?

3. Solve Problem No. 2 if the stress is carried from one-half the elastic limit to the elastic limit each time.

4. If a load of 2000 pounds is suddenly applied to the end of a steel rod 3 feet long and 1.5 inches in diameter, what will be the deformation and the unit-stress developed?

5. If the load in Problem No. 4 is moving horizontally with a velocity of 5 feet per second at the instant of impinging on the rod, what deformation and unit-stress will be developed?

6. If the load in Problem No. 4 is falling with a velocity of 5 feet per second at the instant of impinging on the rod, what will be the deformation and the unit-stress developed?

7. What is the work required to deflect a 2-inch by 4-inch timber beam of 8-ft. span by a central load that will produce a maximum stress equal to 1200 pounds per square inch? Solve this problem for both cases, when the beam is on the edge and when it is lying flat.

8. If a load of 2 tons falls through a distance of ½ foot, and strikes at the center of a 10-in., 25-lb. **I**-beam of 16-ft. span, what deflection and stress will be developed?

9. A structural steel rod is required to support a suddenly applied load of 10,000 pounds. What is the minimum diameter of the rod if permanent set is avoided?

APPENDIX A

CENTROIDS AND MOMENTS OF INERTIA OF AREAS

A1. Such expressions as $\Sigma\, ay$ and $\Sigma\, ay^2$ will occur in finding the stresses developed in beams under load, where a is an element of area and y is the distance of that element from a reference line or axis. It is necessary to be able to evaluate these expressions for the various shapes of cross sections found in beams.

A2. **Centroids of Areas.** The **centroid of an area** is the point at which a very thin homogeneous plate

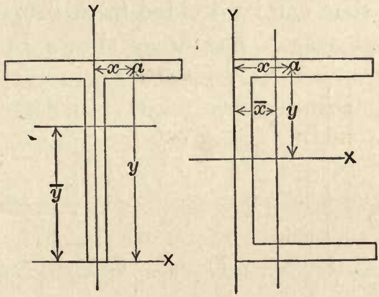


Fig. A1.

of the shape of the area would balance: it is the point at which, if the area were concentrated, its moment about any axis would be equal to the moment of the area as originally distributed. Calling $\bar{y}$ the distance from the X-axis to the centroid of the area A, a an element of the area, and y the distance of that element from the X-axis:

$$A\bar{y} = \Sigma\, ay,$$

$$\bar{y} = \frac{\Sigma\, ay}{A}.$$

And calling $\bar{x}$ the distance of the centroid of the area from the Y-axis, and x the distance of an element of the area from that axis,

$$x = \frac{\Sigma\, ax}{A}.$$

The axes may be chosen arbitrarily (Fig. A_1). For solids the term "centroid" is synonymous with "center of gravity," and the latter term is also frequently used with areas.

A_3. **Axis of Symmetry.** If a straight line can be drawn through an area dividing it into two exactly similar halves, that line is called an **axis of symmetry,** and an area that can be divided in this manner is called a **symmetrical area.** The areas shown in Fig. A_2 are

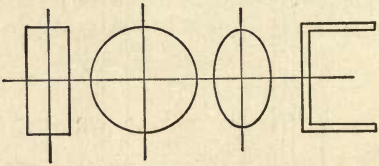


Fig. A_2.

symmetrical areas and the axes shown are axes of symmetry. If there is an axis of symmetry in an area, the centroid is located on that axis. This fact simplifies the solution for locating the centroids of a large number of areas.

A_4. **Centroid of a Triangle.** Imagine the triangle to be made up of a large number of strips of very small width parallel to the base. Each strip may be considered an element of the area. The centroid of any strip *CD*, Fig. A_3, is at the middle point of its length. The centroids of all the other strips parallel to the base come at the middle of their lengths. The line joining the centers of all these strips is a straight line and is called a median.

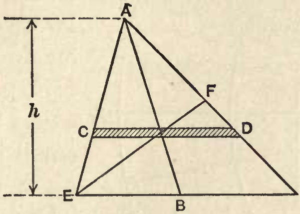


FIG. A_3.

The centroid of the entire triangle falls on the median *AB*, since the centroids of all its elements fall on that line. If the triangle is considered as being made up of strips parallel to another side it is shown by the same reasoning that the centroid of the triangle lies on another median. Therefore, the centroid of a triangle is at the intersection of the medians, which is at a distance of one-third the altitude from the base.

A_5. **CENTROID OF A SECTOR OF A CIRCULAR AREA.** The centroid of a circular sector may be located in the following manner: Let the angle at the center subtended by the radii be 2α, and r be the radius (Fig. A_4). Take the X-axis as the axis of symmetry. Then $\bar{y} = 0$. Consider the sector as being made up of a great number of triangular elements, as OAB. The distance of the centroid of the triangle from O is $\frac{2}{3}r$, and the distance from the Y-axis or x is $\frac{2}{3}r\cos\theta$. Draw the arc CED with radius equal to $\frac{2}{3}r$. The centroids of all elements of the sector fall on this arc. The total area of the sector may be represented by the length of the arc *CED*, and the area *OAB* may be represented in the same way by the arc *GF*. The centroid of the sector would be the same as if the entire area were concentrated on the arc *CED*. Draw *GH* perpendicular to the *X*-axis and *HF* parallel to that axis. Since *FG* is very small it may be taken as a straight line. The angle *FGH* equals θ. The moment of the area of the triangle *OAB* about the *Y*-axis is $x\ (GF) = \frac{2}{3} r \cos\theta\ (GF)$, and the moment of the total sector equals the summation of all such expressions, and is $\Sigma \frac{2}{3} r \cos\theta\ (GF)$. This sum divided by the total area gives $\bar{x}$. The area of the sector equals *CED*, which equals $\frac{2}{3} r\, 2\alpha = \frac{4}{3} r\alpha$,

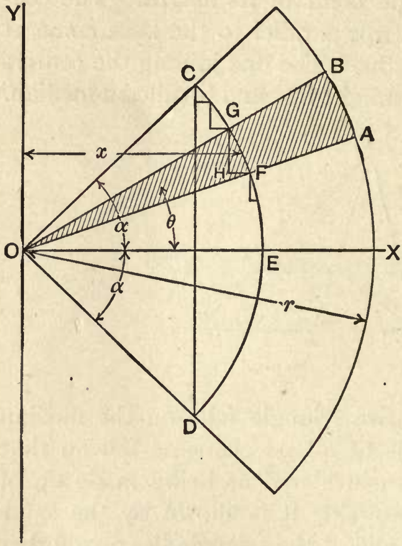


Fig. A4.

$$\bar{x} = \frac{\Sigma \frac{2}{3} r \cos\theta\ (GF)}{\frac{4}{3} r\alpha} = \frac{\Sigma\ (GF) \cos\theta}{2\alpha} = \frac{\Sigma\ (GH)}{2\alpha}.$$

$(GF) \cos\theta = (GH)$, and the summation of all such lengths equals

$$(CD) = 2 \cdot \tfrac{2}{3} r \sin\alpha = \tfrac{4}{3} r \sin\alpha,$$

$$\therefore\ \bar{x} = \frac{\Sigma\ (GF)\cos\theta}{2\alpha} = \frac{\frac{4}{3}\sin\alpha}{2\alpha} = \frac{2}{3} r\, \frac{\sin\alpha}{\alpha}.$$

The angle α must be expressed in radians in applying this formula.

When $\alpha = \frac{\pi}{2}$ the sector is a semicircle and $\bar{x} = \frac{4r}{3\pi}$.

A_6. CENTROIDS OF COMPOSITE AREAS. For such sections as often occur in practice, where they are built up of several different parts, or when the area may be divided into simpler areas, the centroid of the area can be obtained by applying the fundamental formula $\bar{y} = \frac{\Sigma ay}{A}$ to the section. The method is most readily understood from an example. Let it be required to locate the centroid of the channel section shown in Fig. A_5. Divide

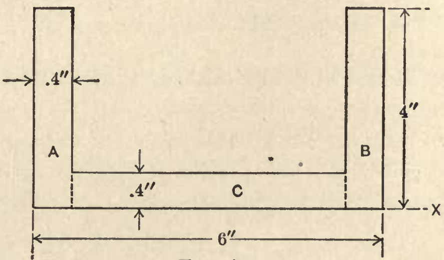


FIG. A_5.

the section into three rectangles *A*, *B*, and *C*. The following tabulated values are then found:

Part.	Area = *a*.	*y*	*ay*
A	4 × .4 = 1.6	2.0	3.200
B	4 × .4 = 1.6	2.0	3.200
C	5.2 × .4 = 2.08	0.2	0.416

$$A = 5.28 \text{ square inches,} \quad \Sigma ay = 6.816,$$

$$\bar{y} = \frac{6.816}{5.28} = 1.29 \text{ inch.}$$

Since the *Y*-axis is an axis of symmetry $\bar{x} = 0$.

Another method easily applied for certain sections results from subtracting moments. The solution of the above example by this method is to consider the whole rectangle 4 inches by 6 inches with the rectangle 3.6 inches by 5.2 inches cut away from the top as indicated

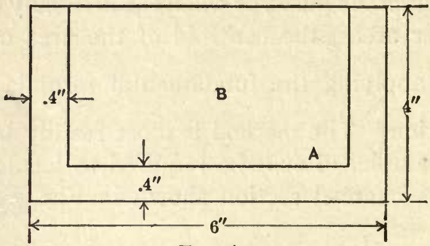


Fig. A_6.

in Fig. A_6. The following table is then obtained:

Part.	Area = *a*.	*y*	*ay*
A	6 × 4 = 24.00	2	48.000
B	5.2 × 3.6 = 18.72	2.2	41.184

$$A = 5.28,\quad \Sigma ay = 6.816,\quad \bar{y} = \frac{6.816}{5.28} = 1.29 \text{ inches.}$$

A_7. **Moment of Inertia.** The **moment of inertia** of an area is the summation of the products obtained by

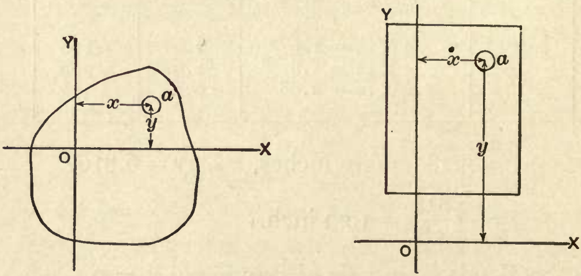


Fig. A_7.

multiplying each elementary part of the area by the square of its distance from an axis. The axis taken is called the **inertia axis.** Thus the moment of inertia of an area shown in Fig. A7 with respect to the X-axis is:

$$I_x = \Sigma\, ay^2,$$

and the moment of inertia with respect to the Y-axis is

$$I_y = \Sigma\, ax^2.$$

These expressions are for the moment of inertia of the area about axes in the plane of the area.

Since the moment of inertia is the product of an area and a length squared, the units in which it is expressed are $L^2 \times L^2 = L^4$, a length to the fourth power.

A8. **THE RADIUS OF GYRATION.** The **radius of gyration** with respect to an axis is defined as the square root of the quotient obtained by dividing the moment of inertia of the area with respect to the same axis by the area. Thus, if I is the moment of inertia and A is the area, the radius of gyration is

$$r = \sqrt{\frac{I}{A}}\cdot$$

It is seen that the radius of gyration gives the position for which a concentration of the area would give the same moment of inertia as is found for the distributed area. The value of r should not be confused with the distance to the centroid of the area.

A9. **POLAR MOMENT OF INERTIA. THE RELATION BETWEEN THE POLAR MOMENT OF INERTIA AND I_x AND I_y.** The moment of inertia of an area about an axis perpendicular to the plane of the area is the **polar moment of inertia** and is obtained by taking the sum of the products formed by multiplying each element of the area by the square of its perpendicular distance to the axis. If the axis is perpendicular to the plane at O in Fig. A8, the distance to an element is

$$\rho = \sqrt{y^2 + x^2}.$$

The polar moment of inertia of the area equals $\Sigma a\rho^2$. If J is the polar moment of inertia of the area about that axis,

$$J = \Sigma a\rho^2 = \Sigma a(y^2 + x^2) = \Sigma ay^2 + \Sigma ax^2,$$

$$\therefore J = I_x + I_y,$$

since $\Sigma ay^2 = I_x$ and $\Sigma ax^2 = I_y$.

Also $\dfrac{J}{A} = \dfrac{I_x}{A} + \dfrac{I_y}{A}$, $\therefore r_z^2 = r_x^2 + r_y^2$.

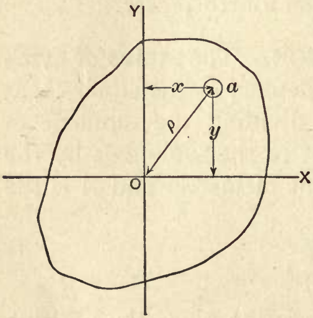


FIG. A8.

The polar moment of inertia of a plane area about an axis perpendicular to the plane of the area equals the sum of the moments of inertia of the area about two rectangular axes in the plane of the area intersecting the given axis. This is the relation between the moments of inertia about three mutually perpendicular axes, two of which lie in the plane of the area.

A10. **RELATION BETWEEN MOMENTS OF INERTIA ABOUT PARALLEL AXES IN THE PLANE OF THE AREA.** In Fig. A9 let O be the centroid of the area, $\bar{I}$ the moment of inertia of the area about the X-axis, and I' the moment of inertia of the area about the X'-axis at a distance $\bar{d}$ from the centroidal axis. Then

$$I' = \Sigma a(y + \bar{d})^2 = \Sigma a(y^2 + 2y\bar{d} + \bar{d}^2),$$

$$I' = \Sigma ay^2 + \Sigma a\bar{d}^2 + 2\bar{d}\,\Sigma ay,$$

$$\Sigma ay^2 = \bar{I},\quad \Sigma a\bar{d}^2 = A\bar{d}^2, \text{ and}$$

$$2\bar{d}\,\Sigma ay = 2\bar{d}A\bar{y} = 0, \text{ since } y = 0,$$

$$\therefore I' = \bar{I} + A\bar{d}^2. \quad \text{And } \frac{I'}{A} = \frac{\bar{I}}{A} + \frac{A\bar{d}^2}{A}, \quad \therefore r^2 = \bar{r}^2 + \bar{d}^2.$$

The moment of inertia of an area about an axis parallel to a centroidal axis in the plane of the area is equal to the

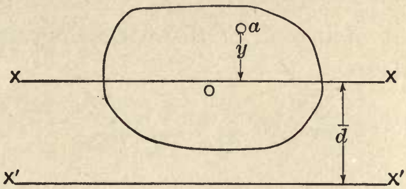


FIG. A9.

moment of inertia about the centroidal axis plus the area times the square of the distance between the two axes.

If the moment of inertia of an area about any axis is given, that for any other parallel axis can be obtained. First, the moment of inertia about the centroidal axis must be obtained by the formula $\bar{I} = I' - A\bar{d}^2$. Second, the moment of inertia about the parallel axis can be obtained by the use of the formula $I'' = \bar{I} + A\bar{d}'^2$.

A11. **The Moment of Inertia of a Parallelogram about a Centroidal Axis in the Plane of the Area.** The inertia axis is taken parallel to opposite sides. b is the breadth of the parallelogram and d is the depth perpendicular to the chosen axis, Fig. A10. Let the area be divided into a large number n, of equal strips parallel to the axis, each strip being taken so small in width that it is an element of the given area. The width of each strip is $\frac{d}{n}$, and the area of each strip is

$$a = \frac{d}{n} b.$$

Let the strip shown be the pth one from the axis in which p is any number up to $\frac{n}{2}$, then the distance from the axis to the element of the area shown is

$$y = p\frac{d}{n}\cdot$$

The moment of inertia of the parallelogram about the centroidal axis is

$$\sum ay^2 = \sum \frac{d}{n} bp^2 \frac{d^2}{n^2} = \sum \frac{db^2p^2}{n^3} = \frac{bd^3}{n^3}\sum p^2.$$

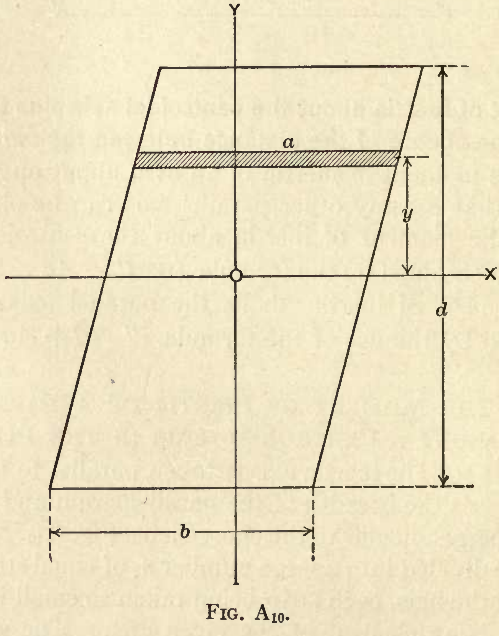


FIG. A10.

To obtain the moment of inertia for the area of the parallelogram above the axis, p must represent all numbers up to $\frac{n}{2}$. The same is true for the area below the axis, therefore,

$$\bar{I}_x = 2\frac{bd^2}{n^3}\sum\left(1^2 + 2^2 + \cdots + p^2 + \cdots + \left(\frac{n}{2}\right)^2\right).$$

From algebra,

$$\sum\left(1^2 + 2^2 + \cdots + \left(\frac{n}{2}\right)^2\right) = \frac{\frac{n}{2}\left(\frac{n}{2}+1\right)(n+1)}{6}$$

$$= \frac{n^3 + 3n^2 + 2n}{24},$$

$$\therefore \bar{I}_x = \frac{bd^3}{12} \times \frac{(n^3 + 3n^3 + 2n)}{n^3},$$

$$\bar{I}_x = \frac{bd^2}{12} \times \left(1 + \frac{3}{n} + \frac{2}{n^2}\right).$$

The greater n is made the more nearly is the true value for I obtained, and when n becomes infinitely great the exact value of the moment of inertia is obtained. For this condition $\frac{3}{n}$ and $\frac{2}{n^2}$ become zero,

$$\therefore \quad \bar{I}_x = \frac{bd^3}{12}.$$

The radius of gyration with respect to the centroidal axis is

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{bd^3}{12} \div bd} = \sqrt{\frac{d^2}{12}} = \frac{d}{2\sqrt{3}}.$$

The moment of inertia of a parallelogram about one of its sides is often needed, and by the application of the formula

$$I' = \bar{I} + A\bar{d}^2,$$

$$I_x = \frac{bd^3}{12} + bd\left(\frac{d}{2}\right)^2 = \frac{bd^3}{12} + \frac{bd^3}{4} = \frac{bd^3}{3}.$$

The corresponding radius of gyration is

$$r = \frac{d}{\sqrt{3}}.$$

The rectangle is the usual form of parallelogram for which the moment of inertia is needed.

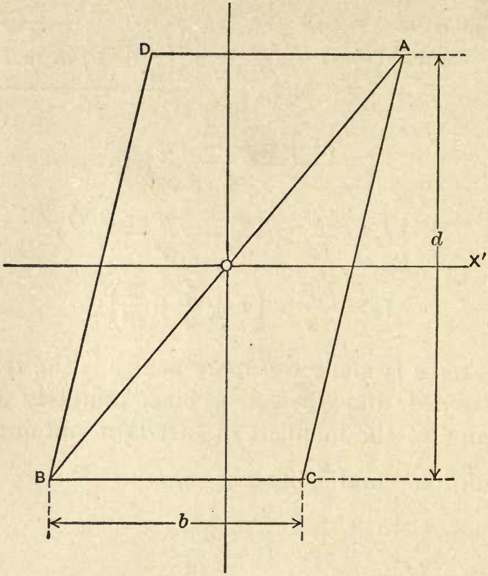


Fig. A11.

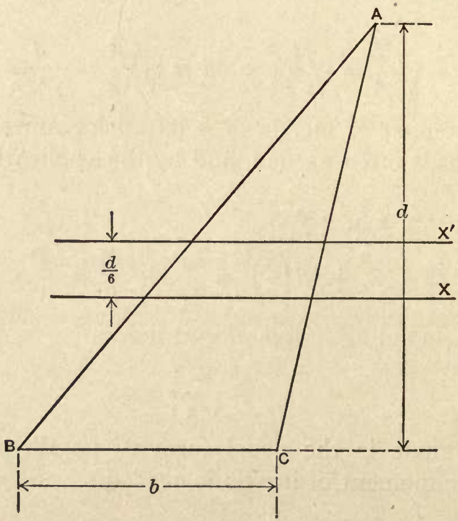


Fig. A12.

A12. **THE MOMENT OF INERTIA OF A TRIANGLE ABOUT ITS CENTROIDAL AXIS.** The moment of inertia of the parallelogram about the X'-axis, Fig. A11, is $\frac{bd^3}{12}\cdot$ The rectangle may be considered as being made up of two triangles ABC and ABD, both equal and similar. Consequently the moments of inertia of the two triangles about the X'-axis are equal. Therefore, the moment of inertia of one of the triangles about that axis is

$$I_x' = \frac{bd^3}{12} \div 2 = \frac{bd^3}{24}.$$

This axis is at the distance $\frac{d}{6}$ from the centroidal axis of the triangle (Fig. A12),

$$\therefore \quad \overline{I}_x = I_x' - A\overline{d}^2,$$

$$\overline{I}_x = \frac{bd^3}{24} - \frac{bd}{2} \times \frac{d^2}{36} = \frac{bd^3}{36}.$$

The corresponding value of the radius of gyration is

$$r = \frac{d}{3\sqrt{2}};$$

b is the base of the triangle and d is the altitude.

A13. **THE MOMENT OF INERTIA OF A CIRCULAR AREA.** Let d be the diameter of the circle. Let the area be divided into a great number, n, of elementary annular strips concentric with the entire area, Fig. A13. The width of each strip will be $\frac{d}{2n}\cdot$ Let the strip shown in Fig. A_{13} be the pth strip from the center, then the radius of this strip is

$$z = \frac{pd}{2n}.$$

The area of the element is

$$a = \frac{\pi pd}{n} \times \frac{d}{2n} = \frac{\pi pd^2}{2n^2}.$$

The polar moment of inertia of the entire area about the axis perpendicular to the area at O is

$$J = \sum az^2 = \sum \frac{\pi p d^2}{2\,n^2}\left(\frac{pd}{2\,n}\right)^2 = \frac{\pi d^4}{8\,n^4}\sum p^3,$$

in which p represents all numbers up to n.

$$\Sigma\, p^3 = \Sigma\,(1^3 + 2^3 + \cdots + p^3 + \cdots + n^3).$$

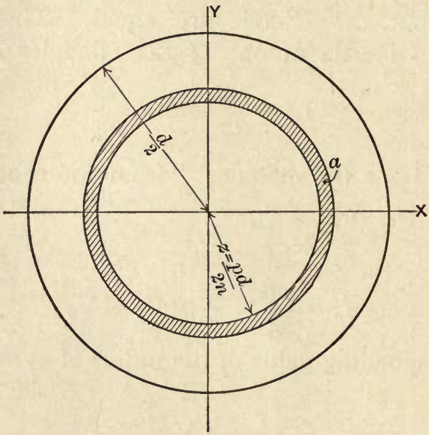


FIG. A13.

From algebra,

$$\Sigma\,(1^3 + 2^3 + \cdots + p^3 + \cdots + n^2)$$
$$= \frac{n^2\,(n+1)^2}{4} = \frac{n^4 + 2\,n^3 + n^2}{4},$$

$$\therefore\quad J = \frac{\pi d^4}{8}\left(\frac{n^4 + 2\,n^3 + n^2}{4\,n^4}\right) = \frac{\pi d^4}{8}\left(\frac{1}{4} + \frac{1}{2\,n} + \frac{1}{4\,n^2}\right).$$

Since n should be made infinitely great to obtain the true moment of inertia, $\frac{1}{2\,n}$ and $\frac{1}{4\,n^2}$ reduce to zero, and

$$J = \frac{\pi d^4}{32},$$

$$\bar{r}_z = \sqrt{\frac{J}{A}} = \frac{d}{2\sqrt{2}}.$$

From Art. A_9 $J = I_x + I_y$ and from the symmetry of the figure $I_x = I_y$,

$$\therefore \quad \bar{I}_x = \bar{I}_y = \frac{J}{2} = \frac{\pi d^4}{64},$$

$$r_x = \bar{r}_y = \frac{d}{4}.$$

I_x is the **moment of inertia of the circle about a diameter.**

A14. **MOMENT OF INERTIA OF COMPOSITE AREAS.** In order to obtain the moment of inertia of a built-up section (composite area) for a given axis the area should be divided into its simpler parts, and the moment of inertia of each part with respect to the given axis obtained. The moment of inertia of the entire area with respect to the axis equals the sum of the moments of inertia of its component parts with respect to the axis. The application can be understood by an example.

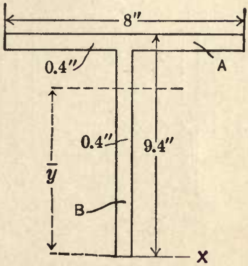


FIG. A14.

Let it be required to determine the moment of inertia and the radius of gyration of a **T**-section 8 inches by 9.4 inches by 0.4 inch with respect to a centroidal axis parallel to the flange of the **T**, Fig. A14.

By taking moments about the X-axis,

$$\bar{y} = \frac{8 \times .4 \times 9.2 + 9 \times .4 \times 4.5}{8 \times .4 + 9 \times .4} = 6.71 \text{ inches.}$$

For the part A,

$$I = \frac{bd^3}{12} + bd \cdot \bar{d}^2 = \frac{8 \times .064}{12} + 3.2 \times 2.49^2$$

$$= .04 + 19.84 = 19.88 \text{ inches.}$$

For the part B,

$$I = \frac{.4 \times 729}{12} + 3.6 \times 2.21^2 = 24.3 + 17.58$$
$$= 41.88 \text{ inches}^4.$$

Therefore, the moment of inertia of the section about the centroidal axis is

$$\overline{I} = 19.88 + 41.88 = 61.76 \text{ inches}^4.$$
$$\overline{r} = \sqrt{\frac{\overline{I}}{A}} = \sqrt{\frac{61.76}{6.8}} = 3.01 \text{ inches.}$$

EXAMPLE

1. Determine the moment of inertia and the radius of gyration with respect to the axis through the base and the centroidal axis, of a channel section, 4 inches by 6 inches by 0.4 inch, Fig. A_{15}.

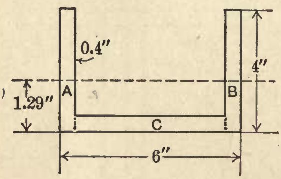


FIG. A_{15}.

The moment of inertia can be obtained for either axis and then transferred to the other, or it can be obtained for each one independently. That for the axis through the base will be obtained, then the transfer to the centroidal axis made.

Part.	Area.	y	$\overline{d}$	$A\overline{d}^2$	$\frac{bd^3}{12}$	I
A	1.6	2.0	2.0	6.4	2.133	8.533
B	1.6	2.0	2.0	6.4	2.133	8.533
C	2.08	0.2	0.2	.083	.027	.11
	5.28			12.883	4.293	17.176

$$I_x = 17.176 \text{ inches}^4\text{, say } 17.18\text{, and } r_x = \sqrt{\frac{17.18}{5.28}} = 1.8 \text{ inches.}$$

$\overline{d} = \overline{y} = 1.29$ inches. (Ex. p. 271.)

$$\overline{I} = I - A\overline{d}^2 = 17.18 - 5.28 \times 1.29^2 = 17.18 - 8.77$$
$$= 8.41 \text{ inches}^4 \text{ and } \overline{r} = \sqrt{\frac{8.41}{5.28}} = 1.26 \text{ inches.}$$

PROBLEMS

1. Find the distance of the centroid of a trapezoid with one base a, the other b, and the altitude h, from the base whose length is a.

$$Ans.\ \bar{y} = \frac{a + 2b}{3a + 3b}h.$$

2. Determine the moment of inertia and radius of gyration with respect to the X-axis and with respect to the Y-axis passing through the centroid of the area shown in Fig. A_{16}.

Ans. $\bar{I}_x = 59.3$ in.^{4}, $\bar{I}_y = 29.8$ in.^{4}, $\bar{r}_y = 1.68$ in.

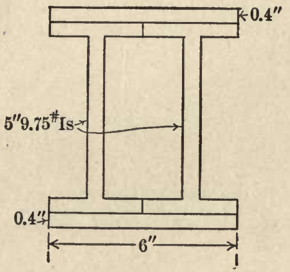


FIG. A16.

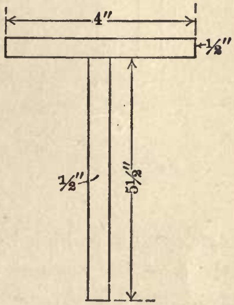


FIG. A17.

3. Locate the centroid and determine the moment of inertia and the radius of gyration with respect to the X and Y axes through the centroid for the T-section shown in Fig. A_{17}.

Ans. $\bar{y} = 4.01$ in., $\bar{I}_x = 17.4$ in.^{4}

4. Calculate the moment of inertia and radius of gyration of a circular area of diameter 4 inches with respect to the diameter and with respect to a tangent. Also find the polar moment of inertia with respect to the center.

5. A section is built up of two 15-in. 33-lb. channels placed back to back. What should be the distance between them to have the moments of inertia of the section equal with respect to the two rectangular axes passing through the centroid of the section? *Ans.* 9.5 in. from back to back.

6. Find the moment of inertia and radius of gyration with respect to the centroidal X and Y axes of the I-section shown in Fig. A_{18}.

7. A girder is built up of four 6-inch by 6-inch by 1-inch

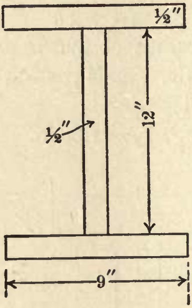


Fig. A_{18}.

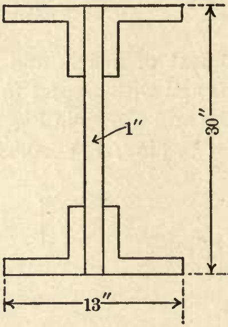


Fig. A_{19}.

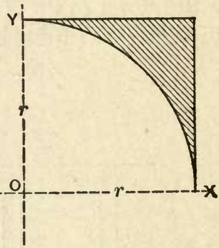


Fig. A_{20}.

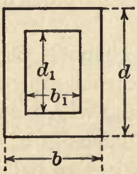
angles and a 30-inch by 1-inch plate; determine the moment of inertia of the section with respect to the centroidal X and Y axes. See Fig. A_{19}.

8. Locate the centroid of the shaded area shown in Fig. A_{20}, and find the moment of inertia with respect to the axis parallel to a side and passing through the centroid.

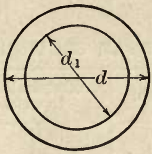
Ans. $\bar{x}=\bar{y}=.776\,r$ in., $I_x=I_y=.1368\,r^4$ in.^{4}, $\bar{I}_x=\bar{I}_y=.0075\,r^4$ in.^{4}

Prove that the moment of inertia of each of the following areas about the centroidal axis and the corresponding radius of gyration are as given:

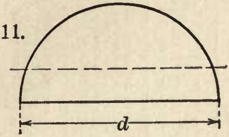
9.



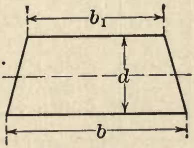
$$\bar{I}_x = \frac{bd^3 - b_1 d_1^3}{12}, \qquad \bar{r}_x = \sqrt{\frac{bd^3 - b_1 d_1^3}{12\,(bd - b_1 d_1)}}.$$

10. 

$$\bar{I}_x = \frac{\pi (d^4 - d_1^4)}{64}, \qquad \bar{r}_x = \frac{\sqrt{d^2 + d_1^2}}{4}.$$

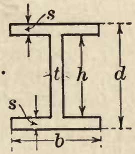
11. 

$$I_x = \left(\frac{9\pi^2 - 64}{1152\pi}\right) d^4, \quad \bar{r}_x = \left(\frac{\sqrt{9\pi^2 - 64}}{12\pi}\right) d.$$

12. 

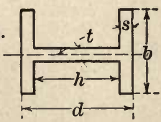
$$\bar{I}_x = \left(\frac{b^2 + 4bb_1 + b_1^2}{36(b + b_1)}\right) d^3,$$

$$\bar{r}_x = \frac{d}{6(b + b_1)} \sqrt{2(b^2 + 4bb_1 + b_1^2)}.$$

13. 

$$\bar{I}_x = \frac{bd^3 - h^3(b - t)}{12},$$

$$\bar{r}_x = \sqrt{\frac{bd^3 - h^3(b - t)}{12[bd - h(b - t)]}}.$$

14. 

$$\bar{I}_x = \frac{2sb^3 + ht^3}{12},$$

$$\bar{r}_x = \sqrt{\frac{2sb^3 + ht^3}{12[bd - h(b - t)]}}.$$

TABLE 21

PROPERTIES OF STANDARD LIGHT STEEL I-BEAM SECTIONS

Depth, Inches.	Weight per foot, Pounds.	Section area, Sq. in.	Axis perpendicular to web.			Axis parallel to web.	
			I Inches⁴.	*I/c* Inches³.	*r* Inches.	*I* Inches⁴.	*r* Inches.
24	80	23.32	2088.0	174.0	9.46	42.86	1.36
20	80	23.73	1467.0	146.7	7.86	45.81	1.39
20	65	19.08	1170.0	117.0	7.83	27.86	1.21
18	55	15.93	795.6	88.4	7.07	21.19	1.15
15	80	23.57	789.1	105.2	5.79	41.31	1.32
15	60	17.67	609.0	81.2	5.87	25.96	1.21
15	42	12.48	441.8	58.9	5.95	14.62	1.08
12	40	11.84	268.9	44.8	4.77	13.81	1 08
12	$31\frac{1}{2}$	9.26	215.8	36.0	4.83	9.50	1.01
10	25	7.37	122.1	24.4	4.07	6.89	0.97
9	21	6.31	84.9	18.9	3.67	5.16	0.90
8	18	5.33	56.7	14.2	3.27	3.78	0.84
7	15	4.42	36.2	10.4	2.86	2.67	0.78
6	$12\frac{1}{4}$	3.61	21.8	7.3	2.46	1.85	0.72
5	$9\frac{3}{4}$	2.37	12.1	4.8	2.05	1.23	0.65
4	$7\frac{1}{2}$	2.21	6.0	3.0	1.64	0.77	0.59
3	$5\frac{1}{2}$	1.63	2.5	1.7	1.23	0.46	0.53

TABLE 22

PROPERTIES OF STANDARD LIGHT STEEL CHANNEL SECTIONS

Depth, Inches.	Weight per foot, Pounds.	Section area, Sq. in.	Axis perpendicular to web.		Axis parallel to web.		
			I Inches⁴.	*r* Inches.	*I* Inches⁴.	*r* Inches.	$\bar{y}$ Inches.
15	33	9.90	312.6	5.62	8.23	0.91	0.79
12	$20\frac{1}{2}$	6.03	128.1	4.61	3.91	0.81	0.70
10	15	4.46	66.9	3.87	2.30	0.72	0.64
9	$13\frac{1}{4}$	3.89	47.3	3.49	1.77	0.67	0.61
8	$11\frac{1}{4}$	3.35	32.3	3.10	1.33	0.63	0.58
7	$9\frac{3}{4}$	2.85	21.1	2.72	0 98	0.59	0.55
6	8	2.38	13.0	2.34	0.70	0.54	0.52
5	$6\frac{1}{2}$	1.95	7.4	1.95	0.48	0.50	0.49
4	$5\frac{1}{4}$	1.55	3.6	1.56	0.32	0.45	0.46
3	4	1.19	1.6	1.17	0.20	0.41	0.44

TABLE 1

WEIGHTS OF VARIOUS MATERIALS USED IN CONSTRUCTION

Material.	Weight, lb. per cu. ft.	Material.	Weight, lb. per cu. ft.
Timber	25 to 45	Sandstone	150
Cast iron	450	Granite	170
Wrought iron	480	Marble	170
Steel	490	Slate	175
Brass	515	Terra cotta, facing	110
Copper, Bronze	550	Terra cotta, fireproofing	50
Aluminum	160	Book tile	60
Brick	100 to 150	Concrete	150
Limestone	165		

TABLE 2

ULTIMATE TENSILE STRENGTH AND ULTIMATE ELONGATION OF MATERIALS

Material.	Ultimate tensile strength, lb. per sq. in.	Ultimate elongation, per cent.
Timber	6,000 to 10,000	1.5
Cast iron	20,000	.3
Wrought iron	50,000	30.0
Structural steel	60,000	25.0 to 30.0
Steel wire	60,000 to 250,000	10.0 to 25.0

TABLE 3

ULTIMATE COMPRESSIVE STRENGTH OF MATERIALS

Material.	Ultimate compressive strength, lb. per sq. in.
Timber	7,000
Cast iron	90,000
Brick	6,000
Brick masonry	1,500
Rich concrete	2,500
Stone	10,000

TABLE 4

ULTIMATE SHEARING STRENGTH OF MATERIALS

Material.	Ultimate shearing strength, lb. per sq. in.
Timber:	
Along grain	400
Across grain	3,000
Cast iron	20,000
Wrought iron	40,000
Structural steel	50,000
Rivet steel	45,000

TABLE 5

ELASTIC LIMIT OF WROUGHT IRON AND STEEL

Material.	Elastic limit, lb. per sq. in.
Wrought iron	25,000
Structural steel	35,000
Hard steel	50,000

TABLE 6

MODULUS OF ELASTICITY

Material.	Modulus of elasticity, lb. per sq. in.
Timber	1,500,000
Cast iron	15,000,000
Wrought iron	25,000,000
Steel	30,000,000

TABLE 7

SHEARING MODULUS OF ELASTICITY

Material.	Shearing modulus of elasticity, lb. per sq. in.
Timber, across grain	400,000
Cast iron	6,000,000
Wrought iron	10,000,000
Steel	12,000,000

TABLE 8

SAFE WORKING STRESSES IN POUNDS PER SQUARE INCH FOR STEADY LOADS

Material.	Tension.	Shear.	Compression. Perpendicular to grain.	Compression. Parallel to grain.	Bending (fiber).
Timber:					
Cedar, white	800	100	180	1,100	1,000
Cypress	600	100	180	1,100	1,000
Elm	1,000	240	300	1,200	1,200
Fir, Washington	1,200	100	300	1,600	1,200
Gum	1,000	200	340	1,300	1,100
Hemlock	800	80	180	1,000	800
Larch	800	120	240	1,200	1,300
Maple, sugar (hard)	1,000	200	800	1,800	1,800
Maple (average)	800	160	500	1,400	1,200
Oak, red	900	160	500	1,200	1,200
white	1,000	200	600	1,750	1,400
Pine, longleaf	1,000	125	240	1,400	1,200
loblolly	....	100	200	1,000	1,000
shortleaf	....	100	200	1,200	1,100
yellow, (Ark., etc.)	800	100	200	1,200	1,000
Spruce	800	100	200	1,200	1,000
Cast iron	3,000	2,500	12,000		6,000
Wrought iron	12,000	9,500	12,000		12,000
Steel, structural	15,000	10,000	12,000		16,000
rivet	....	{ 8,000 / 10,000	18,000 (Bearing)		....
Brickwork (in lime)	....	....	110		....
Brickwork (in Portland cement)	....	....	250		....
Concrete (Portland cement)	....	....	350		....

TABLE 9

COEFFICIENTS OF EXPANSION PER DEGREE FAHR.

Material.	Coefficient of expansion.
Masonry	.0000050
Cast iron	.0000062
Wrought iron	.0000067
Steel	.0000065

TABLE 10

VALUES OF *K*, *K*,′ AND *C* FOR PIPES UNDER EXTERIOR PRESSURE

(For use in the Carman and Carr Formula.)

Material.	*K*.	*K′*.	*C*.
Cold-drawn seamless steel	50,200,000	95,520	2,090
Brass	25,150,000	93,365	2,474
Lap-welded steel	....	83,270	1,025

TABLE 11

EFFICIENCY OF JOINTS

Kind of joint.	Efficiency, per cent.
Single-riveted lap joint	50–65
Double-riveted lap joint	65–75
Single-riveted butt joint	65–75
Double-riveted butt joint	70–80

TABLE 12

RELATIVE STRENGTHS IN SHEAR AND BENDING

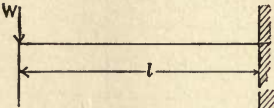
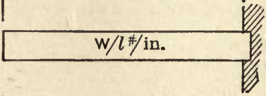
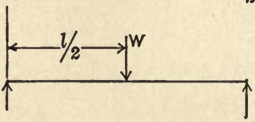
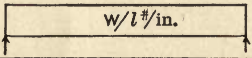
Kind of beam.	Maximum vertical shear.	Maximum bending moment	Relative strength in shear.	Relative strength in bending.
	W	$-Wl$	1	1
	W	$-\frac{Wl}{2}$	1	2
	$\frac{W}{2}$	$\frac{Wl}{4}$	2	4
	$\frac{W}{2}$	$\frac{Wl}{8}$	2	8

TABLE 13

MODULUS OF RUPTURE

Material.	Modulus of rupture lb. per sq. in.
Timber................	7000 to 9000
Cast iron..............	35,000

TABLE 14

MAXIMUM MOMENTS AND MAXIMUM DEFLECTIONS

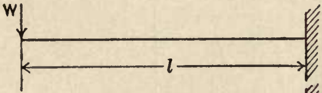
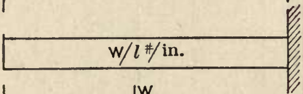
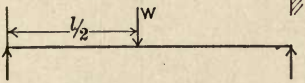
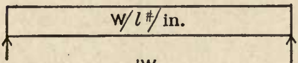
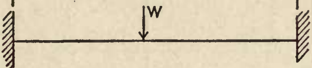
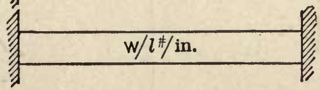
Kind of beam.	Maximum moment M.	Maximum deflection Δ.
	$-Wl$	$-\frac{Wl^3}{3EI}$
	$-\frac{Wl}{2}$	$-\frac{Wl^3}{8EI}$
	$\frac{Wl}{4}$	$-\frac{Wl^3}{48EI}$
	$\frac{Wl}{8}$	$-\frac{5Wl^3}{384EI}$
	$\pm\frac{Wl}{8}$	$-\frac{Wl^3}{192EI}$
	$-\frac{Wl}{12}, +\frac{Wl}{24}$	$-\frac{Wl^3}{384EI}$

TABLE 15

LOAD TO CAUSE A GIVEN MAXIMUM STRESS AND A GIVEN MAXIMUM DEFLECTION

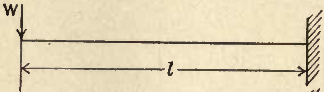
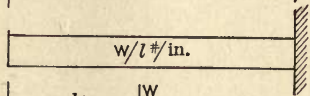
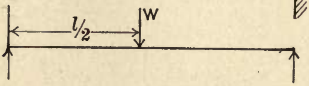
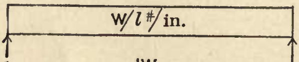
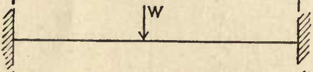
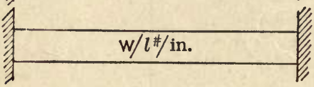
Kind of beam.	Load W to cause stress f.	Load W to cause deflection Δ
	$1 \ \frac{fI}{cl}$	$3 \ \frac{EI}{l^3}\Delta$
	$2 \ \frac{fI}{cl}$	$8 \ \frac{EI}{l^3}\Delta$
	$4 \ \frac{fI}{cl}$	$48 \ \frac{EI}{l^3}\Delta$
	$8 \ \frac{fI}{cl}$	$76\frac{4}{5} \ \frac{EI}{l^3}\Delta$
	$8 \ \frac{fI}{cl}$	$192 \ \frac{EI}{l^3}\Delta$
	$12 \ \frac{fI}{cl}$	$384 \ \frac{EI}{l^3}\Delta$
General type.	$\alpha \frac{fI}{cl}$	$\beta \frac{EI}{l^3}\Delta$

TABLE 16

COEFFICIENTS OF *wl* FOR THE VERTICAL SHEAR AT THE SUPPORTS OF CONTINUOUS BEAMS

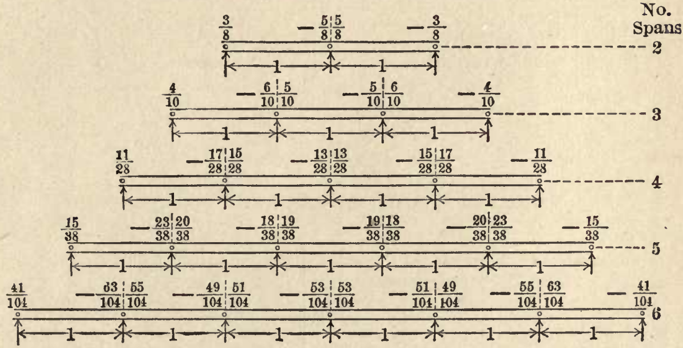


TABLE 17

COEFFICIENTS OF $-wl^2$ FOR THE BENDING MOMENT AT THE SUPPORTS OF CONTINUOUS BEAMS

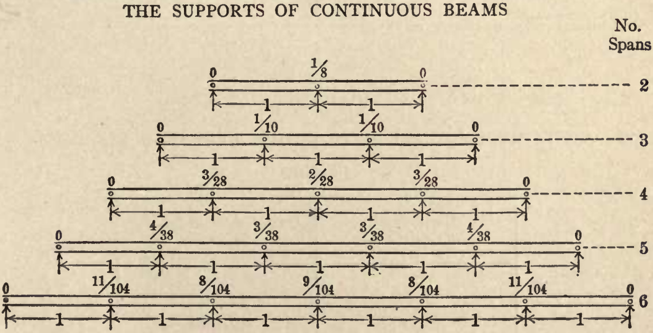


TABLE 18

MAXIMUM ALLOWABLE COMPRESSIVE STRESS IN POUNDS PER SQUARE INCH, CHICAGO BUILDING ORDINANCE, 1910.

Material.	Compressive stress, pounds per square inch.
Steel	14,000
Wrought iron	10,000
Cast iron	10,000

TABLE 19

VALUES OF ϕ USED IN RANKINE'S FORMULA

Conditions of the ends.	Timber.	Cast iron.	Wrought iron.	Steel.
Both ends round	$\frac{4}{3000}$	$\frac{4}{5000}$	$\frac{4}{36,000}$	$\frac{4}{25,000}$
Fixed and round	$\frac{1.78}{3000}$	$\frac{1.78}{5000}$	$\frac{1.78}{36,000}$	$\frac{1.78}{25,000}$
Both ends fixed	$\frac{1}{3000}$	$\frac{1}{5000}$	$\frac{1}{36,000}$	$\frac{1}{25,000}$

TABLE 20

TESTS ON WROUGHT IRON UNDER REPEATED STRESSES

[*Wohler.*]

Number of applications.	Unit-stress producing rupture.
800	52,800
107,000	48,400
450,000	39,000
10,140,000	35,000

TABLE 21

PROPERTIES OF STANDARD LIGHT STEEL I-BEAM SECTIONS

Depth, Inches.	Weight per foot, Pounds.	Section area, Sq. in.	Axis perpendicular to web.			Axis parallel to web.	
			I Inches[4].	I/c Inches[3].	r Inches.	I Inches[4].	r Inches.
24	80	23.32	2088.0	174.0	9.46	42.86	1.36
20	80	23.73	1467.0	146.7	7.86	45.81	1.39
20	65	19.08	1170.0	117.0	7.83	27.86	1.21
18	55	15.93	795.6	88.4	7.07	21.19	1.15
15	80	23.57	789.1	105.2	5.79	41.31	1.32
15	60	17.67	609.0	81.2	5.87	25.96	1.21
15	42	12.48	441.8	58.9	5.95	14.62	1.08
12	40	11.84	268.9	44.8	4.77	13.81	1 08
12	31½	9.26	215.8	36.0	4.83	9.50	1.01
10	25	7.37	122.1	24.4	4.07	6.89	0.97
9	21	6.31	84.9	18.9	3.67	5.16	0.90
8	18	5.33	56.7	14.2	3.27	3.78	0.84
7	15	4.42	36.2	10.4	2.86	2.67	0.78
6	12¼	3.61	21.8	7.3	2.46	1.85	0.72
5	9¾	2.37	12.1	4.8	2.05	1.23	0.65
4	7½	2.21	6.0	3.0	1.64	0.77	0.59
3	5½	1.63	2.5	1.7	1.23	0.46	0.53

TABLE 22

PROPERTIES OF STANDARD LIGHT STEEL CHANNEL SECTIONS

Depth, Inches.	Weight per foot, Pounds.	Section area, Sq. in.	Axis perpendicular to web.		Axis parallel to web.		
			I Inches[4].	r Inches.	I Inches[4].	r Inches.	$\bar{y}$ Inches.
15	33	9.90	312.6	5.62	8.23	0.91	0.79
12	20½	6.03	128.1	4.61	3.91	0.81	0.70
10	15	4.46	66.9	3.87	2.30	0.72	0.64
9	13¼	3.89	47.3	3.49	1.77	0.67	0.61
8	11¼	3.35	32.3	3.10	1.33	0.63	0.58
7	9¾	2.85	21.1	2.72	0.98	0.59	0.55
6	8	2.38	13.0	2.34	0.70	0.54	0.52
5	6½	1.95	7.4	1.95	0.48	0.50	0.49
4	5¼	1.55	3.6	1.56	0.32	0.45	0.46
3	4	1.19	1.6	1.17	0.20	0.41	0.44

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